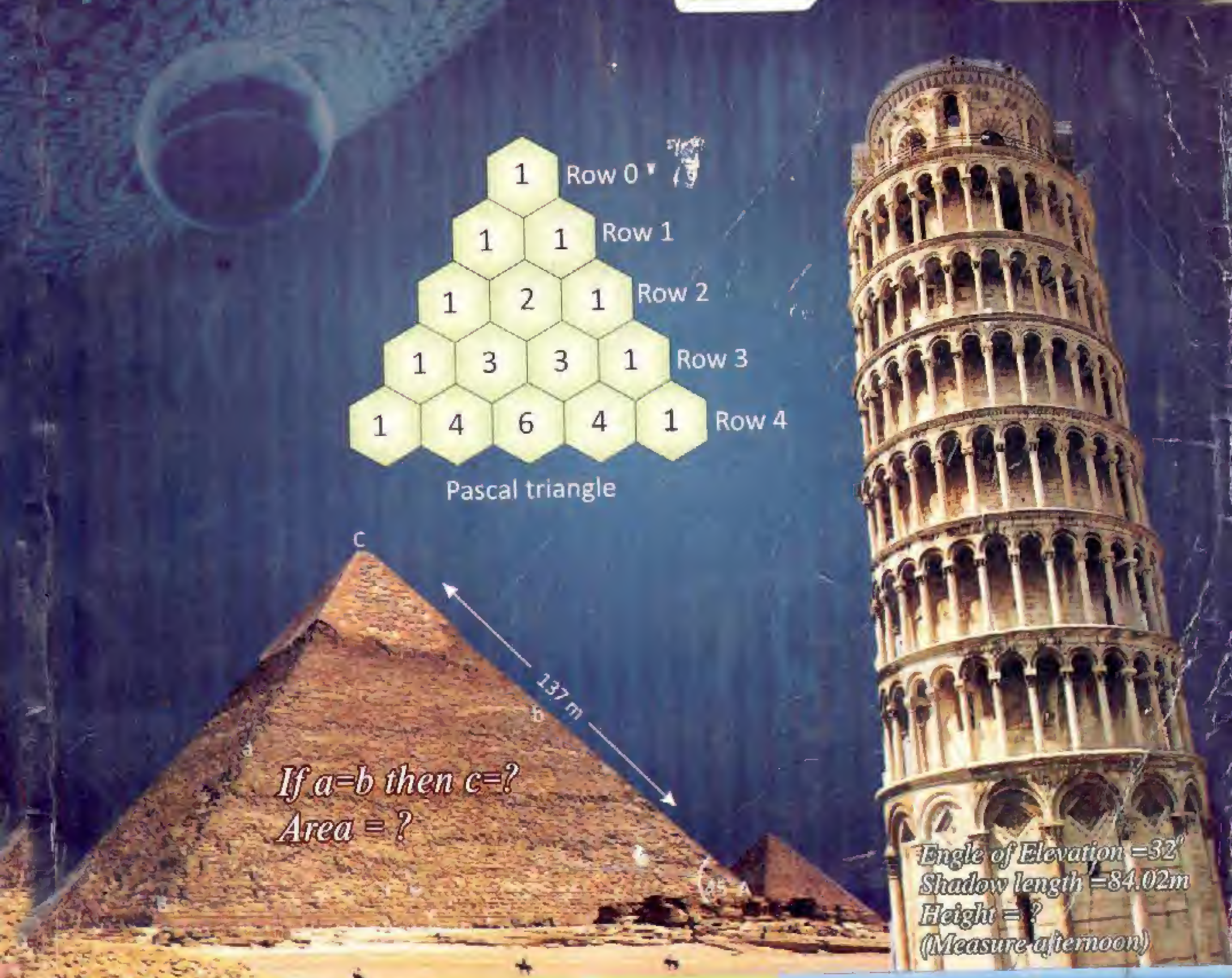
COLLEGE MATHEMATICS

Algebra & Trigonometry

NEW EDITION

11



Farukh Mahmood

Pearl Educational
College Publishing Co.

PREFACE

"If you have knowledge,let the others candle at it"

This book is specially designed by keeping in mind the demand of securing top class marks as well as the difficulties of an average student in understanding of Text Book. A significant feauture of this book is

- All important definitions .
- Formulas in the begening of every exercise.
- Complete and comprehensive notes of every chapter.
- Easy approach towards every solution.
- The questions are supported with comprehensive diagrams
- Each and every important question is highlighted.
- This book is a complete replacement of text book, students need not bother about text book when they have it.

Each chapter is provided with the important questions at its end.This book is a tremendous equalizer with its main focus to save students from any kind of perplexity and preparing them for the examenitions of all the boards of punjab and Federal. Underlying all the aspects, this book will prove to be a great asset, not just for students but for all knowledge seekers.

A special care has been made to avoid mistakes of every kind; therefore this note book has been read repeatedly so that before printing, all sorts of mistakes or shortcomings can be overcome. In this regard, I am highly indebted to Prof. Rafique Bloach, Prof. Nadeem Iqbal, Prof. Nasir Mushtaq, Prof. Javeed Kahoot, Prof. Farooq Khan, Prof. M. Farooq, Prof. Babar Zaheer, Prof. Sadaf Batool, Prof. Hina Sikander for exhaustive proof reading and giving their very valuable directions to keep the book according to the level of the students.

I am highly obliged to my worthy principla **Prof. Shaukat Ali** for his motivation and encouragment to write this book.

It is hoped that this book will serve the purpose well for which it has been compiled. I am a staunch believer of the fact that the students will certainly find a great boosting difference by comparing it with the other books.

Dedication

This book is dedicated to the sacred one Almighty who bestowed knowledge upon me, and endowed me with honour and esteem, and rendered me capacity and ability to toil and labour, no doubt I was ignorant and nameless.

To the Professors

This book will also be beneficial to our worthy teachers as this will make a speedy and quick overview to the lecture.

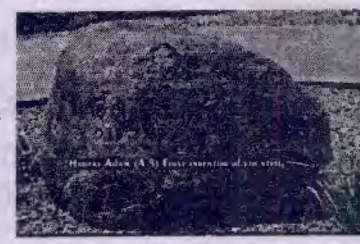
Moreover this will be helpful in pointing out and highlighting all the important definitions and questions.

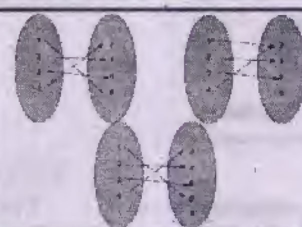
The questions at the end of the chapter are of M.C Qs, short and long questions type. Studying the Concepts reviews the content of the chapter and requires that students write out their answers. "Testing your skills" of the questions.

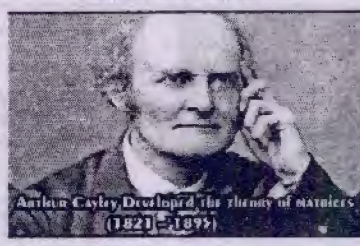
All the convincing comments and patronizingly forwarded Suggestions will be thankfully entertained for making this Book more effective.

Farukh Mahmood

Contents

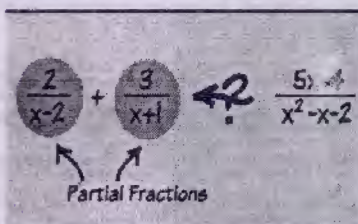






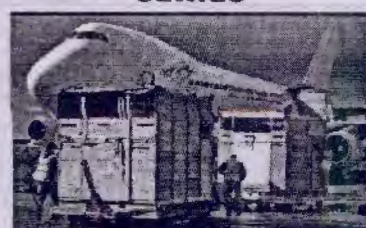
Arthur Cayley Developed the theory of matrices
(1821 - 1895)

ax²+bx+c=0



2/(x-2) + 3/(x+1)
5x-4/(x²-x-2)
Partial Fractions

ARITHMETIC SEQUENCE



Contents

Number System

1

Rational number:

A number which can be written in the form of $\frac{p}{q}$, where p and $q \in z$, $q \neq 0$ called a rational number e.g.; $\frac{1}{2}, \frac{3}{4}, \frac{-7}{2}$ etc. Also 0.21,0.510,0.999 etc are rational numbers as they can be written as $\frac{21}{100}, \frac{510}{1000}, \frac{999}{1000}$, etc.

Irrational number: **Multan 2010**

A number which can not be written in the form of $\frac{p}{q}$, where p and $q \in z$ $q \neq 0$ called an irrational number e.g.; $\sqrt{2}, \sqrt{3}, \sqrt{5}, \pi$ etc. All the square roots with prime number in it are the examples of irrational numbers.

Terminating decimal: **Rawalpindi 2009**

A decimal which has only a finite number of digits in its decimal part, is called a terminating decimal. e.g.; 202.04. 0.000225. 101.25704. 6.25 are examples of terminating decimals.

Since all the terminating decimals can be converted into common fractions as $202.04 = \frac{20204}{100}$ so every terminating decimal is a rational number.

Recurring Decimals: **Rawalpindi 2009**

A decimal in which one or more digits repeat indefinitely is called recurring decimal or a periodic decimal e.g, 1.3333........., 21.134134......... etc are recurring decimals. Such numbers can also be converted into their equivalent common fractions (see Q.6, Ex6.8, chapter6) So every recurring decimal is a rational number.

Non-terminating, Non-recurring Decimals:

A non terminating and non recurring decimal is a decimal which neither terminates nor it is recurring. It is not possible to convert such a decimal into a common fraction. Thus all non terminating and non recurring decimals represent irrational numbers.

For example.

7.3205080 .... (non terminating, non recurring) is irrational.

Example 2 Prove that $\sqrt{2}$ is an irrational number:

Sol. Suppose $\sqrt{2}$ is a rational number. Then it can be written in $\frac{p}{q}$, from. (where $p, q \in z \;\&\; q \neq 0)$ $i.e\ \frac{p}{q} = \sqrt{2}$, where p and q has no common factor.

$\Rightarrow\ p = \sqrt{2}q \Rightarrow p^2 = 2q^2 \rightarrow (i)$ Now R.H.S of (i) is a multiple of 2, therefore L.H.S Must also be a multiple of 2, so let. $P = 2P'$ (P' being an integer) put in (i), then

(i) $\Rightarrow (2p')^2 = 2q^2 \Rightarrow 4p'^2 = 2q^2$

$\Rightarrow 2p'^2 = q^2 \rightarrow (ii)$

Now L.H.S of (ii) is a multiple of 2, then R.H.S of (ii) is a multiple of 2, so let $q = 2q'$ (q' an integer) From the above discussion it is clear that $p = 2p'$ and $q = 2q'$. This shows that p and q have 2 as their common factor which is contradiction to the fact that p & q have no factor in common. Hence our supposition that $\sqrt{2}$ is rational, is wrong. Hence we conclude that is an irrational number.

Example 3 Prove that $\sqrt{3}$ is an irrational number: Lahore 2009

Sol. Suppose $\sqrt{3}$ is a rational number. Then it can be written in $\frac{p}{q}$, from

$(p, q \in z \text{ with } q \neq 0)$ $i.e\ \frac{p}{q} = \sqrt{3} \Rightarrow p = \sqrt{3q}$ (where p , q has no common factor)

$\Rightarrow\ p^2 = 3q^2 \rightarrow (i)\ (squaring)$

Now R.H.S (i) is a multiple of 3, therefore L.H.S must also be a multiple of 3, so let $q = 3p'$ (q' being an integer) From the above discussion it is clear that p and q has 3 as their common factor which is a contradiction to have no fact that in that $\sqrt{3}$ is rational is wrong. Hence we concluding that $\sqrt{3}$ is an irrational number.

Note: Using the above method, we can prove that $\sqrt{2}, \sqrt{7} \ldots\ldots \sqrt{n}$ are irrational numbers where n is prime.

Example 4 (i) a.0=0 **Multan**

Sol. a.0=a.[1+(-1)] additive inverse
=a.1+(-a.1) distributive law
=a+(-a)=0 additive inverse

(ii) ab=0 $\Rightarrow$ a=0 $\vee$ b=0

Sol. given that ab=0

Suppose a $\neq$ 0 then $\frac{1}{a}$ exist

Now $\frac{1}{a}(ab)=\frac{1}{a}.0 \Rightarrow (\frac{1}{a}.a)b=0 \Rightarrow \mathbf{1.b=0} \Rightarrow \mathbf{b=0}$

Similarly if b $\neq 0$ then a=0

Hence if ab=0 then a=0 $\vee$ **b=0**

Example 5

(i) **(-a)b=a(-b)=ab**

Sol. (-a)(b)+ab=(-a+a)b=0(b)=0

So (-a)b and ab are additive inverse

$\therefore$ (-a)b=a(-b)=-ab

(ii) **(-a)(-b)=ab**

Sol. (-a)(-b)-ab=(-a)(-b)+(-ab)= (-a)(-b)+(-a)b=(-a)(-b+b)=-a(0)=0 $\Rightarrow$ (-a)(-b)=ab

Properties of Real numbers:

Addition Properties

Multan 2010

(i) **Closure Property:** for all $a,b \in R, a+b \in R$ in other words sum of two real numbers is real number. **Faisalabad 2009**

(ii) **Commutative Property:** For all $a,b \in R, a+b=b+a$

(iii) **Associative Property:** For all $a,b,c \in R, (a+b)+c=a+(b+c)$

(iv) **Additive identity:** $O \in R$ is the additive identity of the set of real numbers such that $o+a=a+o=a, \ \forall a \in R$

(v) **Additive inverse:** If the sum of two numbers is zero, then two numbers are called additive inverse of each other e.g., additive inverse of 7 is – 7, etc.

Thus for all $a \in R$ such that $a+(-a)=(-a)+a=0$ so "a" and "– a" are inverse of each other.

Multiplication properties:

(i) **Closure Property:** for all $a,b \in \mathbb{R}, a \cdot b \in \mathbb{R}$ in other words product of two real numbers is real number. **Faisalabad 2009**

(ii) **Commutative Property:** For all $a,b \in \mathbb{R}, a \cdot b = b \cdot a$

(iii) **Associative Property:** For all $a,b,c \in \mathbb{R}, (a \cdot b) \cdot c = a \cdot (b \cdot c)$

(iv) **Multiplicative identity:** $1 \in \mathbb{R}$ is the multiplicative identity of the set of real numbers such that $1.a = a.1 = a \ \ \forall a \in \mathbb{R}$

(v) **Multiplicative inverse:** If the product of two numbers is one, then these two numbers are called multiplicative inverse of each other e.g., multiplicative inverse of

7 is $\frac{1}{7}$, etc. thus for all $a \in \mathbb{R}$ there is $\frac{1}{a} \in \mathbb{R}$ such that $a.\frac{1}{a} = \frac{1}{a}.a = 1$ so a and $\frac{1}{a}$ are multiplicative inverse of each other.

Distributive Laws:

Faisalabad 2009

For all $a,b,c \in \mathbb{R}$ $a.(b+c) = a.b + a.c$ (left distributive law)

$(a+b)c = a.c + b.c$ (Right distributive law)

Properties of Equality:

(i) **Reflexive property:** For all $a \in \mathbb{R}, a = a$, i.e, a number is always equal to itself.

(ii) **Symmetric property:** For all $a,b \in \mathbb{R}$ $if\ a = b \Rightarrow b = a$

(iii) **Transitive property:** For all $a,b,c \in \mathbb{R}$. $if\ a = b\ and\ b = c \Rightarrow a = c$

(iv) **Additive property:** For all $a,b,c \in \mathbb{R}$, $if\ a = b\ then\ a + c = b + c$

(v) **Multiplicative property:** For all $a,b,c \in \mathbb{R}$, $if\ a = b\ the\ a.c = b.c$

(vi) **Cancellation property w.r.t "+":** For all $a,b,c \in \mathbb{R}$, $if\ a + c = b + c \Rightarrow a = b$

(vii) **Cancellation property w.r.t "X" :** For all $a,b,c \in \mathbb{R}$, $if\ a.\ c = b.\ c \Rightarrow a = b$

Properties of inequalities

(i) **Trichotomy property:** $\forall a,b \in \mathbb{R}$ $either\ a = b\ or\ a > b\ or\ a < b$ Sargodha 2008

(ii) **Transitive property:** For all $a,b,c \in \mathbb{R}$

$(i)\quad a < b\ and\ b < c \Rightarrow a < c$ $\qquad (ii)\quad a > b\ and\ b > c \Rightarrow a > c$

(iii) **Additive property:** For all $a,b,c \in \mathbb{R}$

$(i)\quad if\ a > b \Rightarrow a + c > b + c$ $\qquad (ii)\quad if\ a < b \Rightarrow a + c < b + c$

(iv) **Multiplicative property:** For all $a,b,c \in R$, $with\ c > 0$

$(i)\ if\ a > b \Rightarrow ac > bc$ $\qquad (ii)\quad if\ a < b \Rightarrow ac < bc$

and for all $a,b, \ \in R\ with\ c < 0$

$(i)\ if\ a < b \Rightarrow ac > bc$ $\qquad (ii)\ if\ a > b \Rightarrow ac < bc$

This shows that if negative number is multiplied to both sides of an inequality then the inequality is reversed.

Note: If reciprocals of both sides of an inequality are taken then the sign of inequality changes e.g., $\frac{1}{2} > \frac{1}{4} \Rightarrow 2 < 4$

Exercise 1.1

1. **Which of the following sets have closure property w.r.t. addition and multiplication?**

(i) $\{0\}$

Sol Addition $0+0=0\in\{0\}$

Multiplication $0\times 0=0\in\{0\}$

$\{0\}$ closed w.r.t '+' and 'x'

(ii) $\{1\}$

Sol Addition $1+1=2\notin\{1\}$

Multiplication $1\times 1=1\in\{1\}$

Not closed w.r.t '+' but closed w.r.t 'x'

(iii) $\{0,-1\}$ **Sargodha 2009, Faisalabad 2008, Multan 2009**

Sol Addition

$0+0=0\in\{0,-1\}$

$(0)+(-1)=-1\in\{0,-1\}$

$(-1)+0=-1\in\{0,-1\}$

$(-1)+(-1)=-2\notin\{0,-1\}$

Not closed w.r.t '+'

Multiplication

$0\times 0=0\in\{0,-1\}$

$0\times(-1)=0\in\{0,-1\}$

$-1\times 0=0\in\{0,-1\}$

$(-1)\times(-1)=1\notin\{0,-1\}$

Not closed w.r.t. 'X'.

(iv) $\{1,-1\}$ **(Sargodha 2009, 2011 Faisalabad 2008, Multan 2008, 2009 Gujranwala 2009)**

Sol Addition $1+1 \quad =2\notin\{1,-1\}$

$1+(-1) \quad =0\notin\{1,-1\}$

$(-1)+1 \quad =0\notin\{1,-1\}$

$(-1)+(-1) \quad =-2\notin\{1,-1\}$

Not closed w.r.t '+'

Multiplication

$1\times 1 \quad =1\in\{1,-1\}$

$1\times(-1) \quad =-1\in\{1,-1\}$

$-1\times 1 \quad =-1\in\{1,-1\}$

$(-1)\times(-1)=1\in\{1,-1\}$

Closed w.r.t. 'X'.

2. **Name the properties used in the following equations. (letters, where used, represent real numbers).**

i. **4 + 9 = 9 + 4**

Commutative w.r.t '+'

ii. $(a+1)+\frac{3}{4}=a+(1+\frac{3}{4})$

Associative property w.r.t '+'

iii. $(\sqrt{3}+\sqrt{5})+\sqrt{7}=\sqrt{3}+(\sqrt{5}+\sqrt{7})$

Associative w.r.t '+'

iv. **4.1 + (-4.1) = 0**

Additive Inverse

v. **1000 x 1 =1000**

Multiplicative Identity

vi. **4.1 + (-4.1) = 0**

Additive Inverse

vii. $a-a \quad =0$

Additive Inverse

viii. $\sqrt{2}\times\sqrt{5} \quad =\sqrt{2}\times\sqrt{5}$

Commutative w.r.t. 'X'

ix. $a(b-c) \quad = ab - ac$

Left Distribution over Subtraction

x. $(x-y)z = xz - yz$

Right Distribution over subtraction

xi. $4\times(5\times 8) = (4\times 5)\times 8$

Associative w.r.t 'X'

xii. $a(a+b-d) = ab + ac - ad$

Left Distribution

3. Name the properties used in the following inequalities:

i. $-3 < -2 \Rightarrow 0 < 1$

Sol (Add 3 both sides)

Additive property

ii. $-5 < 4 \Rightarrow 20 < 16$

Sol (Multiplying b – 4)

Multiplicative property

iii. $1 > -1 \Rightarrow -3 > -5$

Sol (Add – 4)

Additive property

iv. $a < 0 \Rightarrow -a > 0$

Sol (Multiply by – 1)

Multiplicative property

v. $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$ $\boxed{\text{'x' } by \ \frac{1}{ab}}$

Sol Multiplicative property or Inverse.

vi. $a > b \Rightarrow -a < -b$

Sol Multiply by (– 1)

(Multiplicative property)

4. Prove the following rules of addition:

i. $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$

Sol L.H.S$= \frac{a}{c} + \frac{b}{c} = a\times\frac{1}{c} = b\times\frac{1}{c}$

$$= (a+b)\times\frac{1}{c}$$

$$= \frac{a+b}{c} = R.H.S$$

ii. $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

Sol L.H.S $= \frac{a}{b} + \frac{c}{d}$

$$= \frac{a}{b}\times 1 + \frac{c}{d}\times 1$$

$$= \frac{a}{b}\times\left(\frac{d}{d}\right) + \frac{c}{d}\times\left(\frac{b}{b}\right)$$

$$= \frac{ad}{bd} + \frac{bc}{bd}$$

$$= ad\times\frac{1}{bd} + bc\times\frac{1}{bd}$$

$$= (ad+bc)\times\frac{1}{bd}$$

$$= \frac{ad+bc}{bd}$$

5. Prove that $-\frac{7}{12} - \frac{5}{18} = \frac{-21-10}{36}$

Sol L.H.S$= -\frac{7}{12} - \frac{5}{18}$

$$= -\frac{7}{12}\times 1 - \frac{5}{18}\times 1$$

$$= -\frac{7}{12}\times\frac{3}{3} - \frac{5}{18}\times\frac{2}{2}$$

$$= \frac{21}{36} - \frac{10}{36}$$

$$= (-21-10)\times\frac{1}{36}$$

$$= \frac{-21-10}{36} = R.H.S$$

6. Simplify by justifying each step:

(i) $\dfrac{4+16x}{4}$

Sol $\text{L.H.S}=\dfrac{4+16x}{4}$

$=(4+16x)\times\dfrac{1}{4}$

$=(1\times 4+4x\times 4)\times\dfrac{1}{4}$

$=(1+4x)\times\left(4\times\dfrac{1}{4}\right)$ **Left Distribution**

$=(1+4x)\left(4\times\dfrac{1}{4}\right)$

$=(1+4x)(1)$ **Cancellation Law**

$=(1+4x)$ **Multiplicative Identity**

(ii) $\dfrac{\frac{1}{4}+\frac{1}{5}}{\frac{1}{4}-\frac{1}{5}}$

Sol. $\dfrac{\frac{1}{4}+\frac{1}{5}}{\frac{1}{4}-\frac{1}{5}}=\dfrac{\frac{1}{4}\times 1+\frac{1}{5}\times 1}{\frac{1}{4}\times 1-\frac{1}{5}\times 1}$ **'×' Identity**

$=\dfrac{\frac{1}{4}\times\frac{5}{5}+\frac{1}{5}\times\frac{4}{4}}{\frac{1}{4}\times\frac{5}{5}-\frac{1}{5}\times\frac{4}{4}}=\dfrac{\frac{5}{20}+\frac{4}{20}}{\frac{5}{20}-\frac{4}{20}}$ **Closure property**

$=\dfrac{(5+4)\times\frac{1}{20}}{(5-4)\times\frac{1}{20}}$

$=\dfrac{5+4}{5-4}=\dfrac{9}{1}=9$ **Cancellation property.**

(iii) $$\frac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}$$

Sol. $$\frac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}=\frac{\frac{a}{b}\times 1+\frac{c}{d}\times 1}{\frac{a}{b}\times 1-\frac{c}{d}\times 1}$$ Multiplicative Identity

$$=\frac{\frac{a}{b}\times\frac{d}{d}+\frac{c}{d}\times\frac{b}{b}}{\frac{a}{b}\times\frac{d}{d}-\frac{c}{d}\times\frac{b}{b}}$$

$$=\frac{\frac{ad}{bd}+\frac{bc}{bd}}{\frac{ad}{bd}-\frac{bc}{bd}}=\frac{(ad+bc)\times\frac{1}{bd}}{(ad-bc)\times\frac{1}{bd}}$$

$$=\frac{ad+bc}{ad-bc}$$ Cancellation property

(iv) $$\frac{\frac{1}{a}-\frac{1}{b}}{1-\frac{1}{a}.\frac{1}{b}}=\frac{\frac{1}{a}.1-\frac{1}{b}.1}{1.1-\frac{1}{ab}}$$ 'X' Identity

$$=\frac{\frac{1}{a}\cdot\frac{b}{b}-\frac{1}{b}\cdot\frac{a}{a}}{\frac{ab}{ab}.1-\frac{1}{ab}\cdot 1}$$ Cancellation property

$$=\frac{\frac{b}{ab}-\frac{a}{ab}}{\frac{ab}{ab}\cdot 1\times\frac{1}{ab}\cdot 1}=\frac{(b-a)\times\frac{1}{ab}}{(ab-1)\times\frac{1}{ab}}$$ Distribution Law

$$=\frac{b-a}{ab-1}$$ Cancellation Law

Exercise 1.2

1. Verify the addition properties of complex numbers.

i. Closure

Sol Let $a+ib,\ c+id \in C$ then

$(a+ib)+(c+id)=(a+c)+i(b+d) \in C$

ii. Associative

$a+ib,\ c+id,\ e+if \in C$ then

$$\begin{aligned}[(a+ib)+(c+id)]+(e+if) &= [(a+c)+i(b+d)]+(e+if)\\ &=(a+c+e)+i(b+d+f)\\ &=(a+ib)+[(c+e)+i(d+f)]\\ &=(a+ib)+[(c+id)+(e+if)]\end{aligned}$$

iii. Additive Identity

$(0+i0),(a+ib)\in C$

then $(a+ib)+(0+i0)=(a+0)+i(b+0)$

$a+ib \in C$

Also $(0+i0)+(a+ib)$

$=(0+a)+i(0+b)=a+ib\in C$

iv. Additive Inverse

$(a+ib),(-a-ib)\in C$

$(a+ib)+(-a-ib)=(a-a)+i(b-b)$

$0+i0\in C$

Also $(-a-ib)+(a+ib)=(-a+a)+i(a+ib)$

$0+i0\in C$

v. Commutative

$(a+ib),(c+id)\in C$ then

$(a+ib)+(c+id)$

$(a+c)+i(b+d)=(c+a)+i(d+b)$

$=(c+id)+(a+ib)$

2. Verify the multiplication properties of the complex numbers.

i. Close w.r.t. 'X'

Sol $(a+ib),(c+id)\in C$ then

$$\begin{aligned}(a+ib)(c+id) &= ac+iad+ibc+i^2bd\\ &=ac+i(ad+bc)-bd\\ &=(ac-bd)+i(ad+bc)\in C\end{aligned}$$

ii. **Associative w.r.t. 'X'**

$(a+ib),(c+id),(e+if)\in C$

$$then\left[(a+ib)(c+id)\right](e+if)=(ac+i^2bd+ibc+iad)(e+if)$$
$$=\left[(ac-bd)+i(bc+ad)\right](e+if)$$
$$=\left[e(ac-bd)-f(bc+ad)\right]+i\left[f(ac-bd)+e(bc+ad)\right]$$
$$=\left[aec-ebd-fbc-fad\right]+i\left[afc-fbd+ebc+ead\right]$$
$$=\left[a(ec-df)-b(cf+de)\right]+i\left[a(cf+de)+b(ec-df)\right]$$
$$=(a+ib)\left[(ec-df)+i(cf+de)\right]$$
$$=(a+ib)\left[(c+id)(e+if)\right]$$

iii. **Identity**

$(a+b),(1+i0)\in C$ *then* $(a+ib)(1+i0)$

$$=a+0+ib+0=a+ib\in C$$

iv. **Inverse**

$$(a+ib),\left(\frac{a}{a^2+b^2}-\frac{ib}{a^2+b^2}\right)\in C\ then$$

$$(a+ib).\left(\frac{a}{a^2+b^2}-\frac{ib}{a^2+b^2}\right)=(a+ib)\frac{(a-ib)}{a^2+b^2}$$
$$=\frac{a^2-(ib)^2}{a^2+b^2}=\frac{a^2+b^2}{a^2+b^2}=1=1+0i$$

$$Also\ \left(\frac{a}{a^2+b^2}-\frac{ib}{a^2+b^2}\right)(a+ib)$$
$$=\frac{(a-ib)(a+ib)}{a^2+b^2}=\frac{a^2+b^2}{a^2+b^2}$$
$$=1=1+0i$$

v. **Commutative**

Sol $(a+ib),(c+id)\in c$

$$(a+ib)(c+id)=(ac-bd)+i(ad+bc)$$
$$=(ca-db)+i(da+cb)$$
$$=(c+id)(a+ib)$$

3. **Verify the distribution law of complex numbers.**

$$\boldsymbol{(a,b)\left[(c,d)+(e,f)\right]=(a,b)(c,d)+(a,b)(e,f)}$$

Sol Distribution law is $(a,b)\left[(c,d)+(e,f)\right]=(a,b)(c,d)+(a,b)(e,f)$

$$L.H.S = (a,b)[(c,d)+(e,f)]$$
$$= (a,b)[(c,d)+(e,f)]$$
$$= [a(c+e-b(d+f), a(d+f)+(c+e)]$$
$$= (ac+ae-bd-df), (ad+af)+(bc+be)$$
$$= (ac-bd, ad+bc)+(ae-bf, af+be)$$
$$= (a,b)(c,d)+(a,b)(e,f)$$
$$= R.H.S$$

4. Simplify the following:

Note: $i = \sqrt{-1}$

$\Rightarrow i^2 = -1$

i. i^9

Sol $i^9 = i^8 \times i$

$$= (i^2)^4 \times i$$
$$= (-1)^4 \times i$$
$$= 1 \times i = i$$

ii. i^{14}

Sol $i^{14} = (i^2)^7 = (-1)^7 = -1$

iii. $(-i)^{19}$

Sol $(-i)^{19} = -i^{19} = -i^{18}.i = -(i^2)^9.i = -(-1)^9.i = -(-1).i = i$

iv. $(-1)^{\frac{-21}{2}}$

Sol

$$(-1)^{\frac{-21}{2}} = (i^2)^{\frac{-21}{2}} = (i)^{-21} = \frac{1}{i^{21}} = \frac{1}{i^{20} \times i} = \frac{1}{(i^2)^{10} \times i}$$

$$= \frac{1}{(-1)^{10} \times i} = \frac{1}{1 \times i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$$

5. Written in terms of i

i. $\sqrt{-1}b$

Sol $\sqrt{-1}b = ib$

ii. $\sqrt{-5}$

Sol $\sqrt{-5} = \sqrt{(-1)(5)} = \sqrt{-1}\sqrt{5} = i\sqrt{5} = \sqrt{5}\,i$

iii. $\sqrt{\dfrac{-16}{25}}$

Sol $\sqrt{\dfrac{-16}{25}}=\sqrt{(-1)(\dfrac{16}{25})}=\sqrt{-1}\sqrt{\dfrac{16}{25}}=i\dfrac{4}{5}$

iv. $\sqrt{\dfrac{1}{-4}}$

Sol $\sqrt{\dfrac{1}{-4}}=\sqrt{\dfrac{1}{(-1)4}}=\sqrt{(-1)}\sqrt{\dfrac{1}{4}}=i\times\dfrac{1}{2}=\dfrac{i}{2}$

6. **(7,9) + (3,-5)**

Sol $(7,9)+(3,-5)=(7+3,9-5)$

$=(10,4)$

7. **(8, −5) − (−7,4)**

Sol $=(8-(-7),-5-4)=(8+7,-5-4)=(15,-9)$

8. **(2,6) (3,7)** **Multan 2009**

Sol $=(2+6i)(3+7i)$

$=6+14i+18i+42i^2$

$=6+32i+42(-1)$

$=6+32i-42$

$=-36+32i$

$=(-36,32)$

9. **(5, − 4) (− 3, − 2)**

Sol $=(5-4i)(-3-2i)$

$=-15-10i+12i+8i^2$

$=-15+2i-8$

$=-23+2i$

$=(-23,2)$

10. **(0, 3) (0, 5)**

Sol $=(0+3i)(0+5i)$

$=(3i)(5i)$

$=15i^2=15(-1)$

$=-15$

$=(-15,0)$

11: **(2, 6) ÷ (3, 7)**

Sol $\frac{(2,6)}{(3,7)} = \frac{2+6i}{3+7i} = \frac{2+6i}{3+7i} \times \frac{3-7i}{3-7i}$

$= \frac{(2+6i)(3-7i)}{(3)^2-(7i)^2} = \frac{6-14i+18i-42i^2}{9-(-49)}$

$= \frac{6+4i-42(-1)}{9+49} = \frac{6+4i+42}{58}$

$= \frac{48+4i}{58} = \frac{48}{58} + i\frac{4}{58}$

$= \frac{24}{29} + i\frac{2}{29} = (\frac{24}{29}, \frac{2}{29})$

12. $(5,-4) \div (-3,-8)$ **Faisalabad 2009**

Sol $= \frac{(5,-4)}{(-3,-8)} = \frac{5-4i}{-3-8i}$

$= \frac{5-4}{-3-8i} \times \frac{-3+8i}{-3+8i} = \frac{(5-4i)(-3+8i)}{(-3-8i)(-3+8i)}$

$= \frac{-15+40i+12i-32i^2}{(-3)^2-(8i)^2}$

$= \frac{-15+52i-32i^2}{9-64(-1)}$

$= \frac{-15+52i+32}{9+64} = \frac{17+52i}{73}$

$(\frac{17}{73}, \frac{52}{73})$

13. **Prove that the sum as well as the product of any two conjugate complex numbers is a real number.** **Federal 2008**

Sol *Let* $Z = x + iy$

Conjgate $= \overline{Z} = x - iy$

Sum $= Z + \overline{Z}$

$= x + iy + x - iy$

$= 2x$ *is real*

Product $= Z\,\overline{Z}$

$= (x+iy)(x-iy)$

$= x^2 - (iy)^2 = x^2 - (-y^2) = x^2 + y^2$ *is real*

14. Find the multiplicative inverse of each of the following numbers:

(i) (– 4,7) **Faisalabad 2008, Multan 2008**

Sol $$\textit{Multiplicative Inverse}=\frac{1}{(-4,7)}$$

$$=\frac{1}{-4+7i}\times\frac{-4-7i}{-4-7i}$$

$$=\frac{-4-7i}{(-4)^2-(7i)^2}=\frac{-4-7i}{16-(-49)}$$

$$=\frac{-4-7i}{16+49}=\frac{-4-7i}{65}=(\frac{-4}{65},\frac{-7}{65})$$

(ii) $(\sqrt{2},-\sqrt{5})$ **Sargodha 2007, 2010, Gujranwala 2009**

Sol $$\textit{Multiplicative Inverse}=\frac{1}{(\sqrt{2},-\sqrt{5})}$$

$$=\frac{1}{\sqrt{2}-\sqrt{5}i}=\frac{1}{\sqrt{2}-\sqrt{5}i}\times\frac{\sqrt{2}+i\sqrt{5}}{\sqrt{2}+i\sqrt{5}}$$

$$=\frac{\sqrt{2}+i\sqrt{5}}{(\sqrt{2})^2-(\sqrt{5}i)^2}=\frac{\sqrt{2}+i\sqrt{5}}{2-(-5)}$$

$$=\frac{\sqrt{2}+i\sqrt{5}}{2+5}=\frac{\sqrt{2}+i\sqrt{5}}{7}=\frac{\sqrt{2}}{7}+\frac{\sqrt{5}}{7}i$$

$$=\left(\frac{\sqrt{2}}{7},\frac{\sqrt{5}}{7}\right)$$

(iii) (1, 0)

Sol $$\textit{Multiplicative Inverse}=\frac{1}{(1,0)}$$

$$=\frac{1}{1+0i}\times\frac{1-0i}{1-0i}=\frac{1-0i}{(1)^2-(i0)^2}$$

$$=\frac{1-0i}{1-0}=1-i0=(1,0)$$

15. Factorize the following:

(i) a^2+4b^2 **Sargodha 2008, Multan 2009, 2010**

Sol $$=a^2-(-4b^2)=(a^2)-(i^2 4b^2)=(a)^2-(2bi)^2$$

$$=(a-2bi)(a+2bi)$$

(ii) $\mathbf{9a^2+16b^2}$ *S* arg*odha* 2008, *Faisalabad* 2007

Sol $=9a^2+16b^2=9a^2-(-16b^2)=9a^2-(i^2 16b^2)$

$$=9a^2-(i4b)^2=(3a)^2-(i4b)^2$$

$$=(3a-4bi)(3a+4bi)$$

(iii) $\mathbf{3x^2+3y^2}$

Sol $=3x^2+3y^2=3(x^2+y^2)$

$$=3\left[(x)^2-(-y^2)\right]$$

$$=3(x)^2-(iy)^2)$$

$$=3(x-iy)(x+iy)$$

16. Separate into real and imaginary parts (write as a simple complex number):

(i) $\mathbf{\dfrac{2-7i}{4+5i}}$

Sol $\dfrac{2-7i}{4+5i}=\dfrac{2-7i}{4+5i}\times\dfrac{4-5i}{4-5i}$

$$=\frac{(2-7i)(4-5i)}{(4)^2-(5i)^2}=\frac{8-10i-28i+35i^2}{16-(-25)}$$

$$=\frac{8-38i-35}{16+25}=\frac{-27-38i}{41}=\frac{-27}{41}-i\frac{38}{41}$$

(ii) $\mathbf{\dfrac{(-2+3i)^2}{1+i}}$

Sol $\dfrac{(-2+3i)^2}{1+i}=\dfrac{4-12i+9i^2}{1+i}\times\dfrac{1-i}{1-i}$

$$=\frac{(4-12i+9(-1))(1-i)}{(1+i)(1-i)}$$

$$=\frac{(4-12i-9)(1-i)}{(1)^2-(i)^2}=\frac{(-5-12i)(1-i)}{1-(-1)}$$

$$=\frac{-5+5i-12i+12i^2}{1+1}$$

$$=\frac{-5-7i+12(-1)}{2}=\frac{-5-7i-12}{2}$$

$$=\frac{-17-7i}{2}=\frac{-17}{2}-\frac{7}{2}i$$

(iii) $\dfrac{i}{1+i}$

Sol $\dfrac{i}{1+i}=\dfrac{i}{1+i}\times\dfrac{1-i}{1-i}=\dfrac{i-i^2}{(1)^2-(i)^2}$

$=\dfrac{i-(-1)}{1-(-1)}=\dfrac{i+1}{2}=\dfrac{1}{2}+\dfrac{i}{2}$

Example 1: Find the Module of the following complex numbers.

i. $1-i\sqrt{3}$ ii. 3 iii. $-5i$ iv. $3+4i$

Solution:

(i) $Let\ Z=1-i\sqrt{3}$

Faisalabad 2009, Sargodha 2010

$\therefore |Z|=\sqrt{(1)^2+(-\sqrt{3})^2}$

$=\sqrt{1+3}=2\ Ans$

(ii) $Let\ Z=3$

$Z=3+0i$

$\therefore |Z|=\sqrt{(3)^2+(0)^2}=3\ Ans$

(iii) $Let\ Z=-5i$

$Z=0-5i$

$\therefore |Z|=\sqrt{(0)^2+(-5)^2}=5\ Ans$

(iv) $Let\ Z=3+4i$

$\therefore |Z|=\sqrt{(3)^2+(4)^2}$

$=\sqrt{25}=5\ Ans$

Example 2: (Federal board)

$If\ Z_1=2+i,\quad Z_2=3-2i$

$Z_3=1+3i,\ then\ express$

$\dfrac{\overline{Z_1 Z_3}}{Z_2}=\dfrac{\overline{(2+i)}\,\overline{(1+3i)}}{3-2i}=\dfrac{(2-i)(1-3i)}{3-2i}$

$\dfrac{2-6i-i+3i^2}{3-2i}=\dfrac{2-7i-3}{3-2i}$

$\dfrac{-1-7i}{3-2i}\times\dfrac{3+2i}{3+2i}$

$$\frac{-3-2i-21i-14i^2}{9-4i^2}=\frac{-3-23i+14}{9+4}$$

$$\frac{11-23i}{13}=\frac{11}{13}-\frac{23}{13}i \quad Ans$$

Example 4: **Express the complex number $1+i\sqrt{3}$ in polar form.**

Sargodha 2011, Fasaiabad 2007

Solution: Put $r\cos\theta=1\rightarrow(i)$ & $r\sin\theta=\sqrt{3}\rightarrow(ii)$

Squaring & adding (i) & (ii)

$$r^2\cos^2\theta+r^2\sin^2\theta=(1)^2+(\sqrt{3})^2$$

$$r^2(\cos^2\theta+\sin^2\theta)=1+3$$

$$r^2=4$$

$$r=2$$

Dividing (ii) by (i)

$$\frac{r\sin\theta}{r\cos\theta}=\frac{\sqrt{3}}{1}$$

$$tan\theta=\sqrt{3}$$

$$\theta \quad =Tan^{-1}(\sqrt{3})$$

$$\theta \quad =60^o$$

$$Thus\ 1+i\sqrt{3}=r(Cos\theta+i\sin\theta)$$

$$=2(Cos60^o+i\sin60) \quad Ans$$

State Demoiver,s Theorem: **Lahore 2009**

Statement:

$$(Cos\theta+i\sin\theta)^n=Cos(n\theta)+i\sin(n\theta)$$

Example 5: Find out real and imaginary parts of each of the following complex numbers.

(i) $(\sqrt{3}+i)^3$ **Federal 2009**

(ii) $\left(\dfrac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^5$

Solution (i):

$Let \quad rcos\theta=\sqrt{3}$, & $r\sin\theta=1\ where$

$$r^2cos^2\theta+r^2\sin^2\theta=(\sqrt{3})^2+(1)^2$$

$$r^2(cos^2\theta+\sin^2\theta)=3+1$$

$$r^2 = 4$$
$$or \quad = 2$$

also $\frac{r\sin\theta}{r\cos\theta} = \frac{1}{\sqrt{3}}$

$$tan\theta = \frac{1}{\sqrt{3}}$$
$$\theta = tan^{-1}(\frac{1}{\sqrt{3}})$$
$$\theta = 30^o$$
$$(\sqrt{3}+1)^3 = \left[r(\cos\theta + i\sin\theta)\right]^3 = r^3(\cos\theta + i\sin\theta)^3$$
$$= 2^3(\cos\theta + i\sin\theta)^3 = 8(\cos 3(30^0) + i\sin 3(30^0)$$ **By demoiver's theorem.**
$$= 2^3(\cos 30^o + i\sin 30^o)^3 = 8\left[\cos 90^o + i\sin 90^o\right]$$
$$= 8\left[0 + i\,.\,1\right] = 0 + 8i$$

$Real \ Part = 0$

$Imaginary \ Part = 8$

Solution (ii):

$$Let \quad r_1 cos\theta_1 = 1 \,\&\, r_1\sin\theta = -\sqrt{3}$$
$$r_1 = \sqrt{(1)^2 + (-\sqrt{3})^2}$$
$$r_1 = \sqrt{1+3} = 2$$
$$\frac{r_1\sin\theta_1}{r_1\cos\theta_1} = \frac{-\sqrt{3}}{1}$$
$$\tan\theta_1 = -\sqrt{3} \quad \theta_1 = Tan^{-1}(-\frac{\sqrt{3}}{1}) = -60^o$$
$$Also \quad Let \quad r_2 cos\theta_2 = 1 \,\&\, r_2\sin\theta = \sqrt{3}$$
$$\Rightarrow r_2 = \sqrt{(1)^2 + (\sqrt{3})^2}$$
$$r_2 = \sqrt{4} = 2$$
$$and \quad \theta_2 = Tan^{-1}(\frac{\sqrt{3}}{1}) = 60^o$$
$$So \quad \left(\frac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^5 = \frac{\left[(\cos(-60^0) + i\sin(-60^0))\right]^5}{\left[(\cos 60^0 + i\sin 60^0)\right]^5}$$

$$=\left[\frac{\cos(-60^0)+i\sin(-60^0)}{(\cos 60^0+i\sin 60^0)^5}\right]^5$$

$$=\left[\cos(-60^0)+i\sin(-60^0)\right]^5\left[\cos(60^0)+i\sin(60^0)\right]^{-5}$$

$$=\left[\cos(-300^0)+i\sin(-300^0)\right]\left[\cos(-300^0)+i\sin(-300^0)\right] \text{ By demaiver,s theorem}$$

$$=\left[\cos(300^0)-i\sin(300^0)\right]\left[\cos 300^0-i\sin 300^0\right]=\left[\cos 300^0-i\sin 300^0\right]^2 \quad (i)$$

$$as\ \cos 300^0=\cos\left[3\times 90+30\right]=\sin 30=\tfrac{1}{2}$$

$$\sin 300^o=\sin\left[3\times 90^o+30^o\right]=-\cos 30^o=-\frac{\sqrt{3}}{2}$$

$$=(\frac{1}{2}+i\frac{\sqrt{3}}{2})^2 \quad (i)become$$

$$\left(\frac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^5=\frac{1}{4}+2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}i\right)+\left(\frac{\sqrt{3}}{2}i\right)^2$$

$$=\frac{1}{4}+\frac{\sqrt{3}}{2}i+\frac{3}{4}i^2=\frac{1}{4}+\frac{\sqrt{3}}{2}i+\frac{3}{4}(-1)=\frac{1}{4}-\frac{3}{4}+\frac{\sqrt{3}}{2}i$$

$$=\frac{1-3}{4}+\frac{\sqrt{3}}{2}i=-\frac{2}{4}+\frac{\sqrt{3}}{2}i$$

$$=\frac{-1}{2}+\frac{\sqrt{3}}{2}i$$

Theorems If z,z_1,z_2 be any complex numbers then show that

(i) $|Z|=|-Z|=|\overline{Z}|=|-\overline{Z}|$

Sol. $Let\ Z=a+ib\Rightarrow|Z|=\sqrt{a^2+b^2}$ (1)

$Also\ \overline{Z}=a-ib\Rightarrow|Z|=\sqrt{(a)^2+(-b)^2}=\sqrt{a^2+b^2}$ (2)

$-Z=-a-ib\Rightarrow|-Z|=\sqrt{(-a)^2+(-b)^2}=\sqrt{a^2+b^2}$ (3)

$-\overline{Z}=-a+ib\Rightarrow|-Z|=\sqrt{(-a)^2+(b)^2}=\sqrt{a^2+b^2}$ (4)

From (1),(2),(3) & (4) we have.

$|Z|=|-Z|=|\overline{Z}|=|-\overline{Z}|$

(ii) $(\overline{\overline{Z}}) = Z$ **Multan 2009**

Sol. $Let \quad Z = a + ib \rightarrow (1)$

$\Rightarrow \quad \overline{Z} = a - ib$

$\Rightarrow \quad \overline{\overline{Z}} = a + ib \rightarrow (2)$

From (1) & (2) we have $(\overline{\overline{Z}}) = Z$

(iii) $Z\ \overline{Z} = \left|\overline{Z}\right|^2$ **Lahore 2009**

Sol. $Let \quad Z = a + ib \Rightarrow \overline{Z} = a - ib$

$\text{L.H.S} = Z.\ \overline{Z} = (a+ib)(a-ib) = (a^2) - (ib)^2$

$= a^2 - i^2b^2$

$= a^2 + b^2 \qquad \rightarrow (i)$

$\text{R.H.S} = |Z|^2 = (\sqrt{a^2+b^2})^2 = a^2 + b^2 \qquad \rightarrow (ii)$

L.H.S = R.H.S

(iv) $\overline{Z_1 + Z_2} = \overline{Z}_1 + \overline{Z}_2$ **Sargodha 2008**

Sol. $Let\ Z_1 = a + ib, \qquad Z_2 = c + id$

$\overline{Z}_1 = a - ib, \qquad \overline{Z}_2 = c - id$

$Now\ \ Z_1 + Z_2 = (a+ib) + (c+id)$

$= (a+c) + i(b+d)$

$\Rightarrow \overline{Z_1 + Z_2} = (a+c) - i(b+d) \qquad \rightarrow (i)$

$\text{Also} \quad \overline{Z}_1 + \overline{Z}_2 = (a - ib) + (c - id)$

$= (a+c) - i(b+c)$

From (1) & (2) we have $\overline{Z_1 + Z_2} = \overline{Z}_1 + \overline{Z}_2 \qquad \rightarrow (ii)$

(v) $\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\overline{Z}_1}{\overline{Z}_2}$ **Federal 2008, Sargodha 2009, Faisalabad 2008**

Sol. $Let\ Z_1 = a + ib, \qquad Z_2 = c + id$

$\overline{Z}_1 = a - ib, \qquad \overline{Z}_2 = c - id$

Now $\left(\frac{Z_1}{Z_2}\right) = \frac{a+ib}{c+id} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id}$

$\frac{ac - iad + ibc - i^2bd}{(c)^2 - (id)^2}$

$$\left(\frac{Z_1}{Z_2}\right)=\frac{ac-iad+ibc+bd}{c^2-i^2d^2}=\frac{(ac+bd)-i(ad-bc)}{c^2+d^2}$$

$$\Rightarrow \overline{\left(\frac{Z_1}{Z_2}\right)}=\frac{(ac+bd)+i(ad-bc)}{c^2+d^2} \qquad \rightarrow (1)$$

Again $\left(\frac{\overline{Z}_1}{\overline{Z}_2}\right)=\frac{a-ib}{c-id}=\frac{a-ib}{c-id}\times\frac{c+id}{c+id}$

$$=\frac{ac+iad-ibc-i^2bd}{(c)^2-(id)^2}=\frac{ac+iad-ibc+bd}{c^2-i^2d^2}$$

$$\frac{\overline{Z}_1}{\overline{Z}_2}=\frac{(ac+bd)+i(ad-bc)}{c^2+d^2} \qquad \rightarrow (2)$$

From (1) & (2) we have $\overline{\left(\frac{Z_1}{Z_2}\right)}=\frac{\overline{Z}_1}{\overline{Z}_2}$

(vi) $|Z_1.Z_2|=|Z_1||Z_2|$

Sol. $Let\ Z_1=a+ib,\ Z_2=c+id$

$$\Rightarrow \ |Z_1|=\sqrt{a^2+b^2},\ |Z_2|=\sqrt{c^2+d^2}$$

Now L.H.S $|Z_1 . Z_2|=|(a+ib)(c+id)|$

$$=|ac+iad+ibc+i^2bd|$$

$$=|ac+iad+ibc-bd|$$

$$=|(ac-bd)+i(ad+bc|$$

$$=\sqrt{(ac-bd)^2+(ad+bc)^2}$$

$$=\sqrt{a^2c^2+b^2d^2-\cancel{2\,acbd}+a^2d^2+b^2c^2+\cancel{2\,acbd}}$$

$$=\sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2}=\sqrt{a^2c^2+a^2d^2+b^2c^2+b^2d^2}$$

$$=\sqrt{a^2(c^2+d^2)+b^2(c^2+d^2)}$$

$$=\sqrt{(a^2+b^2)(c^2+d^2)}$$

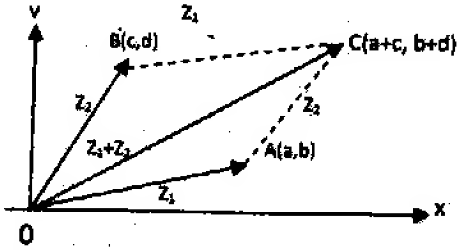
$$=\sqrt{a^2+b^2}\,.\sqrt{c^2+d^2}$$

$$=|Z_1|\,.\,|Z_2|=R.H.S$$

(vii) $|Z_1|-|Z_2| \le |Z_1+Z_2| \le |Z_1|+|Z_2|$

Sol. *Let* $Z_1 = a+ib,\ \ Z_2 = c+id$

then $Z_1+Z_2 = (a+c)+i(b+d)$

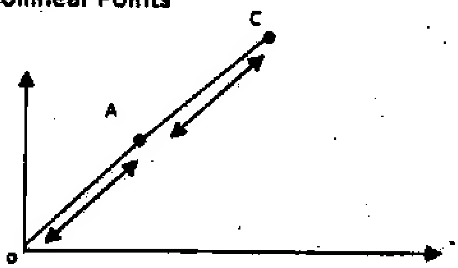


$|Z_1| = |\overrightarrow{OA}|, \qquad |Z_2| = |\overrightarrow{OB}|, \qquad |Z_1+Z_2| = |\overrightarrow{OC}|$

$|\overrightarrow{OA}| + |\overrightarrow{AC}| > |\overrightarrow{OC}|$

$|Z_1|+|Z_2| > |Z_1+Z_2| \qquad \rightarrow (1)$

For Collinear Points



$|\overrightarrow{OA}| + |\overrightarrow{AC}| = |\overrightarrow{OC}| \qquad \rightarrow (2)$

$|Z_1|+|Z_2| = |Z_1+Z_2|$

By (1) & (2) $|Z_1|+|Z_2| \ge |Z_1+Z_2| \qquad \rightarrow (3)$

Now

$|Z_1| = |Z_1+Z_2-Z_2|$

$|Z_1| \le |Z_1+Z_2| + |-Z_2|$

$|Z_1| \le |Z_1+Z_2| + |Z_2|$

$|Z_1|-|Z_2| \le |Z_1+Z_2| \qquad \rightarrow (4)$

By (3) & (4)

$|Z_1|-|Z_2| \le |Z_1+Z_2| \le |Z_1|+Z_2|$ *Proved*

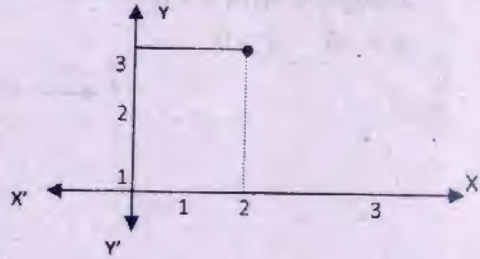
Exercise 1.3

1. Graph the following numbers on the complex plane:

(i) $2+3i$

Sol. $2+3i$ *Compare with* $x+iy$

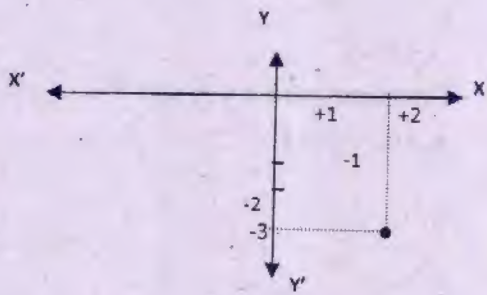
Here $x=2,\ y=-3$



(ii) $2-3i$

Sol. $2-3i$ *Compare with* $x+iy$

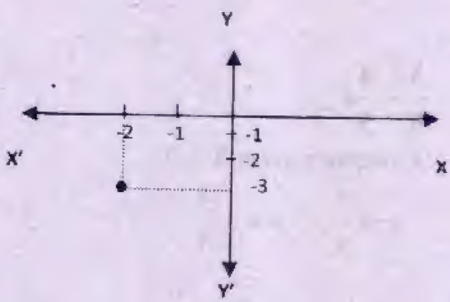
$x=2,\quad y=-3$



(iii) $-2-3i$

Sol. *Compare with* $x+iy$

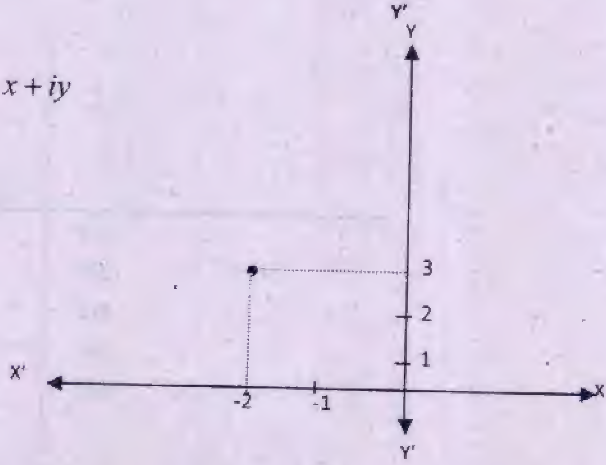
$x=-2,\quad y=-3$



(iv) $-2+3i$

Sol. $-2+3i$ *Compare with* $x+iy$

$x=-2,\quad y=3$

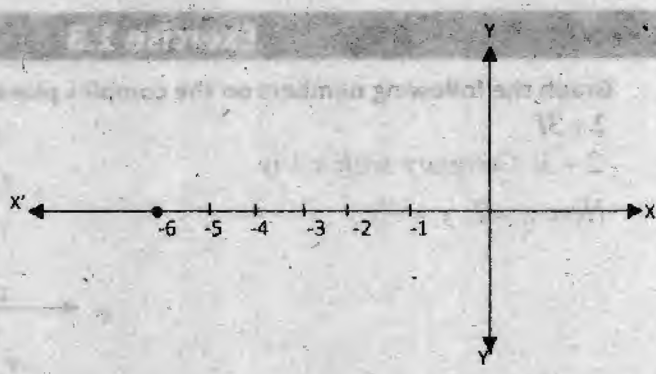


(v) -6

Sol. $-6=-6+0i$

Compare with $x+iy$

$x=-6, \quad y=0$

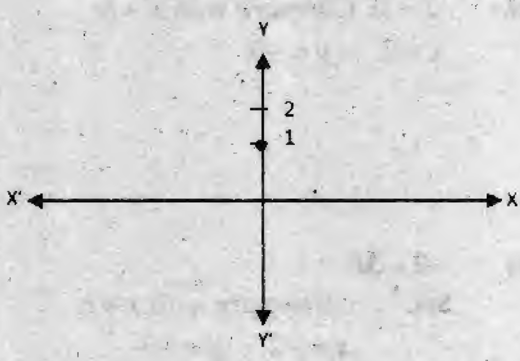


(vi) i

Sol. $i=0+i$

Compare with $x+iy \Rightarrow$

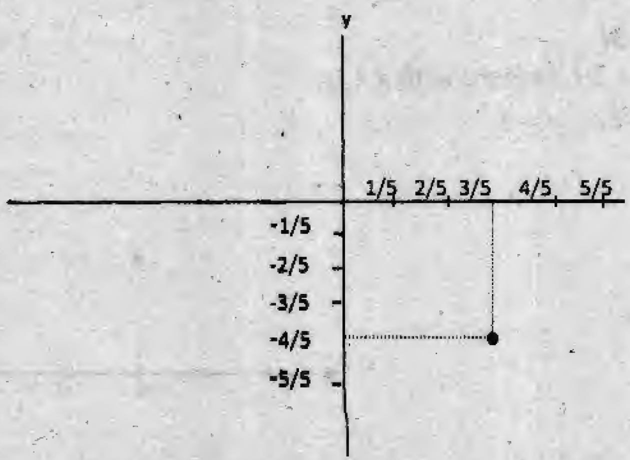
$x=0, \quad y=1$



(vii) $\frac{3}{5}-\frac{4}{5}i$

Sol. *Compare with* $x+iy$

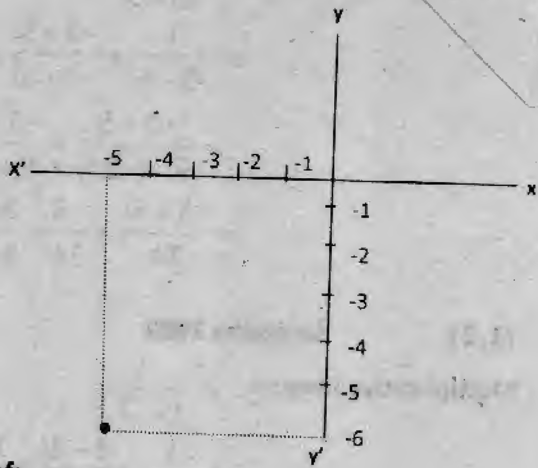
$x=\frac{3}{5}, \quad y=-\frac{4}{5}$



(viii) $-5-6i$

Sol. *Compare with* $x+iy$

$x=-5, \quad y=-6$



2. **Find the multiplicative inverse of:**

(i) $-3i = 0-3i$ **Faisalabad 2008**

Multiplicative inverse $=\dfrac{1}{0-3i}$

$$=\frac{1}{0-3i}\times\frac{0+3i}{0+3i}=\frac{0+3i}{0-(9i^2)}$$

$$=\frac{3i}{-(-9)}=\frac{3i}{9}=\frac{i}{3}$$

(ii) $1-2i$

Multiplicative inverse $=\dfrac{1}{1-2i}$

$$=\frac{1}{1-2i}\times\frac{1+2i}{1+2i}$$

$$=\frac{1+2i}{(1)^2-(2i)^2}=\frac{1+2i}{1-(-4)}=\frac{1+2i}{1+4}$$

$$=\frac{1+2i}{1+4}=\frac{1+2i}{5}$$

$$=\frac{1}{5}+\frac{2i}{5}$$

$-3-5i$

Multiplicative inverse $= \dfrac{1}{-3-5i}$

$$= \frac{1}{-3-5i} \times \frac{-3+5i}{-3+5i} = \frac{-3+5i}{(-3)^2-(5i)^2}$$

$$= \frac{-3+5i}{9-(-25)} = \frac{-3+5i}{9+25}$$

$$= \frac{-3+5i}{34} = \frac{-3}{34} + \frac{5i}{34}$$

(iv) $(1,2)$ **Sargodha 2009**

Multiplicative inverse $= \dfrac{1}{(1,2)}$

$$= \frac{1}{1+2i} \times \frac{1-2i}{1-2i} = \frac{1-2i}{1-2i}$$

$$= \frac{1-2i}{(1)^2-(2i)^2} = \frac{1-2i}{1-(-4)}$$

$$= \frac{1-2i}{1+4} = \frac{1-2i}{5} = \frac{1}{5} - \frac{2i}{5}$$

$$= (\frac{1}{5}, -\frac{2}{5})$$

3. Simplify:

(i) i^{101} **Lahore 2009, Multan 2010**

Sol. $i^{101} = i^{100}.i = (i^2)^{50} \times i = (-1)^{50} \times i = 1 \times i = i$

(ii) $(-ai)^4,\ a \in \mathbb{R}$

Sol. $(-ai)^4 = a^4 i^4 = a^4 (i^2)^2 = a^4(-1)^2 = a^4 1 = a^4$

(iii) i^{-3}

Sol. $\dfrac{1}{i^3} = \dfrac{1.i}{i^3.i} = \dfrac{i}{i^4} = \dfrac{i}{(i^2)^2} = \dfrac{i}{(-1)^2} = i$

(iv) i^{-10} **Rawalpindi 2009**

Sol. $= \dfrac{1}{i^{10}} = \dfrac{1}{(i^2)^5} = \dfrac{1}{(-1)^5} = \dfrac{1}{-1} \times \dfrac{1}{-1} = -1$

4. Prove that $\overline{Z} = Z$ if Z is real

Sol. *Suppose* $Z = a + ib \quad (i) \quad \Rightarrow \overline{Z} = a - ib$

Given $\overline{Z} = Z$

$a - ib = a + ib \Rightarrow a - a = ib + ib \Rightarrow 0 = 2ib$

$\Rightarrow b = 0$

(i) *become*

$Z = a + ib \Rightarrow Z = a + 0 \Rightarrow Z = a \Rightarrow Z \text{ is real}$

So Z is real conversely suppose that Z is real.

So $Z = a \quad \rightarrow (i)$

$\Rightarrow \overline{Z} = \overline{a}$

$\overline{Z} = a \quad \rightarrow (ii)$

Compare (II) and (III)

$Z = \overline{Z}$ Hence proved.

5. Simplify by expressing in the form $a + bi$

(i) $\mathbf{5 + 2\sqrt{-4}}$

Sol. $5 + 2\sqrt{-4} = 5 + 2\sqrt{(-1)4}$

$$= 5 + 2i\sqrt{4} = 5 + 2i(2) = 5 + i4$$

(ii) $\mathbf{(2 + \sqrt{-3})(3 + \sqrt{-3})}$

Sol.
$$= (2 + i\sqrt{3})(3 + i\sqrt{3})$$
$$= 6 + 2i\sqrt{3} + 3i\sqrt{3} + i^2\sqrt{3}\sqrt{3}$$
$$= 6 + 5i\sqrt{3} + (-1)(3) = 6 - 3 + 5\sqrt{3}i$$
$$= 3 + 5\sqrt{3}i$$

(iii) $\mathbf{\dfrac{2}{\sqrt{5} + \sqrt{-8}}}$

Sol.
$$\frac{2}{\sqrt{5} + \sqrt{-8}} = \frac{2}{\sqrt{5} - i\sqrt{8}} \times \frac{\sqrt{5} - i\sqrt{8}}{\sqrt{5} - i\sqrt{8}}$$
$$= \frac{2(\sqrt{5} - i\sqrt{8})}{(\sqrt{5})^2 - (i\sqrt{8})^2} = \frac{2(\sqrt{5} - i\sqrt{8})}{5 - (-8)}$$
$$= \frac{2(\sqrt{5} - i\sqrt{8})}{5 + 8} = \frac{2\sqrt{5}}{13} - i\frac{2\sqrt{8}}{13}$$

(iv) $\mathbf{\dfrac{3}{\sqrt{6} - \sqrt{-12}}}$

Sol.
$$\frac{3}{\sqrt{6} - \sqrt{-12}} = \frac{3}{\sqrt{6} - i\sqrt{12}} \times \frac{\sqrt{6} + i\sqrt{12}}{\sqrt{6} + i\sqrt{12}}$$

$$= \frac{3\sqrt{6} + i3\sqrt{12}}{(\sqrt{6})^2 - (i\sqrt{12})^2} = \frac{3\sqrt{6} + i3\sqrt{12}}{6 - (-12)}$$

$$= \frac{3\sqrt{6} + i3\sqrt{12}}{6 + 12} = \frac{3(\sqrt{6} + i\sqrt{12})}{18}$$

$$= \frac{\sqrt{6}}{6} + i\frac{\sqrt{4\times 3}}{6} = \frac{\sqrt{6}}{\sqrt{6}\sqrt{6}} + i\frac{2\sqrt{3}}{6}$$

$$= \frac{1}{\sqrt{6}} + \frac{i\sqrt{3}}{3} = \frac{1}{\sqrt{6}} + \frac{i\sqrt{3}}{\sqrt{3}\sqrt{3}}$$

$$= \frac{1}{\sqrt{6}} + \frac{i}{3}$$

6. Show that $\forall z \in C$

(i) $\boldsymbol{Z^2 + \bar{Z}^2}$ ***is a real number*** **Faisalabad 2007**

Sol. *Take* $Z = a + ib$ *then* $\bar{Z} = a - ib$

Now $Z^2 + \bar{Z}^2 = (a+ib)^2 + (a-ib)^2$

$= a^2 + 2iab + (ib)^2 + a^2 - 2iab + (ib)^2$

$= a^2 - b^2 + a^2 - b^2 = 2a^2 - 2b^2$ which is real.

(ii) $\boldsymbol{(Z - \bar{Z})^2}$ ***is a real number***

Sol. *Take* $Z = a + ib$

then $\bar{Z} = a - ib$

Now $\left[Z - \bar{Z}\right]^2 = \left[(a+ib) - (a-ib)\right]^2$

$= \left[a + ib - a + ib\right]^2$

$= (2ib)^2 = 4i^2b^2 = -4b^2$

Which is real.

7. Simplify the following

(i) $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$

Sol.

$$= (-\frac{1}{2})^3 + 3(\frac{-1}{2})^2(\frac{\sqrt{3}}{2}i) + 3(\frac{-1}{2})(\frac{\sqrt{3}}{2}i)^2 + (\frac{\sqrt{3}}{2}i)^3$$

$$= -\frac{1}{8} + 3(\frac{1}{4})(\frac{\sqrt{3}}{2}i) - (\frac{3}{2})(\frac{-3}{4}) + \frac{3\sqrt{3}}{8}(-i)$$

$$= \frac{-1}{8} + \cancel{\frac{3\sqrt{3}}{8}}i + \frac{9}{8} - \cancel{\frac{3\sqrt{3}i}{8}}$$

$$= \frac{-1}{8} + \frac{9}{8} = \frac{-1+9}{8} = \frac{8}{8}$$

$$= 1 \quad Ans$$

(ii) $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3$

Sol. $(-\frac{1}{2} - \frac{\sqrt{3}}{2}i)^3 = (-\frac{1}{2})^3 + 3(-\frac{1}{2})^2(\frac{-\sqrt{3}}{2}i) + 3(-\frac{1}{2})(\frac{-\sqrt{3}}{2}i)^2 + (\frac{-\sqrt{3}}{2}i)^3$

$$= \frac{-1}{8} + 3(\frac{1}{4})(-\frac{\sqrt{3}}{2}i) - \frac{3}{2}(-\frac{3}{4}) + (-\frac{3\sqrt{3}}{8}i^3)$$

$$= \frac{-1}{8} + \frac{3}{4}(-\frac{\sqrt{3}}{2}i) + \frac{9}{8} + (-\frac{3\sqrt{3}}{8}(-i)) = \frac{-1}{8} - \cancel{\frac{3\sqrt{3}}{8}}i + \frac{9}{8} + \cancel{\frac{3\sqrt{3}}{8}}i$$

$$= \frac{-1}{8} + \frac{9}{8} = \frac{9}{8} - \frac{1}{8} = \frac{9-1}{8} = \frac{8}{8} = 1$$

(iii) $(-\frac{1}{2} - \frac{\sqrt{3}}{2}i)^{-2}(-\frac{1}{2} - \frac{\sqrt{3}}{2}i)$

Sol. $(-\frac{1}{2} - \frac{\sqrt{3}}{2}i)^{-2}(-\frac{1}{2} - \frac{\sqrt{3}}{2}i) = (-\frac{1}{2} - \frac{\sqrt{3}}{2}i)^{-2+1} = (-\frac{1}{2} - \frac{\sqrt{3}}{2}i)^{-1}$

$$\frac{1}{-\frac{1}{2} - \frac{\sqrt{3}}{2}} \times \frac{-\frac{1}{2} + \frac{\sqrt{3}}{2}i}{-\frac{1}{2} + \frac{\sqrt{3}}{2}i)} = \frac{-\frac{1}{2} + \frac{\sqrt{3}}{2}i}{(-\frac{1}{2})^2 - (\frac{\sqrt{3}}{2}i)^2} = \frac{\frac{-1}{2} + \frac{\sqrt{3}}{2}i}{\frac{1}{4} - (-\frac{3}{4})}$$

$$= \frac{\frac{-1}{2} + \frac{\sqrt{3}}{2}i}{\frac{1}{4} + \frac{3}{4}} = \frac{\frac{-1}{2} + \frac{\sqrt{3}}{2}i}{\frac{\cancel{4}}{\cancel{4}}} = \frac{-\frac{1}{2} + \frac{\sqrt{3}}{2}i}{1} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

(iv) $(a + bi)^2$

Sol. $(a + ib)^2 = a^2 + 2abi + (ib)^2 = a^2 + 2abi - b^2$

(v) $(a+bi)^{-2}$

Sol. $(a+ib)^{-2}=\dfrac{1}{(a+ib)^2}=\dfrac{1}{a^2+(ib)^2+2abi}=\dfrac{1}{(a^2-b^2)+2abi}\times\dfrac{(a^2-b^2)-2abi}{(a^2-b^2)-2abi}$

$$=\frac{(a^2-b^2)-2abi}{(a^2-b^2)^2-(2abi)^2}=\frac{(a^2-b^2)-2abi}{a^4+b^4-2a^2b^2-(-4a^2b^2)}$$

$$=\frac{(a^2-b^2)-2abi}{a^4+b^4-2a^2b^2+4a^2b^2}=\frac{(a^2-b^2)-2abi}{a^4+b^4+2a^2b^2}=\frac{(a^2-b^2)-2abi}{(a^2+b^2)^2}$$

$$=\frac{a^2-b^2}{(a^2-b^2)^2}-\frac{2abi}{(a^2+b^2)^2}$$

Note $(a+b)^3=a^3+b^3+3ab(a+b)$
$=a^3+b^3+3a^2b+3ab^2$

(vi) $(a+bi)^3$

Sol.
$$(a+ib)^3=a^3+3a^2(bi)+3a(bi)^2+(bi)^3$$
$$=a^3+3a^2bi+3a(-b^2)+i^3b^3$$
$$=a^3+3a^2bi-3ab^2-ib^3=(a^3-3ab^2)+i(3a^2b-b^3) \quad \boxed{i^3=-i}$$

(vii) $(a-ib)^3$

Sol.
$$(a-ib)^3=(a^3+(-bi))^3=a^3+3a^2(-ib)+3a(-ib)^2+(-ib)^3$$
$$=a^3-3a^2bi+3a(-b^2)-i^3b^3$$
$$=a^3-3a^2bi-3ab^2-(-i)b^3$$
$$=a^3-3a^2bi-3ab^2+ib^3=(a^3-3ab^2)+i(b^3-3a^2b)$$

(viii) $(3-\sqrt{-4})^{-3}$

Federal 2007

Sol. $(3-\sqrt{-4})^{-3}=(3-i\sqrt{4})^{-3}=(3-2i)^{-3}=\dfrac{1}{(3-2i)^3}$

$$=\frac{1}{(3)^3-(3)^2(2i)+3(3)(2i)^2-(2i)^3}$$

$$=\frac{1}{27-54i+9(-4)-(-i8)}=\frac{1}{27-54i-36+8i}$$

$$=\frac{1}{-9-46i}\times\frac{-9+46i}{-9+46i}=\frac{-9+46i}{(-9)^2-(46i)^2}$$

$$=\frac{-9+46i}{81-(-2116)}=\frac{-9+46i}{81+2116}=\frac{-9+46i}{2197}$$

$$=\frac{-9}{2197}+\frac{46}{2197}$$

TEST YOUR SKILLS Marks: 50

Q # 1. Select the Correct Option **(10)**

i. The property used in inequality $a<0 \Rightarrow -a>0$ is:

a)	Additive	b)	Transitive
c)	Multiplicative	d)	Trichotomy

ii. Multiplicative inverse of $(1,0)$ is:

a)	$(-1,0)$	b)	$(0,1)$
c)	$(0,-1)$	d)	$(1,0)$

iii. Union of Rational and Irrational Numbers is set of

a)	Real numbers	b)	Integers
c)	Whole numbers	d)	Complex numbers

iv. Factors of $9a^2+16b^2$ are

a)	$(3a+4b)(3a-4b)$	b)	$(3a+4ib)(3a-4ib)$
c)	$(3ai+4b)(3ai-4b)$	d)	$(\sqrt{3}a+4ib)(\sqrt{3}a-4ib)$

v. $\frac{22}{7}$ is

a)	Rational numbers	b)	Irrational numbers
c)	Whole numbers	d)	Natural numbers

vi. The number $\sqrt{2}$ is

a)	Natural	b)	Rational
c)	Irrational	d)	Integer

vii. $(-i)^{19}$ equal to

a)	1	b)	-1
c)	i	d)	$-i$

viii. The numbers 0.142857142857.................. is

a)	Natural	b)	Integer
c)	Rational	d)	Irrational

ix. The number $\sqrt{16}$ is called:

a)	Natural	b)	Integer
c)	Rational	d)	Irrational

x. Multiplicative identity in complex number is:

a)	$(1,0)$	b)	$(0,1)$
c)	$(0,0)$	d)	$(1,1)$

Q # 1. Short Questions:

i. Does the $\{0,-1\}$ posses closure property w.r.t '+' & '×'
ii. Find multiplicative inverse of the complex number $(1,2)$
iii. Define Recurring decimal and terminating decimal:
iv. Prove that $\overline{Z} = Z$ iff Z is real.
v. State De Mouvre's Theorem.
vi. Prove that $Z\overline{Z} = |Z|^2 \; \forall \; Z \in C$
vii. Show that $\sqrt{3}$ is an irrational number.
viii. Simplify i^{101}
ix. What is Closure Law of addition in the set of real numbers.
x. Find modulus of $1-\sqrt{3}i$
xi. Simplify $(5,4)\div(-3,-8)$
xii. $\forall Z_1, Z_2 \in C$ show that $\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\overline{Z_1}}{\overline{Z_2}}$
xiii. Find Multiplicative Inverse of $-3i$
xiv. Express $1+i\sqrt{3}$ in polar form.
xv. Simplify $(-1)^{-21/2}$
xvi. For a Real number a,b show that $a(-b) = -ab$
xvii. Factorize $9a^2+16b^2$
xviii. $\forall Z_1, Z_2 \in C$ Show that $\overline{Z_1+Z_2} = \overline{Z_1}+\overline{Z_2}$
xix. State Trichotomy property
xx. Factorize a^2+4b^2

Sets Functions and Groups 2

Set:

Well defined collection of distinct objects is called a set. Well defined, we mean an object that we can separate easily from other objects.

The object in a set are called elements or members of a set Capital letters A, B, C, D, ... are used as names of sets small letters a, b, c, d, ... elements of sets.

Different ways of describing a set:

There are three different ways to describe a set.

i. **Descriptive method**: A method by which a set is described in words

For example. N=The set of all natural number.

ii. **Tabular method**: In this form, we have to write all the elements of a set within the brackets. For example; the set of all natural numbers can be written as:

$$N=\{1,2,3,4,5,6,7..........\}$$

iii. **Set-builder form**: In this form, we use a letter or symbol for an arbitrary element of set and also write the property that is common to all element. For example; the set of natural number. Can be written as $N=\{x \mid x \text{ is any natural numbers}\}$

Some different sets of numbers:

i. **N** = set of all natural numbers = $\{1,2,3,4..........\}$ = set of all +ve integers = Z^{+}

ii. **W** = set of all whole number = $\{0,1,2,3,4..........\}$ = set of non negative integers.

iii. **Z** = set of all integers = $\{0,\pm1,\pm2,\pm3,..........\}$

iv. **Z'**= set of all – ve integers = $\{-1,-2,-3,-4..........\}$

v. **O**= set of all odd integers = $\{\pm1,\pm3,\pm5,..........\}$

vi. **E** = set of all even integers = $\{0,\pm2,\pm4,..........\}$

vii. **Q** = set of all rational numbers = $\left\{x \mid x=\frac{p}{q} \text{ where } p,q\in Z \text{ and } q\neq 0\right\}$

viii. **Q'** = set of all irrational numbers = $\left\{x \mid x\neq\frac{p}{q} \text{ where } p,q\in Z \text{ and } q\neq 0\right\}$

Order of a set Number of elements in a set is called its order: **Lahore 2009**

Membership of a set: The symbol used for a member ship of a set is $\in$ is read as "belongs to" Thus **a** $\in A$ means a is an element of a set A or a belongs to A. If a is not an element of set A. It is written as **a** $\notin A$.

Equal Sets: Two sets A and B are said to be equal sets if each element of one set is an element of other set, written as **A = B.**

Equivalent sets: Two sets are said to be equivalent if one-to–one correspondence can be established between them

Example: If $A=\{1\ 2\ 3\}$; $B=\{a\ b\ c\}$

Then one–to–one correspondence between **A&B** can be established as under:

$$A=\{1\ 2\ 3\}$$
$$\uparrow\uparrow\uparrow$$
$$B=\{a\ b\ c\}$$

Singleton Set: A set having one element is called singleton set.

Null Set: A set having zero number of element is called null set or empty set. It is denoted by $\phi=\{\ \}$

Finite Set: A set having finite number of elements.

Infinite Set: A set having infinite numbers of elements.

Sub Set: If each element of set **A** is also an element set **B**. Then **A** is called subset of **B** written as $A\subseteq B$ and in such a case **B** is called **SUPER SET** of **A**.

Note: (i) Empty Set "ϕ" is subset of every set.

(ii) Every Set is sub set of itself.

Power Set:

The set of all subset of set **A** is called power set of **A**, defined by **P(A).**

Note: Power Set of empty set is not empty.

Proper subset: **Faisalabad 2009**

If **A** is subset of **B** and contains at least on element which is not in **A** than **A** in called proper subset of **B** donated by $A\subset B$

Improper subset:

If **A** is subset of **B** and **A=B** then **A** is Improper subset of **B** its follow that every set is improper subset of its self.

EXERCISE 2.1

1. Write the following sets in set builder notation:

i. $\{1,2,3,......1000\}$

Sol $\{x|x \in N \wedge x \leq 1000\}$

ii. $\{0,1,2,........100\}$

Sol $\{x|x \in W \wedge x \leq 100\}$

iii. $\{0,\pm1,\pm2,........\pm1000\}$

Sol $\{x|x \in Z \wedge -1000 \leq x \leq 1000\}$

iv. $\{0,-1,-2,........-500\}$

Sol $\{x|x \in Z \wedge -500 \leq x \leq 0\}$

v. $\{100,101,102,........400\}$

Sol $\{x|x \in N \text{ and } 100 \leq x \leq 400\}$

vi. $\{-100,-101,-102,........-500\}$

Sol $\{x|x \in Z \wedge -500 \leq x \leq -100\}$

vii. $\{Peshawar, Lahore, Quetta, Karachi\}$

Sol $\{x|x \text{ is a provincial capital of Pakistan}\}$

viii. $\{January, June, July\}$

Sol $\{x|x \text{ is month of Calender year beginning with } J\}$

ix. **The set of all odd natural numbers.**

Sol $\{x|x \text{ is an odd natural number}\}$

x. **The set of all rational numbers.**

Sol $\{x|x \in Q\}$

xi. **The Set of all real numbers between 1 and 2.**

Sol $\{x|x \in \mathbb{R} \wedge 1 < x < 2\}$

xii. **The set of all integers between -100 and 1000**

Sol $\{x|x \in Z \wedge -100 < x < 1000\}$

2. Write each of the following sets in the descriptive and tabular forms:

i. $\{x|x\ N \wedge x \leq 10\}$

Sol **Tabular Forms:** $\{1,2,3,4,............10\}$

Des. Form: set of first ten natural numbers.

ii. $\{x|x \in N \wedge 4 < x < 12\}$

Sol **Tabular Forms:** $\{5,6,7,............11\}$

Des. Form: set of natural numbers between 4 and 12.

iii. $\{x|x \in Z \wedge -5 < x < 5\}$

Sol **Tabular Forms:** $\{-4,-3,-2,-1,0,1,2,3,4\}$

Des. Form: set of all integers between -5 and 5.

iv. $\{x|x \in E \wedge 2 < x \leq 4\}$

Sol **Tabular Forms:** $\{4\}$

Des. Form: set of even numbers between 2 and 5.

v. $\{x|x \in P \wedge x < 12\}$

Sol **Tabular Forms:** $\{2,3,5,7,11\}$

Des. Form: set of prime numbers between 1 and 12.

vi. $\{x|x \in O \wedge 3 < x < 12\}$

Sol **Tabular Forms:** $\{5,7,9,11\}$

Des. Form: set of odd integers between 3 and 12.

vii. $\{x|x \in E \wedge 4 \leq x \leq 10\}$

Sol **Tabular Forms:** $\{4,6,8,10\}$

Des. Form: The Set of even integers from 4 to 10.

viii. $\{x|x \in E \wedge 4 < x < 6\}$

Sol **Tabular Forms:** $\{\ \}$

Des. Form: The Set of even integers between 4 and 6.

ix. $\{x|x \in O \wedge 5 \leq x \leq 7\}$ **Rawalpindi 2009**

Sol **Tabular Forms:** $\{5,7\}$

Des. Form: The Set of odd integers from 5 up to 7.

x. $\{x|x \in O \wedge 5 < x < 7\}$

Sol **Tabular Forms:** $\{\ \}$

Des. Form: The Set of odd integers between 5 and 7.

xi. $\{x | x \in N \wedge x+4=0\}$

Sol **Tabular Forms:** $\{\ \}$

Des. Form: The Set of natural numbers x, satisfying x + 4 = 0

xii. $\{x | x \in Q \wedge x^2 = 2\}$ **Multan 2010**

Sol **Tabular Forms:** $\{\ \}$

Des. Form: The Set of rational numbers x, satisfying $x^2 = 2$

xiii. $\{x | x \in R \wedge x = x\}$

Sol **Tabular Forms:** $\mathbb{R}$

Des. Form: The Set of real numbers x, satisfying $x = x$

$x = x$ is satisfying by all real numbers.

xiv. $\{x | x \in Q \wedge x = -x\}$

Sol **Tabular Forms:** $\{0\}$

Des. Form: The Set of rational numbers satisfying $x = -x$

$\therefore x = -x \Rightarrow 2x = 0 \ or \ x = 0$

xv. $\{x | x \in \mathbb{R} \wedge x \neq 2\}$

Sol **Tabular Forms:** $\mathbb{R} - \{2\}$

Des. Form: The Set of real numbers x, except 2

xvi. $\{x | x \in \mathbb{R} \wedge x \notin Q\}$

Sol **Tabular Forms:** Q'

Des. Form: The Set of real numbers x, which are not rational so it will set of irrational numbers.

3. **Which of the following sets are finite and which of these are infinite?**

i. **The set of students of your class.**

Sol Finite

ii. **The set of all schools in Pakistan.**

Sol Finite

iii. **The set natural numbers between 3 and 10.**

Sol Finite

iv. **Set of rational numbers between 3 and 10.**

Sol Infinite

v. **The set of real numbers between 0 and 1.**

Sol Infinite

vi. **The set of rationales between 0 and 1.**

Sol Infinite

vii. **The set of whole numbers between 0 and 1.**

Sol Finite

viii. **The set of all leaves of trees of Pakistan.**

Sol Infinite

ix. $P(N)$:

Sol Infinite

x. $P(a,b,c)$

Sol Finite

xi. $\{1,2,3,4................\}$

Sol Infinite

xii. $\{1,2,3...........,100,000,0000\}$

Sol Finite

xiii. $\{x|x \in R \wedge x \neq x\}$

Sol Finite

xiv. $\{x|x \in R \wedge x^2 = -16\}$

Sol Finite

xv. $\{x|x \in Q \wedge x^2 = 5\}$

Sol Finite

xvi. $\{x|x \in Q \wedge 0 \leq x \leq 1\}$

Sol Infinite

4. **Write two proper subsets of each of the following sets:**

i. $\{a,b,c\}$

Sol $\{a\},\{b\}$

ii. $\{0,1\}$

Sol $\{0\},\{1\}$

iii. N

Sol $N = \{1,2,.............\}$

$\{1\},\{2\}$

iv. Z

Sol $Z = \{0,\pm1,\pm2...........\}$

$\{1\},\{2\}$

v. $\mathbb{R}$

Sol $\mathbb{R} = set\ of\ real\ numbers$

$\{1\},\{2\}$

vi. W

Sol $W = set\ of\ whole\ numbers$

$\{1\},\{2\}$

vii. $\{x|x \in Q \wedge 0 \leq x \leq 2\}$

Sol $\{1\},\{2\}$

5. **Is there any set which has no proper subset? If so name the set.** **Lahore 2009**

Sol Yes, ϕ is set which has no proper subset.

6. **What is the difference between $\{a,b\}$ and $\{\{a,b\}\}$ Faisalabad 2008, Sargodha 2009**

Sol $\{a,b\}$ is a set with o elements and $\{\{a,b\}\}$ is set with one element $\{a,b\}$

7. **Which of the following sentences are true and which of them are false?**

i. $\{1,2\} = \{2,1\}$

Sol True

ii. $\phi \subseteq \{\{2,1\}\}$

Sol True

iii. $\{a\} \subseteq \{\{a\}\}$

Sol False

iv. $\{a\} \in \{\{a\}\}$

Sol True

v. $a \in \{\{a\}\}$

Sol False

vi. $\phi \in \{\{a\}\}$

Sol False

8. What is the number of elements of the power set of the each of the following sets?

i. $\{\ \}$

Sol Power set of $\{\ \}$ has elements $= 2^o = 1$

ii. $\{0,1\}$

Sol Power set of $\{0,1\}$ has elements $= 2^2 = 4$

iii. $\{1,2,3,4,5,6,7\}$

Sol Power set of $\{1,2,3,4,5,6,7\}$ has elements $= 2^7 = 128$

iv. $\{0,1,2,3,4,5,6,7\}$

Sol Power set of $\{0,1,2,3,4,5,6,7\}$ has elements $= 2^8 = 256$

v. $\{a,\{b,c\}\}$

Sol Power set of $\{a,\{b,c\}\}$ has elements $= 2^2 = 4$

vi. $\{\{a,b\},\{b,c\},\{d,c\}\}$

Sol Power set of $\{\{a,b\},\{b,c\},\{d,c\}\}$ has elements $= 2^3 = 8$

9. Write down the power set of each of the following sets:

Sol (i) $\{9,11\}$ *Power set is* $\{\phi\{9\},\{11\},\{9,11\}\}$

(ii) $\{+,-,\times,\div\}$ **Sargodha 2010**

Power set is $\{\phi, \{+\},\{-\},\{\times\},\{\div\},\{+,-\},\{+,\times\},\{+,\div\},\{-,\times\}\{-,\div\},\{\times,\div\}$
$\{+,-,\times\},\{+,-,\div\},\{+,\times,\div\},\{-,\times,\div\},\{+,-,\times,\div\}\}$

(iii) $\{\phi\}$

Sol Power set of $\{\phi\}$ *is* $= \{\phi,\{\phi\}\}$

(iv) $\{a,\{b,c\}\}$ **Sargodha 2009**

Sol Power set $= \{\phi\{a\},\{b,c\},\{a\{b,c\}\}$

10. Which pair of sets are equivalent? Which of them are also/equal?

i. $\{a,b,c\},\{1,2,3\}$

Sol Equivalent

ii. **The set of the first 10 whole numbers, $\{0,1,2,3..........9\}$**

Sol Equal

iii. **Set of angles of a quadrilateral ABCD, set of the sides of the same quadrilateral**

Sol Equivalent

iv. **Set of the sides of a hexagon ABCDEF, set of the angles of the same hexagon:**

Sol Equivalent

v. $\{1,2,3,4,............\},\{2,4,6,8,............\}$

Sol Equivalent

vi. $\{1,2,3,4,............\},\left\{1,\frac{1}{2}\frac{1}{3}\frac{1}{4}............\right\}$

Sol Equivalent

vii. $\{5,10,15..........5555\},\{5,10,15,20............\}$

Sol Neither equivalent nor equal sets.

Union of two Sets:

Union of two sets A and B, denoted by $A\cup B$ is the set of all elements, which belongs to A or B: symbolically;

$$A\cup B=\{x|x\in A\vee x\in B\}$$

Example: If $A=\{1,2,3,\};B=\{2,3,4,5\}$, *then* $A\cup B=\{1,2,3,4,5\}$

Intersection of two sets:

A and B denoted by $A\cap B$, is the set of all elements, which belong to both A and B:

symbolically; $A\cap B=\{x|x\in A\wedge x\in B\}$

Example: If $A=\{1,2,3,\};B=\{2,3,4,5\}$, *then* $A\cap B=\{2,3\}$

Disjoint Sets:

If intersection of two set A and B is empty. Then sets A and B are called Disjoint Sets.

Example: $O\cap E=\phi$ Where 'O' is set of odd integers 'E' is even.

Overlapping Sets:

If the intersection of two sets A and B is non-empty but neither is subset of the other, then such sets are called overlapping Sets.

Example: Let $A=\{1,2,3,4\};B=\{3,4,5,6\};A\cap B=overlapping\ \ set=\{3,4\}$

Complement of a Set:

If U is universal set, then U/A or U – A is called Complement of A, denoted by A' *or* A^c. *Thus* $A'=A^c=U-A$

Symbolically $A'=\{x|x\in U\wedge x\notin A\}$

Example: If $U=N$, *then* $E'=O$ *and* $O'=E$

Difference of Two Sets:

The difference $A-B$ or A/B of two sets A and B is the set of elements which belong to A but do not belong to B.

Symbolically $A-B=A/B=\{x|x\in A\wedge x\notin B\}$

Example: If $A=\{1,2,3,4,5\};B=\{4,5,6,7\};A-B=\{1,2,3\}$

Note: $A-B\neq B-A$ *because* $B-A=\{6.7\}$

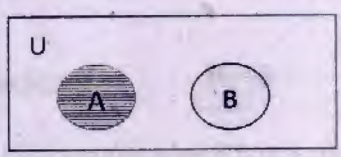
Venn Diagram:

(named by "**JOHN VENN**" The English Logician and Mathematician **(1834-83) A.D** (it is the picture representation of given sets in the form of rectangle and circles). In Venn Diagram, rectangular region represents universal set U and circular region represent given sets.

Venn Diagrams of given Sets.

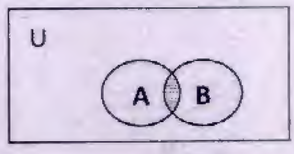
1. $A\cup B$

When A and B are disjoint sets OR when $A\cap B=\phi$



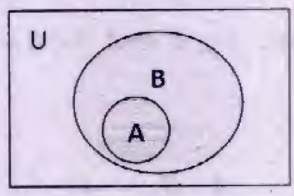
2. $A\cup B$

When A and B are overlapping set OR when $A\cap B\neq\phi$



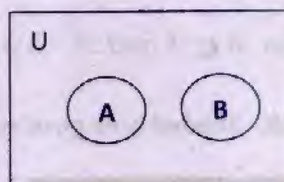
3. $A\cup B$

When $A\subseteq B$



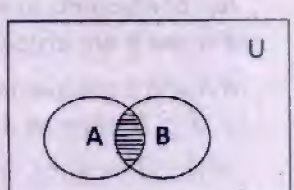
4. $A\cap B$

When A and B are Disjoint Set i.e $A\cap B=\phi$



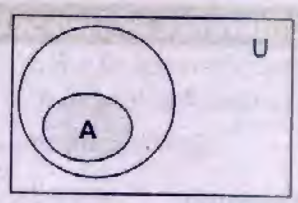
5. $A\cap B$

When A and B are overlapping sets i.e. $A\cap B\neq\phi$



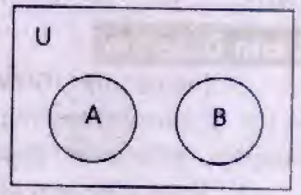
6. $A \cap B$

When $A \subseteq B$



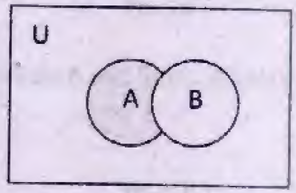
7. $A - B = A / B$ when $A \cap B = \varphi$

When A and B are Disjoint sets i.e., $A \cap B = \phi$



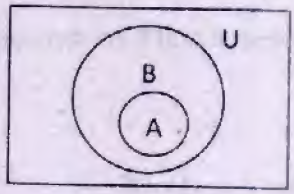
8. $A - B$ **Faisalabad 2008**

When A and B over lapping sets i.e when $A \cap B \neq \phi$



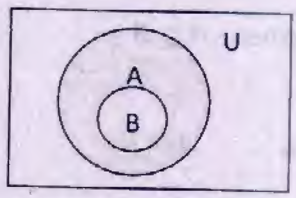
9. $A - B$

When $A \subseteq B$ and $A - B = \phi$



10. $A - B$

When $B \subseteq A$ and $A - B \neq \phi$



Note: Shaded area gives required region or required result

Number of elements:

(i) No. of elements in set A is denoted by n (A).

(ii) If A and B are disjoint sets then $n(A \cup B) = n(A) + n(B)$

(iii) If A and B are overlapping sets, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(iv) If $A \subseteq B$; then $n(A \cup B) = n(B)$ and $n(A \cap B) = n(A)$

(v) $n(A - B) = n(A) - n(A \cap B)$

(vi) $n(B - A) = n(B) - n(A \cap B)$

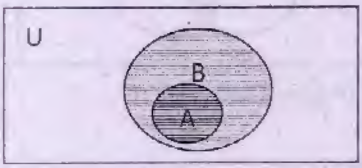
EXERCISE 2.2

1. Exhibit $A \cup B$ and $A \cap B$ by Venn Diagrams in the following cases:

i. $A \subseteq B$

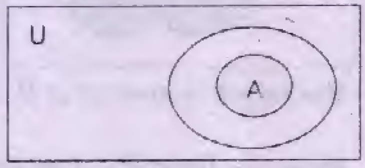
Sol $A \cup B$: *when* $A \subseteq B$

Dotted region represents $A \cup B$



$A \cap B = ?$ *when* $A \subseteq B$

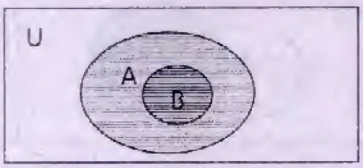
Doted region represents $A \cap B$



ii. $B \subseteq A$

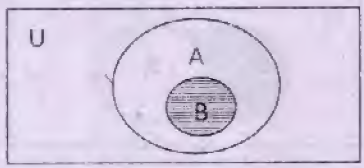
Sol: $A \cup B$: *when* $B \subseteq A$

Dotted region shows $A \cup B$



$A \cap B = ?$ *when* $B \subseteq A$

Doted region gives $A \cap B$

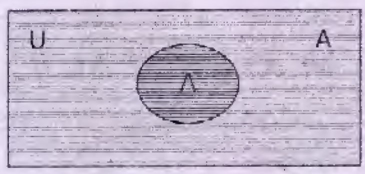


iii. $A \cup A'$

Sol : $A \cup A' = ?$

Dotted region represents

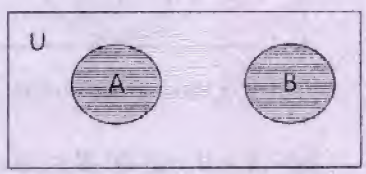
$A \cup A' = U$



iv. **A and B are Disjoints sets.**

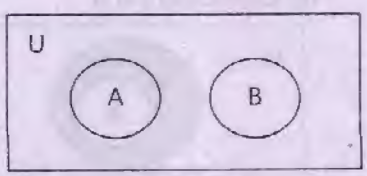
Sol : $A \cup B = ?$ When A and B are disjoint sets.

Shaded region represents $A \cup B$



v. $A \cap B = ?$ **when A and B are disjoint sets.**

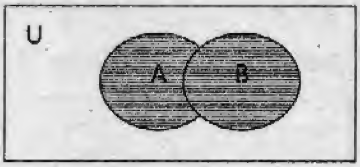
Sol Blank region represents $A \cap B$. Because according to the condition $A \cap B = \varnothing$



vi. A and B are over lapping Sets.

Sol $A \cup B = ?$

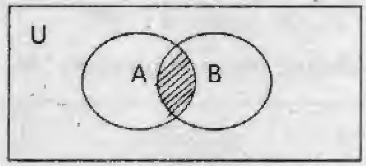
When A and B are overlapping sets.



Shaded region gives $A \cup B$

$A \cap B = ?$.

When A and B are overlapping sets.

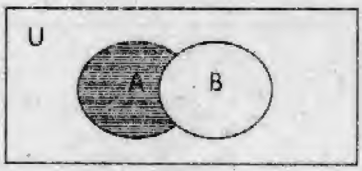


Shaded region gives $A \cap B$

2. Show $A-B$ and $B-A$ by Venn Diagrams when:

i. (a) If A and B are overlapping

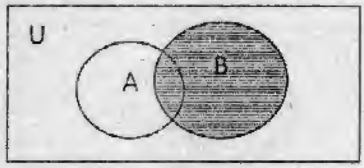
Sol $A-B=?$



Shaded region gives $A-B$

(b) If A and B are overlapping set, then

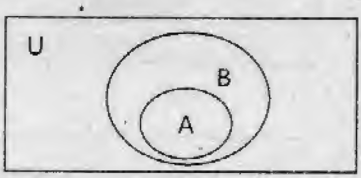
$B-A=?$



Shaded region gives $B-A$

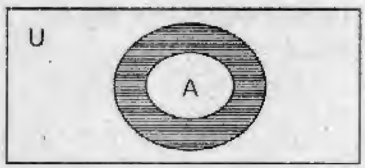
ii. (a) $A-B=?$ *If* $A \subseteq B$

Sol



Which is Venn diagram of $A-B$

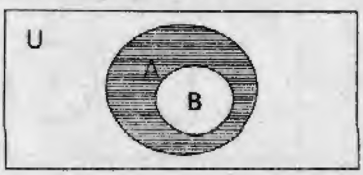
(b) $B-A=?$ *If* $A \subseteq B$



Which is Venn diagram of $B-A$

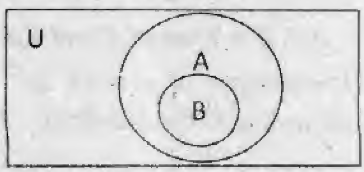
iii. (a) $A-B=?$ *If* $B \subseteq A$

Sol Its Venn diagram is



(b) $B-A=?$ *If* $B \subseteq A$

Its Venn diagram is



3. **Under what conditions on A and B are the following statements true?**

i. $A \cup B = A$

Sol. $If\ B \subseteq A$

ii. $A \cup B = B$

Sol. $If\ A \subseteq B$

iii. $A - B = A$

Sol. $If\ A \cap B = \phi$

iv. $A \cap B = B$

Sol. $If\ B \subseteq A$

v. $n(A \cup B) = n(A) + n(B)$

Sol. If A and B is are disjoint sets.

vi. $n(A \cap B) = n(A)$

Sol. $If\ A \subseteq B$

vii. $A - B = A$

Sol. If A and B disjoint or $A \cap B = \phi$

viii. $n(A \cap B) = 0$

Sol. $If\ A \cap B = \phi$

ix. $A \cup B = U$ **Multan 2009**

Sol. $If\ B = A'\ or\ B' = A$

x. $A \cup B = B \cup A$

Sol. It is always true.

xi. $n = (A \cap B) = n(B)$

Sol. $If\ B \subseteq A$

xii. $U - A = \phi$

Sol. $If\ U = A$

4. Let
$U = \{1, 2, 3 \ldots\ldots\ldots\ldots 10\}$, $A = \{2, 4, 6, 8, 10\}$, $B = \{1, 2, 3, 4, 5\}$ *and* $C = \{1, 3, 5, 7, 9\}$ **List the numbers of each of the following sets.**

i. A^c

Sol. $A^c = U - A = \{1, 2, 3 \ldots\ldots 10\} - \{2, 4, 6, 8, 10\} = \{1, 3, 5, 7, 9\} = C$

ii. B^c

Sol. $B^c = U - B = \{1, 2, 3 \ldots\ldots 10\} - \{1, 2, 3, 4, 5\} = \{6, 7, 8, 9, 10\}$

iii. $A \cup B$

Sol. $A \cup B = \{2, 4, 6, 8, 10\} U \{1, 2, 3, 4, 5\}$

$\Rightarrow A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$

iv. $A-B$

Sol. $A-B=\{2,4,6,8,10\}-\{1,2,3,4,5\}$

$or\ A-B=\{6,8,10\}$

v. $A\cap C$

Sol. $A\cap C=\{2,4,6,8,10\}\cap\{1,2,3,4,5\}$

$A\cap C=\{\ \}=\phi$

vi. $A^c\cup C^c$

Sol. $A^c\cup C^c=\{1,3,5,7,9\}\cup\{2,4,6,8,10\}$

$A^c\cup C^c=\{1,2,3,4,..........10\}$

vii. $A^c\cup C$

Sol. $A^c\cup C=\{1,3,5,7,9\}\cup\{1,3,5,7,9\}$

$A^c\cup C=\{1,3,5,7,9\}$

viii. U^c

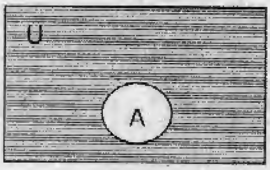
Sol. $U^c=U-U$

$=\{1,2,3,.............10\}-\{1,2,3,...........10\}=\phi$

5. **Using Venn diagrams, If necessary, find the single sets of equal to the following**

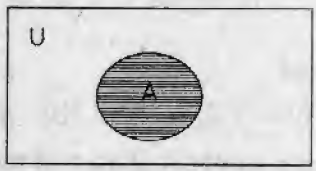
i. A'

Sol. $\therefore A'=U-A$



ii. $A\cap U$

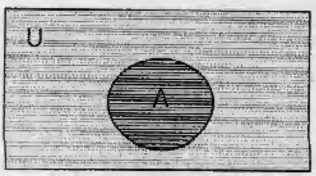
Sol. $A\cap U=$ set of common elements =A



iii. $A\cup U$

Sol. $\therefore A\cup U=U$

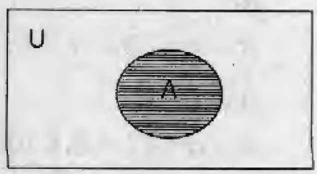
Shaded region shows $A\cup U$



iv. $A\cup\phi$

Sol. $A\cup\phi=A$

Shaded region shows $A\cup\phi$



v. $\phi \cap \phi$

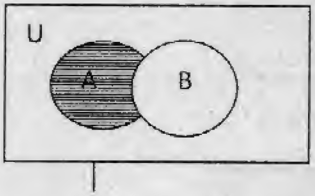
Sol $\phi \cap \phi = \{\ \}$

6. **Use Venn diagram to verify the following:**

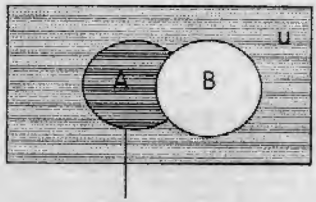
i. $\boldsymbol{A - B = A \cap B'}$

Sol. From Venn diagram of $A - B$ *and* $A \cap B'$

We see that $A - B = A \cap B'$



$A - B$

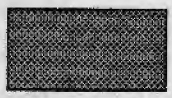


$A \cap B^c$

ii. $\boldsymbol{(A - B)^c \cap B = B}$

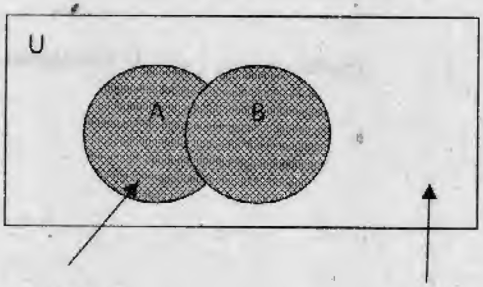
Sol. Use diagram to verify $(A - B)^c \cap B = B$

Case-I When A and B are overlapping

Here $A - B =$ 

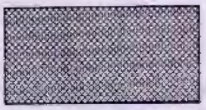
$(A - B)^c =$ 

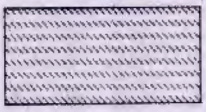
From Venn diagram $(A - B)^c \cap B =$

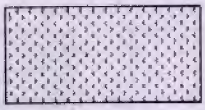


$A - B$ $(A - B)^c$

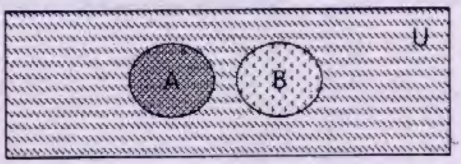
Case-II: When A and B are disjoint sets ; then

$A-B=$ 

$(A-B)^c=$ 

$(A-B)^c\cap B=$ 

From Venn diagram $(A-B)^c\cap B=B$



PROPERTIES OF UNION AND INTERSECTION

(Sargodha 2008, Lahore 2009)

i.	$A\cup B=B\cup A;$	Commutative property of union.
ii.	$A\cap B=B\cap A;$	Commutative property of Intersection
iii.	$(A\cup B)\cup C=A\cup(B\cup C);$	Associative property of union
iv.	$(A\cap B)\cap C=A\cap(B\cap C);$	Associative property of Intersection
v.	$A\cup(B\cap C)=(A\cup B)\cap(A\cup C);$	Distributive property of union over Intersection **(Faisalabad 2009)**
vi.	$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$	Distributive property of intersection over union
vii.	$\left.\begin{array}{l}(A\cup B)'=A'\cap B'\\(A\cap B)'=A'\cup B'\end{array}\right\}$	De Morgan's Laws. **(Faisalabad 2008)**

Exercise 2.3

1. Verify the commutative properties of union and intersection for the following pairs of sets:

i.(a) $A \cup B = B \cup A$

$A = \{1,2,3,4,5\}$, $B = \{4,6,8,10\}$

Sol. $A \cup B = \{1,2,3,4,5\} \cup \{4,6,8,10\} = \{1,2,3,4,5,6,8,10\} \rightarrow 1$

$B \cup A = \{4,6,8,10\} \cup \{1,2,3,4,5\} = \{1,2,3,4,5,6,8,10\} \rightarrow 2$

From 1 & 2 $A \cup B = B \cup A$

i.(b) $A \cap B = B \cap A$

$A \cap B = \{1,2,3,4,5\} \cap \{4,6,8,10\} = \{4\} \longrightarrow 1$

$B \cap A = \{4,6,8,10\} \cap \{1,2,3,4,5\} \Rightarrow B \cap A = \{4\} \rightarrow 2$

From 1 and 2

$A \cap B = B \cap A$ proved

ii. N

Sol. N = set of natural numbers Z = set of integers

Given sets are N and Z then

$$Z \cup N = Z$$
$$N \cup Z = Z$$
$$N \cap Z = N$$
$$Z \cap N = N$$

$So \quad N \cup Z = Z \cup N$

$and \quad N \cap Z = Z \cap N$

iii. $A = \{x | x \in \mathbb{R} \wedge x \geq 0\}$ *and* $B = \mathbb{R}$

Sol. $A \cup B = \{x | x \in \mathbb{R} \wedge x \geq 0\} \cup \mathbb{R} = \mathbb{R}$

$B \cup A = \mathbb{R} \cup \{x | x \in \mathbb{R} \wedge x \geq 0\} = \mathbb{R}$

$A \cap B = \{x | x \in \mathbb{R} \wedge x \geq 0\} \cap \mathbb{R}$

$= \{x | x \in \mathbb{R} \wedge x \geq 0\}$

$B \cap A = \mathbb{R} \cap \{x | x \in \mathbb{R} \wedge x \geq 0\}$

$= \{x | x \in \mathbb{R} \wedge x \geq 0\}$

$A \cup B = B \cup A$

$and \quad A \cap B = B \cap A$

2. Verify the properties for the sets A,B and C given below:

i. Associative Law of Union

$A=\{1,2,3,4\}$

$B=\{3,4,5,6,7,8\}$

$C=\{5,6,7,9,10\}$

Sol. Associative Law of union $A\cup(B\cup C)=(A\cup B)\cup C$

L.H.S$=A\cup(B\cup C)$

$=\{1,2,3,4\}\cup[\{3,4,5,6,7,8,\}\cup\{5,6,7,9,10\}]$

$=\{1,2,3,4\}\cup\{3,4,5,6,7,8,9,10\}$

$=\{1,2,3,4,5,............10\} \rightarrow 1$

R.H.S$=(A\cup B)\cup C)$

$=[\{1,2,3,4\}\cup\{3,4,5,6,7,8,\}]\cup\{5,6,7,9,10\}$

$=\{1,2,3,4,5,6,7,8\}\cup\{5,6,7,8,9,10\}$

$=\{1,2,3,4,5,............10\} \rightarrow 2$

From 1 and 2;

$A\cup(B\cup C)=(A\cup B)\cup C$

ii. Associativity of intersection

Sol. $A\cap(B\cap C)=(A\cap B)\cap C$

L.H.S$=A\cap(B\cap C)$

$=\{1,2,3,4\}\cap\{3,4,5,6,7,8,\}\cap\{5,6,7,9,10\}$

$=\{1,2,3,4\}\cap\{5,6,7\}$

$=\{\ \} \rightarrow 1$

R.H.S$=(A\cap B)\cap C)$

$=[\{1,2,3,4\}\cap\{3,4,5,6,7,8,\}]\cap\{5,6,7,9,10\}$

$=\{3,4\}\cap\{5,6,7,9,10\}=\{\ \} \rightarrow 2$

From 1 and 2

$A\cap(B\cap C)=(A\cap B)\cap C$

iii. Distributivity of union over intersection

Sol. $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

L.H.S$=A\cup(B\cap C)$

$=\{1,2,3,4\}\cup[\{3,4,5,6,7,8,\}\cap\{5,6,7,9,10\}]$

$=\{1,2,3,4\}\cup\{5,6,7,\}=\{1,2,3,4,5,6,7\} \rightarrow 1$

R.H.S $= (A\cup B)\cap(A\cup C)$

$= [\{1,2,3,4\}\cup\{3,4,5,6,7,8,\}]\cap[\{1,2,3,4\}\cup\{5,6,7,9,10\}]$

$= \{1,2,3,4,5,6,7,8\}\cap\{1,2,3,..........7,9,10\} = \{1,2,3,.........7\} \rightarrow 2$

From 1 and 2 L.H.S = R.H.S.

iv. Distributativty of $\cap$ over $\cup$

Sol. $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

L.H.S $= A\cap(B\cup C)$

$= \{1,2,3,4\}\cap[\{3,4,5,6,7,8,\}\cup\{5,6,7,9,10\}]$

$= \{1,2,3,4\}\cap\{3,4,5,6,7,8,9,10\}$

L.H.S $= \{3,4\} \rightarrow 1$

R.H.S $= (A\cap B)\cup(A\cap C)$

$= [\{1,2,3,4\}\cap\{3,4,5,6,7,8,\}]\cup[\{1,2,3,4\}\cap\{5,6,7,9,10\}]$

$= \{3,4\}\cup\{\ \} = \{3,4\} \rightarrow 2$

From 1 and 2 we get.

L.H.S = R.H.S

Part ii. $A=\phi;\ B=\{0\};\ C=\{0,1,2\}$

Sol. Given $A=\phi;\ B=\{0\};\ C=\{0,1,2\}$ then

(a) Associativity of union; $A\cup(B\cup C)=(A\cup B)\cup C \quad \rightarrow I$

Putting value in 1, we get $\phi\cup[\{0\}\cup\{0,1,2\}]=[(\phi\cup\{0\})]\cup\{0,1,2\}$

$\Rightarrow \phi\cup\{0,1,2\}=\{0\}\cup\{0,1,2\}$

$\{0,1,2\}=\{0,1,2\}$

L.H.S = R.H.S

b. Associtivity of Intersection $A\cap(B\cap C)=(A\cap B)\cap C \quad \rightarrow I$

Sol. Putting values in 1, we get

$\phi\cap[\{0\}\cap\{0,1,2\}]=\{(\phi\cap\{0\}\}\cap\{0,1,2\}$

$\phi\cap\{0\}=\phi\cap\{0,1,2\}$

$\phi=\phi$

L.H.S = R.H.S

c. Distributive Law of $\cup$ over $\cap$

Sol. $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

L.H.S $= A\cup(B\cap C)$

$= \phi\cup[(\{0\}\cap\{0,1,2\})]$

$= \phi \cup \{0\} = \{0\} \quad \to 1$

R.H.S $= (A \cup B) \cap (A \cup C)$

$= [(\{\ \} \cup \{0\})] \cap [(\{\ \} \cup \{0,1,2\})]$

$= \{0\} \cap \{0,1,2\} = \{0\} \quad \to 2$

From 1 and 2; L.H.S = R.H.S

d. Distributive Law of $\cap$ over $\cup$

Sol. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

L.H.S $= A \cap (B \cup C)$

$= \phi \cap [(\{0\} \cup \{0,1,2\})]$

$= \phi \cap \{0,1,2\} = \phi \quad \to 1$

R.H.S $= (A \cap B) \cup (A \cap C)$

$= [(\phi \cap \{0\})] \cup [(\phi \cap \{0,1,2\})]$

$= \phi \cup \phi = \phi \quad \to 2$

From 1 and 2

L.H.S = R.H.S

Part-iii. N, Z, Q

Sol. Given $N \le Z \le Q$

$N = \{1,2,3,4,..........\}$

$Z = \{0, \pm1, \pm2, \pm3, ...\}$

$Q = Set\ of\ rational\ numbers$

a. Associativity of Union

Sol. $N \cup (Z \cup Q) = (N \cup Z) \cup Q$

$N \cup Q = Z \cup Q (\because N \le Z \le Q)$

$Q = Q$

L.H.S = R.H.S ved

b. Associativity of Intersection

Sol. $N \cap (Z \cap Q) = (N \cap Z) \cap Q$

$\Rightarrow N \cap Z = N \cap Q (\because N \le Z \le Q)$

$N = N$

$\Rightarrow$ L.H.S = R.H.S proved

c. Distributivity of $\cup$ over $\cap$

Sol. $N \cup (Z \cap Q) = (N \cup Z) \cap (N \cup Q)$

$\Rightarrow N \cup Z = Z \cap Q (\because N \le Z \le Q)$

$Z = Z$

$\Rightarrow$ L.H.S = R.H.S proved

d. **Distributivity of $\cap$ over $\cup$**

Sol. $N \cap (Z \cup Q) = (N \cap Z) \cup (N \cap Q)$

$\Rightarrow N \cap Q = N \cup N (\because N \leq Z \leq Q)$

$N = N$

$\Rightarrow$ L.H.S = R.H.S proved

3. **Verify De Morgan's Laws for the following sets:**

$U = \{1, 2, 3 \ldots\ldots\ldots, 20\}$, $A = \{2, 4, 6, \ldots\ldots\ldots, 20\}$ *and* $B = \{1, 3, 5, \ldots\ldots\ldots, 19\}$

Sol.(i) **We have to prove** $(A \cap B)' = A' \cup B'$

L.H.S $= (A \cap B)'$

Where $A \cap B = \{2, 4, 6, \ldots\ldots\ldots, 20\} \cap \{1, 3, 5, \ldots\ldots\ldots, 19\}$

$A \cap B = \phi = \{\ \}$

$(A \cap B)' = U - (A \cap B) = U - \phi = U \rightarrow 1$

R.H.S $= A' \cup B'$

Where $A' = U - A$

$= \{1, 2, 3, \ldots\ldots\ldots, 20\} - \{2, 4, 6, \ldots\ldots\ldots, 20\}$

$A' = \{1, 3, 5, \ldots\ldots\ldots, 19\}$

$B' = U - B$

$= \{1, 2, 3, \ldots\ldots\ldots, 20\} - \{1, 3, 5, \ldots\ldots\ldots, 19\}$

$B' = \{2, 4, 6, \ldots\ldots\ldots 20\}$

$A' \cup B' = \{1, 3, 5, \ldots\ldots\ldots 19\} \cup \{2, 4, 6, \ldots\ldots\ldots 20\}$

$A' \cup B' = \{1, 2, 3, 4, \ldots\ldots\ldots, 20\} = U \rightarrow 2$

From 1 and 2

L.H.S = R.H.S

ii. **We have to prove that $(A \cup B)' = A' \cap B'$**

Sol. L.H.S $= (A \cup B)'$

$A \cup B = \{2, 4, 6, \ldots\ldots\ldots, 20\} \cup \{1, 3, 5, \ldots\ldots\ldots 19\} = \{1, 2, 3, \ldots\ldots 20\} = U$

$(A \cup B)' = U - (A \cup B) = U - U = \phi \rightarrow 1$

R.H.S $= A' \cap B'$

Where $A' = U - A$

$A' = \{1, 2, 3, 4, \ldots\ldots\ldots, 20\} - \{2, 4, 6, \ldots\ldots\ldots, 20\}$

$A' = \{1, 3, 5, \ldots\ldots\ldots, 19\}$

$B' = U - B = \{1, 2, 3, \ldots\ldots\ldots, 20\} - \{1, 3, 5, \ldots\ldots\ldots 19\} = \{2, 4, 6, \ldots\ldots\ldots, 20\}$

$A' \cap B' = \phi \rightarrow 2$

From 1 and 2 $\Rightarrow$ L.H.S = R.H.S

4. Let U = The set of the English alphabet;

$A = \{x | x \text{ is a vowel}\}, \; B = \{y | y \text{ is a consonant}\}$

Verify De Morgan's Laws for these sets.

Sol. We want to prove that

$(A \cup B)' = A' \cap B'$

L.H.S $= (A \cup B)'$

Now $(A \cup B) = \{x | x \text{ is a vowel}\} \cup \{y | y \text{ is a consonant}\}$.

$A \cup B =$ Set of English alphabet $= \cup$

$(A \cup B)' = U - (A \cup B) = U - U = \{ \; \} \quad \rightarrow 1$

R.H.S $= A' \cap B'$

Where $A' = U - A = U - \{x | x \text{ is a vowel}\}$

$A' = \{y | y \text{ is a consonant}\}$

and $B' = U - B = U - \{y | y \text{ is a consonant}\}$

$B' = \{x | x \text{ is a vowel}\}$

Then $A' \cap B' = \{y | y \text{ is a consonant}\} \cap \{x | x \text{ is a vowel}\}$

$A' \cap B' = \{ \; \} \quad \rightarrow 2$

From 1 and 2

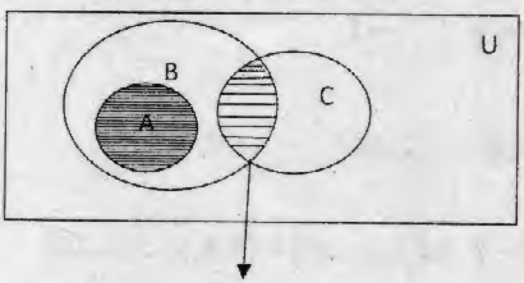
L.H.S = R.H.S

5. With the help of Venn diagram, verify the two distributive properties in the following cases w.r.t union and intersection.

i. (a) $A \subseteq B, A \cap C = \varnothing$ **and B and C are overlapping.**

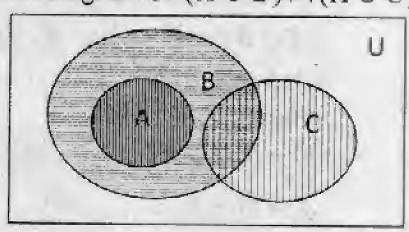
Sol. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

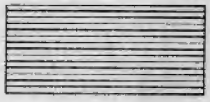
Venn diagram of $A \cup (B \cap C)$



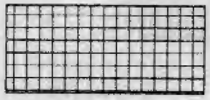
$B \cap C$

Venn diagram of $(A \cup B) \cap (A \cup C)$



$A \cup B =$ 

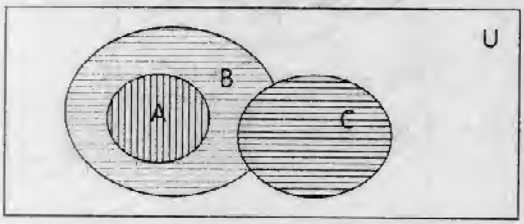
$A \cup C =$ 

$(A \cup B) \cap (A \cup C) =$
$A \cup (B \cap C) =$ 

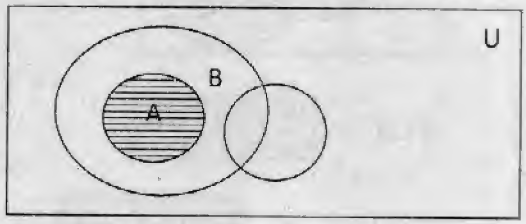
From Venn diagram. It is clear that $A \cup (B \cap A) = (A \cup B) \cap (A \cup C)$

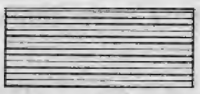
(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

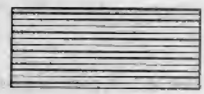
Sol. Venn diagram of $A \cap (B \cup C)$

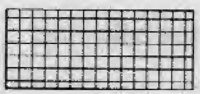


Venn diagram of $(A \cap B) \cup (A \cap C)$

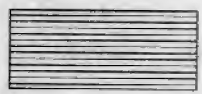


$B \cup C =$ 

$A \cap B =$ 

$A \cap (B \cup C) =$ 

$A \cap C =$ no thing is common.

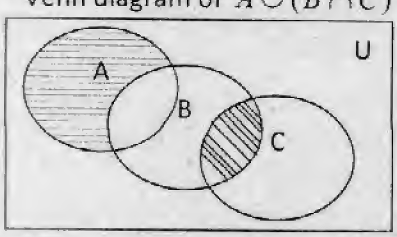
$\therefore (A \cap B) \cup (A \cap C) =$ 

From Venn diagram. It is clear that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

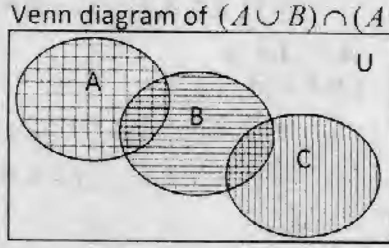
ii. (a) **A and B and are the overlapping, B and C are overlapping but A and C are disjoint.**

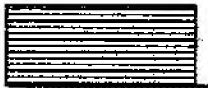
Sol. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

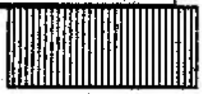
Venn diagram of $A \cup (B \cap C)$

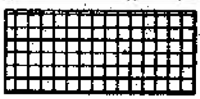


Venn diagram of $(A \cup B) \cap (A \cup C)$



$A \cup B =$ 

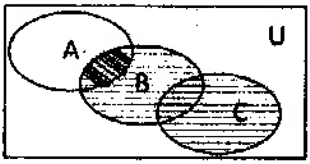
$A \cup C =$ 

$(A \cup B) \cap (A \cup C) =$ 

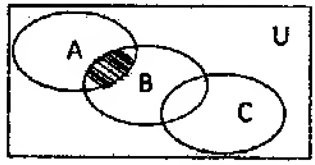
From Venn diagram obviously $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

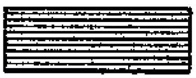
(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

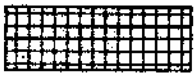
Sol. Venn diagram of $A \cap (B \cup C)$

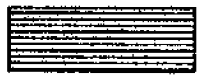


Venn diagram of $(A \cap B) \cup (A \cap C)$



$B \cup C =$ 

$A \cap (B \cup C) =$ 

$A \cap B =$ 

$A \cap C =$ no common elements

From Venn Diagram it is clear that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6. **Taking any set, say $A = \{1,2,3,4,5\}$ verify the following:**

i. $A \cup \varnothing = A$

Sol. L.H.S $= A \cup \varnothing$

$= \{1,2,3,4,5\} \cup \varnothing$

$= \{1,2,3,4,5\} = A =$ R.H.S

ii. $A \cup A = A$

Sol. L.H.S $= A \cup A = A$

$= \{1,2,3,4,5\} \cup \{1,2,3,4,5\}$

$= \{1,2,3,4,5\} = A =$ R.H.S

iii. $A \cap A = A$

Sol. L.H.S $= A \cap A$

$= \{1,2,3,4,5\} \cap \{1,2,3,4,5\}$

$= \{1,2,3,4,5\} = A =$ R.H.S

7. **If $U=\{1,2,3,4,5,\ldots\ldots\ldots,20\}$ and $A=\{1,3,5,\ldots\ldots\ldots,19\}$ verify the following:**

i. $A\cup A'=U$ **Mulatan 2008**

Sol. L.H.S $=A\cup A'=\{1,2,3,4,5,\ldots\ldots\ldots,20\}-\{1,3,5,\ldots\ldots\ldots,19\}$

Where $A'=U-A=\{2,4,6,\ldots\ldots\ldots,20\}$

L.H.S $=A\cup A'=\{1,3,5,\ldots\ldots\ldots,19\}\cup\{2,4,\ldots\ldots\ldots,20\}$

$=\{1,2,3,4,5,\ldots\ldots\ldots,20\}=U$

ii. $A\cap U=A$

Sol. L.H.S $=A\cap U$

$=\{1,3,5,\ldots\ldots\ldots,19\}\cap\{1,2,3,4,\ldots\ldots\ldots,20\}$

$=\{1,3,5,\ldots\ldots\ldots,19\}=A=R.H.S$

iii. $A\cap A'=\varnothing$ **Faisalabad 2007**

Sol. $A\cap A'=\varnothing;$

L.H.S $=A\cap A'$

$=\{1,3,5,\ldots\ldots\ldots,19\}\cap\{2,4,6,\ldots\ldots\ldots,20\}$

$=\{\ \}=\varnothing=R.H.S$

8. From suitable properties of union and intersection deduce the following results:

i. $A\cap(A\cup B)=A\cup(A\cap B)$

Sol. L.H.S $=A\cap(A\cup B)$

$=(A\cap A)\cup(A\cap B)$ (using Distributive law)

$=A\cup(A\cap B)\ \because A\cap A=A$

$=A\cup(A\cap B)$

$=R.H.S$

ii. $A\cup(A\cap B)=A\cap(A\cup B)$

Sol. L.H.S $=A\cup(A\cap B)$

$=(A\cup A)\cap(A\cup B)$ (Distributive law)

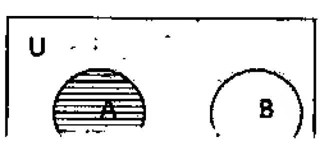
$=A\cap(A\cup B)\ \because A\cup A=A$

$=R.H.S$

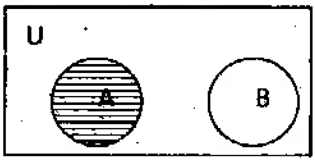
9. **Using Venn diagrams, verify the following results:**

i. $A\cap B'=A$ *if* $A\cap B=\varnothing$

Sol. New if $A\cap B'=A$ then we have to show that $A\cap B=\varnothing$ in Venn diagram



$A \cap B' = A$ is shaded in Venn diagram. This is possible only if A and B are disjoint $\Rightarrow A \cap B = \varnothing$ Conversly suppose that $A \cap B = \varnothing$ i.e A and B are disjoint we have to show that $A \cap B' = A$



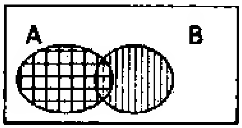
$A \cap B' = A$
Since $A \cap B = \varnothing$
$\Rightarrow A \cap B' = A$ as shaded in Venn Diagram.

ii. $(A - B) \cup B = A \cup B$

Sol. Given sets are $(A - B) \cup B$ and $A \cup B$

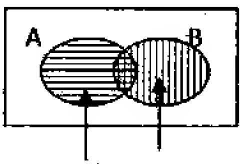
Their Venn diagrams show.

$(A - B) \cup B = A \cup B$



iii. $(A - B) \cap B$

Sol. Given sets are $A - B$ and B. Their Venn diagrams show $(A - B) \cap B$



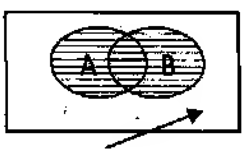
$(A - B) \cap B$

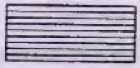
$(A - B) \cap B = \varnothing$

($\because$ no elements is common is shaded region of $A - B$ & B)

iv. $A \cup B = A \cup (A' \cap B)$

Sol. L.H.S $= A \cup B$

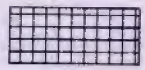


$A \cup B =$ 

R.H.S $= A \cup (A' \cap B)$

$A' =$

$A' \cap B =$ 

$A \cup (A' \cap B) =$  OR 

Logic:

Logic is the discipline that deals with the methods of reasoning. OR logic provides rules and techniques for finding that given argument is valid.

Uses of logic:

1. Logical reasoning is used in Mathematic to proves theorems.
2. In Computer Science to verify the correctness of programs.
3. In Physical science to draw conclusion from experiments.

Statement:

It is a declarative sentence that is either true (T) or false (F) but not both.

Examples:

i. Earth is round

ii. $2 + 3 = 5$

are statements

iii. Do you speak English? It is question. So it is not statement.

Proportional Variables

The letters $(p, q, r............)$ that can be replaced by statements are called proportional variables.

e.g P = It is raining q = It is cold.

Induction:

To draw general conclusions from limited number of observations. Or experiences is called Induction.

Example:

A person gets penicillin injection once or twice and experiences reaction soon afterwards. He generalizes that he is allergic to penicillin.

Deduction:

To draw general conclusion from well knows facts is called deduction.

Example:

All men are mortal. We are men. Therefore, we are all mortal.

Logical Connectives:

Symbols that are used to combine statements or proportional variables.

LIST OF SYMBOLS

Symbol	How to be read	Symbolic expression	How to be read.
$\sim$	not	$-p$	not p (Negation of p)
$\wedge$	and	$p \wedge q$	p and q
$\vee$	Or	$p \vee q$	p or q
$\rightarrow$	If........ then implies	$p \rightarrow q$	If p then q p implies q
$\leftrightarrow$	Is equivalent to If and only it.	$p \leftrightarrow q$	p if and only if q p is equivalent to q

Compound Statement:

Two or more sentences are connected to form a compound statement. e. g "It is raining and it is cold" is a compound statement in which p = it is raining; q = It is cold.

Then r = It is raining and it is called compound statement.

Truth Table:

A table to drives truth values of a given compound statement in terms of its component parts is called Truth Table.

Negation:

It is denoted so $\sim$ p means "not p"

Truth Table

P	$\sim$ P
T	F
F	T

Conjunction:

Conjunction of two statements p and q is true only if both p and q are true other wise false. It is denoted by $p \wedge q$

Truth Table

P	Q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction:

Disjunction of two statements **p** and **q** is denoted by $p \vee q$ (**p** or **q**).

Disjunction i.e $p \vee q$ is false only when both **p** and **q** are false, other wise true.

Truth Table

P	Q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional statement:

The statement " $p \rightarrow q$ " is called a conditional statement OR implication of **p** and **q** In a conditional statement. **P** is called **Hypothesis or anticident** and **"q"** is called Conclusion or consequent.

$p \rightarrow q$ is false only when **p** is true and **q** is false.($p \rightarrow q$) otherwise true

Truth Table

P	Q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional:

The statement $p \leftrightarrow q$ is called bi-conditional. It is written $p \leftrightarrow q$ and $q \leftrightarrow p$ It is also called equivalent " $p \leftrightarrow q$ " read as **p** if and only if **q**. If **p** and **q** both are same, then $p \leftrightarrow q$ is true otherwise false.

Truth Table

P	Q	$p \rightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Converse & contrapositive of conditional statement:

Let $p \rightarrow q$ be a given conditional statement, then.

i. $q \rightarrow p$ is called Converse of $p \rightarrow q$

ii. $\sim p \rightarrow \sim q$ is called inverse of $p \rightarrow q$

iii. $\sim q \rightarrow \sim p$ is called contra positive of $p \rightarrow q$

Truth table: of converse, inverse and contra positive of given conditional

p	q	$\sim p$	$\sim q$	Given conditional $p \to q$	Converse $q \to p$	Inverse ~ $\sim p \to \sim q$	Contra positive $\sim q \to \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Imp. note:

Truth table shows that conditional and contra positive are equivalent. So any Theorem may be proved by proving its contra positive.

Converse and inverse are equivalent to each other.

Example:

Prove that in any universal the empty set φ is subset of any set A.

Proof:

Let U is universal set. Then $\forall x \in U, x \in \varnothing \to x \in A$

Here p (Hypothesis) $= x \in \varnothing$, is false and q (conclusion) $= x \in A$.

$\because$ Conditional $(p \to q)$ is false only when p is true and q is false and $p \to q$ is true in all other cases. Implies that Conditional $x \in A$ is true $\Rightarrow \varnothing \subseteq A$ (any set)

Tautology: **Faisalabad 2009, Sargodha 2011**

A statement which is true for all possible values of variable involved in it is called a Tautology.

Contradiction or Absurdity: **Faisalabad 2009**

A statement which is always false is called contradiction. Or absurdity.

Contingency:

A statement which can be true or false depending upon the truth values of variable in it is called contingency.

Quantifier:

The word or symbol, which convey the idea of quantity or numbers called quantifier.

In mathematics two types of quantifier are generally used.

i. Symbol " $\forall$ "mean for all is called UNIVERSAL QUANTIFIRE.

ii. Symbol " $\exists$ "mean there exist is called EXISTENTIAL QUANTIFIRE.

EXERCISE 2.4

1. Truth table: of converse, inverse and contra positive of given conditional

Part	Conditional	Converse	Inverse	Contra positive
i.	$\sim p \to q$	$q \to \sim p$	$p \to \sim q$	$\sim q \to p$
ii.	$q \to p$ **(Multan 2009)**	$p \to q$	$\sim q \to \sim p$	$\sim p \to \sim q$
iii.	$\sim p \to \sim q$ **(Sgd 2008,09)**	$\sim q \to \sim p$	$p \to q$	$q \to p$
iv.	$\sim q \to \sim p$	$\sim p \to \sim q$	$q \to p$	$p \to q$

2. Construct truth table for the following statement:

i. $(p \to \sim p) \vee (p \to q)$

Sol.

p	q	$\sim p$	$p \to \sim p$	$p \to q$	$(p \to \sim p) \vee (p \to q)$
T	T	F	F	T	T
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	T	T	T

ii. $(p \wedge \sim p) \to q$ **Multan 2009**

Sol. Truth table of $(p \wedge \sim q) \to q$

p	q	$\sim p$	$p \wedge \sim p$	$(p \wedge \sim q) \to q$
T	T	F	F	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

III. $\sim (p \to q) \leftrightarrow (P \wedge \sim q)$

Sol. Truth table of $\sim (p \to q) \leftrightarrow (P \wedge \sim q)$

p	q	$p \to q$	$\sim (p \to q)$	$\sim q$	$p \wedge \sim q$	$\sim (p \to q) \leftrightarrow (P \wedge \sim q)$
T	T	T	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	F	T	F	T	F	T

3. Show that each of the following statements is a tautology:

i. $(p \wedge q) \rightarrow p$ **Multan 2008, Faisalabad 2008,**

Sol. Truth table

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

ii. $p \rightarrow (p \vee q)$ **Faisalabad 2007, Lahore 2009, Sargodha 2010**

Sol.

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

iii. $\sim(p \rightarrow q) \rightarrow p$ **Multan 2009, Rawalpindi 2009**

Sol.

p	q	$p \rightarrow q$	$\sim(p \rightarrow q)$	$\sim(p \rightarrow q) \rightarrow p$
T	T	T	F	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

iv. $\sim q \wedge (p \rightarrow q) \rightarrow \sim p$

Sol.

p	q	$\sim q$	$p \rightarrow q$	$\sim q \wedge (p \rightarrow q)$	$\sim p$	$\sim q \wedge (p \rightarrow q) \rightarrow \sim p$
T	T	F	T	F	F	T
T	F	T	F	F	F	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T

Since all the values of $\sim q \wedge (p \rightarrow q) \rightarrow \sim p$ are true. So $\sim q \wedge (p \rightarrow q) \rightarrow \sim p$ is a tautology.

4. Determine whether each of the following is a tautology, a contingency or an absurdity:

i. $p\wedge\sim p$ **Multan 2008,**

Sol. $p\wedge\sim p$

p	$\sim p$	$p\wedge\sim p$
T	F	F
F	T	F

$\therefore$ all value of $p\wedge\sim p$ are false. So it is absurdity.

ii. $p\rightarrow(q\rightarrow p)$

Sol.

p	q	$q\rightarrow p$	$p\rightarrow(q\rightarrow p)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

iii. $q\vee(\sim q\vee p)$ **Multan 2010, Sargodha 2008**

Sol.

p	q	$\sim q$	$\sim q\vee p$	$q\vee(\sim q\vee p)$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	T

5. Prove that $p\vee(\sim p\wedge\sim q)\vee(p\wedge q)=p\vee(\sim p\wedge\sim q)$ **Sargodha 2008**

Sol. Truth table

p	q	$\sim p$	$\sim q$	$\sim p\wedge\sim q$	$p\wedge q$	$p\vee(\sim p\wedge\sim q)$ R.H.S	$p\vee(\sim p\wedge\sim q)\vee(p\wedge q)$ L.H.S
T	T	F	F	F	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	F	F	F	F
F	F	T	T	T	F	T	T

Since last two columns are same.

$\therefore p\vee(\sim p\wedge\sim p)\vee(p\wedge q)=p\vee(\sim p\wedge\sim q)$ proved.

Examples: Give logical proofs of following theorems:

i. $(A \cup B)' = A' \cap B'$

Sol. Its logical form is $\sim(p \vee q) = \sim p \wedge \sim q$

$\because$ no. of variable = 2

$\because$ no. of rows of truth table = $2^2 = 4$

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

$\because$ last two columns are same

$\because \sim(p \vee q) = \sim p \wedge \sim q$

$\Rightarrow (A \cup B)' = A' \cap B'$ proved.

ii. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ **Faisalabad 2007,**

Sol. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

$\because$ no. of variable = 3

$\because$ no. of rows = $2^3 = 8$

p	q	r	$p \wedge q$	$p \wedge r$	$q \vee r$	L.H.S $p \wedge (q \vee r)$	R.H.S $(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	F	T	T	F	F
F	T	F	F	F	T	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

$\because$ last two columns are same

$\Rightarrow \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ proved.

EXERCISE 2.5

Convert the following theorems to logical form and prove them by constructing truth tables:

1. $(A \cap B)' = A' \cup B'$ **Faisalabad 2008**

Sol. Its logical form is $\sim (p \wedge q) = \sim p \vee \sim q$

$\because$ '2' variable, so rows = $2^2 = 4$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

$\therefore$ last two columns are same.

$\therefore \sim (p \wedge q) = \sim p \vee \sim q$

$\Rightarrow (A \cap B)' = A' \cup B'$ proved.

2. $(A \cup B) \cup C = A \cup (B \cup C)$

Sol. Its logical form is $(p \vee q) \vee r = p \vee (q \vee r)$

$\because$ no. of variables = 3

$\because$ no. of rows of truth tables $2^3 = 8$

p	q	r	$p \vee q$	$q \vee r$	$(p \vee q) \vee r$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

$\because$ last two columns are same

$\Rightarrow (p \vee q) \vee r = p \vee (q \vee r)$

$\Rightarrow (A \cup B) \cup C = A \cup (B \cup C)$ proved.

3. $(A\cap B)\cap C = A\cap(B\cap C)$ **Federal**

Sol. Its logical form is $(p\wedge q)\wedge r = p\wedge(q\wedge r)$

$\because$ no. of variables = 3

$\because$ no. of rows of truth tables $2^3 = 8$

p	q	r	$p\wedge q$	$q\wedge r$	$(p\wedge q)\wedge r$	$p\wedge(q\wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

$\because$ last two columns are same

$\Rightarrow (p\wedge q)\wedge r = p\wedge(q\wedge r)$

$\Rightarrow (A\cap B)\cap C = A\cap(B\cap C)$ proved.

4. $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$ **Faisalabad 2007**

Sol. Its logical form is $p\vee(q\wedge r) = (p\vee q)\wedge(p\vee r)$

$\because$ no. of variables = 3

$\because$ no. of rows = $2^3 = 8$

p	q	r	$p\vee q$	$q\wedge r$	$p\vee r$	$p\vee(q\wedge r)$	$(p\vee q)\wedge(p\vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	T	F	T	T	T
T	F	F	T	F	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

$\because$ last two columns are same

$\Rightarrow p\vee(q\wedge r) = (p\wedge q)\wedge(p\vee r)$

$\Rightarrow A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$ proved.

Binary Relation: **Sargodha 2009**

Let A and B be two non empty sets. Then any subset of Cartesian product $A\times B$ is called Binary relation or simply Relation from A to B.

Examples:

Let $A = \{1,2\}$ *and* $B = \{a,b\}$; Then

$A \times B = \{(1,a),(1,b),(2,a),(2,b)\}$, Then.

$R = \{(1,a),(2,b)\}$, is called Relation from A to B.

Domain R:

Set of 1st element of ordered pairs in R is called Domain R.

Range R:

Set of 2nd elements of ordered pairs in R is called Range R.

Function: **Rawalpindi 2009**

Let A and B be two non empty sets.

if

i. F is relation from A to B i.e F is a subset of $A \times B$

ii. Domain F = A

iii. No two ordered pairs of F have same 1st elements.

Then F is called a function from A to B and is written as F: $A \to B$ denoted $y = f(x)$;

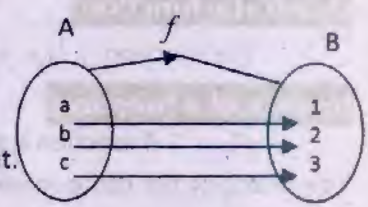
Example:

Let $A = \{a,b,c\}; B = \{1,2,3\}$

then let F is a relation A to B, such that

$A \times B = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3),(c,1),(c,2),(c,3)\}$;

Now since $f = \{(a,1);(b,2),(c,2)\}$;



i. f is subset of $A \times B$

Dom f = A

ii. No two ordered pairs of f have same 1st element.

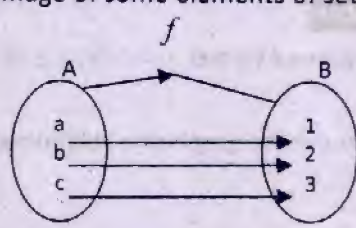
$\Rightarrow$ 'f' is a function from A to B.

Onto function: **Lahore 2009**

(Subjective function) a function $f : A \to B$ is said to be onto function if Range f =B.

Range of f:

i.e Every element of Set B is the image of some elements of set A, as shown in fig.2

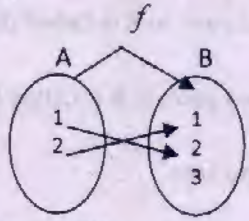


Into function: **Multan 2008, 2009**

A function $f: A \to B$ is said to be into function if Rang $\neq$ B or Range $f \subset B$ as shown in fig.1

One-one function: **Multan 2008**

A function $f: A \to B$ is called (1 – 1) function if different elements of A has different images in B as shown in fig.3.



Bijective function: **Multan 2009**

(Range f = B and 1 – 1) A function f which is both one-one and onto is called Bijective function.

Injective function:

(Range $f \neq B$ and 1 – 1) A function f which is both one-one and into is called Injective function.

Linear function:

The function $f\left\{(x,y)\middle| y = mx + c\right\}$ is called linear function. Where $y = mx + c$ is straight line.

Quadratic function:

The function $f\left\{(x,y)\middle| y = ax^2 + bx + c\right\}$ is called quadratic function.

Inverse of a function:

(i). If function is given in tabular form. Then its inverse function is obtained by interchanging the components of each ordered pairs. e.g. of $f = \{(1,2),(3,4)\}$, then $f^{-1} = \{(2,1),(4,3)\}$

Identity function:

The function $f = \left\{(x,y)\middle| y = x\right\}$ is called identity function.

Square root function:

The function defined by the $y = \sqrt{x}; x \geq 0$ is square root function.

Verticalline test:

If a vertical line cut the graph of a relation at a single point. Then such relation is called function.

EXERCISE 2.6

1. For $A=\{1,2,3,4\}$, find the following relation in A. State the domain and range of each relation. Also draw the graph of each.

i. $\{(x,y)|\,y=x\}$

Multan 2009

Sol. $R=\{(x,y)|\,y=x\}$

$A=\{1,2,3,4\}$

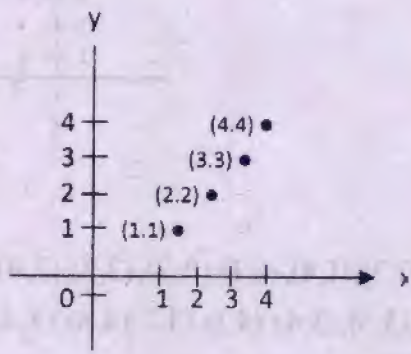
$\Rightarrow A\times A=\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),$
$(3,1),(3,2),(3,3),(3,4),(4,1),(4,2)(4,3),(4,4)\}$

According to the condition

$R=\{(1,1),(2,2),(3,3),(4,4)\}$

Dom. $R=\{1,2,3,4\}=A$

Range $R=\{1,2,3,4\}=A$



ii. $\boldsymbol{R=\{(x,y)|\,y+x=5\}}$

Sol. $\therefore A=\{1,2,3,4\}$

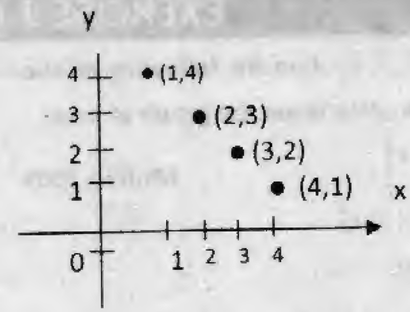
$A\times A=\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),$
$(3,1),(3,2),(3,3),(3,4),(4,1),(4,2)(4,3),(4,4)\}$

According to the condition

$R=\{(1,4),(2,3),(3,2),(4,1)\}$

Dom. $R=\{1,2,3,4\}$

Range $R=\{1,2,3,4\}$



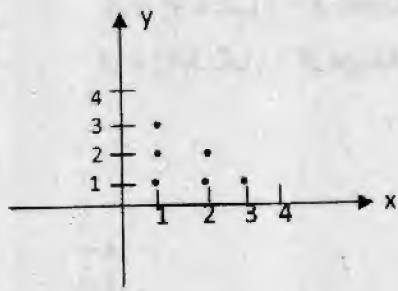
iii. $\{(x,y)|x+y<5\}$

Sol. $\therefore A=\{1,2,3,4\}$

$$A\times A=\begin{Bmatrix}(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),\\(3,1),(3,2),(3,3),(3,4),(4,1),(4,2)(4,3),(4,4)\end{Bmatrix}$$

According to the condition

$R=\{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\}$



iv. $\{(x,y)|x+y>5\}$

Sol. $\therefore A=\{1,2,3,4\}$

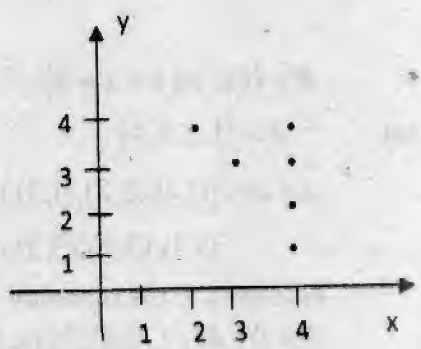
$$A\times A=\begin{Bmatrix}(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),\\(3,1),(3,2),(3,3),(3,4),(4,1),(4,2)(4,3),(4,4)\end{Bmatrix}$$

According to the condition

$R=\{(2,4),(3,3),(4,4),(3,4),(4,2),(4,3)\}$

Range $R=\{2,3,4\}$

Domain $R=\{2,3,4\}$



2. Repeat Q.1 When A = R, the set of real numbers. Which of the real lines are functions.

i. $\{(x,y)|\, y=x\}$

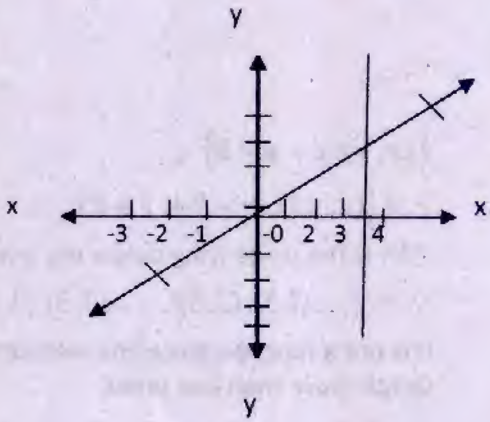
Sol. $A\times A = R\times R = \{(x,y)|\, y,x\in R\}$

$r=\{(x,y)|\, x=y\}$

Or $r=\{.........,(-1-1),(0,0),(1,1),(..........\}$

Here Domain = R & Range = R

This relation is function, because any vertical line cut only at one point.

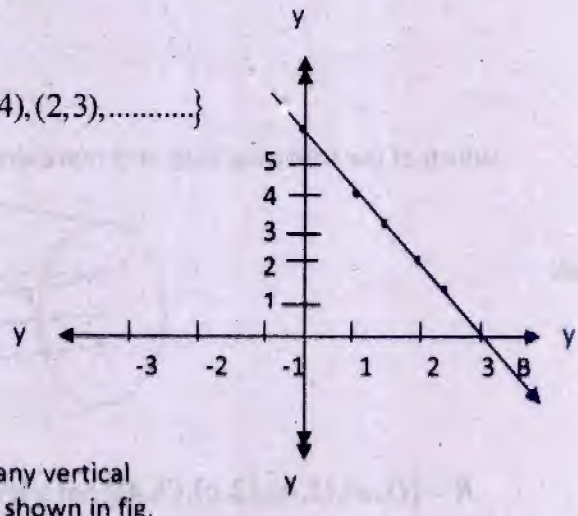


ii. $\{(x,y)|\, x+y=5\}$

Sol. $A\times A = R\times R = \{(x,y)|\, x,y\in R\}$

$\because \quad r=\{(x,y)|\, x+y=5\}$

$\because \quad r=\{.........,(-1,6),(0,5),(1,4),(2,3),..........\}$



$\Rightarrow$ Domain Range = R

This relation is function, because any vertical line will cut it only at one point as shown in fig.

iii. $\{(x,y)|x+y>5\}$

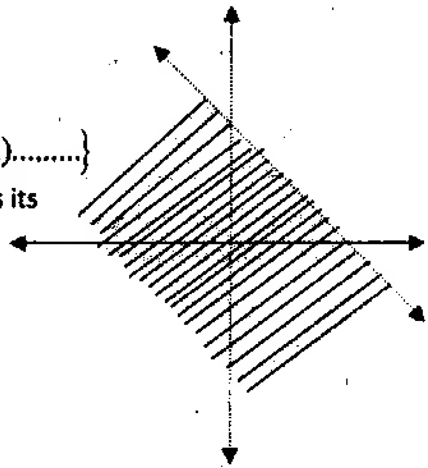
Sol. $So\ \ A\times A = R\times R = \{(x,y)|x,y\in R\}$

$\because \quad r=\{(x,y)|x+y<5\}$

$\Rightarrow$It is the plane lying below the line $x+y=5$

$\because \quad \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1)........\}$

It is not a function Since any vertical line meets its Graph more than one point.



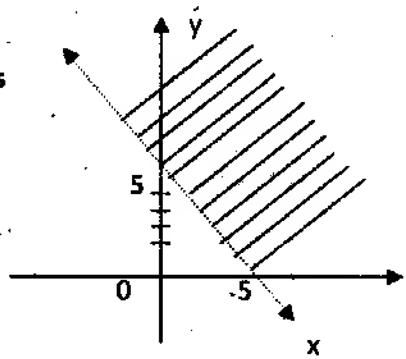
iv. $\{(x,y)|x+y<5\}$

Sol. $r=\{(x,y)|x,y>5;x,y\in R\}$

$\Rightarrow$It is the plane lying below the line $x+y=5$

$\because \quad \{.....(1,5),(2,5),.........(3,3),(3,4),.......\}$

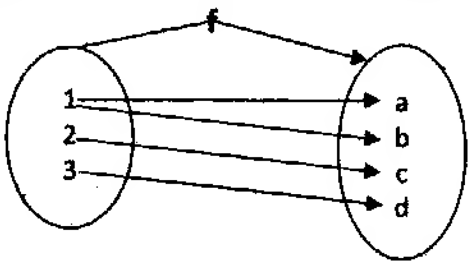
It is not a function Since any vertical line meets its Graph more than one point.



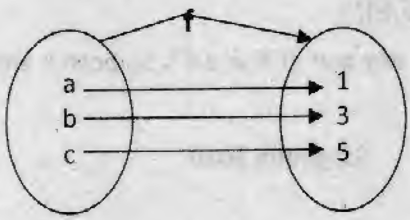
3. Which of the following diagrams represent functions and of which type?

i.

Sol.



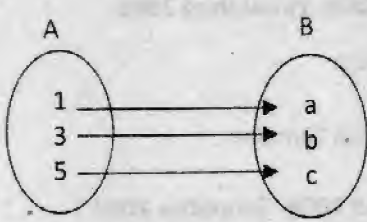
$R=\{(1,a),(1,b),(2,c),(3,d)\}$ not a function. Since there are two ordered pairs that have same 1st element.

ii. 

$$\Rightarrow R = \{(a,1),(b,3),(c,5)\}$$

Sol. $\because$ both condition are satisfied. So is a function.

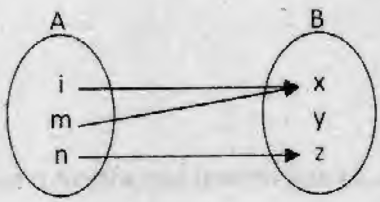
$\because$ R is one – one and onto so R is bijective function also.

iii. 

$$\Rightarrow R = \{(1,a),(2,b),(3,c)\}$$

Sol. $\therefore$ (i) different element has different images so is one-one.

(ii) Range R = B so is onto $\Rightarrow$ function (1 – 1) & on to i.e. bijective function.

iv. 

$$\Rightarrow R = \{(i,x),(m,x),(n,z)\}$$

Sol. $\because$ (i) No two ordered pairs of R have same 1st element.

(ii) Domain R = A

$\Rightarrow$ 'R' is a function from A to B.

4. Find the inverse of each of the following relations. Tell whether each relation and its inverse is a function or not:

i. $\{(2,1),(3,2),(4,3),(5,4),(6,5)\}$ **Multan 2008, 2010, Sargodha 2011**

Sol. $R = \{(2,1),(3,2),(4,3),(5,4),(6,5)\}$

$$\Rightarrow R^{-1} = \{(1,2),(2,3),(3,4),(4,5),(5,6)\}$$

Since no first element is repeated in any pair of R and R^{-1}. So both R and R^{-1} are functions.

ii. $\{(1,3),(2,5),(3,7),(4,9),(5,11)\}$ **Sargodha 2010**

Sol. $R = \{(1,3),(2,5),(3,7),(4,9),(5,11)\}$

$$\Rightarrow R^{-1} = \{(3,1),(5,2),(7,3),(9,4),(11,5)\}$$

Since no first element is repeated in any pair of R and R^{-1} So both R and R^{-1} are functions.

iii. $\{(x,y)|\, y = 2x+3; x \in R\}$ **Multan 2008, Faisalabad 2009**

Sol. $R = \{(x,y)|\, y = 2x+3; x \in R\}$ It is a function.

$R^{-1} = \left\{(x,y)|\, y = \dfrac{x-3}{2}, x \in R\right\}$ It is also a function.

iv. $\{(x,y)|\, y^2 = 4ax; x \geq 0\}$ **Faisalabad 2008, Sargodha 2009**

Sol. $R = \{(x,y)|\, y^2 = 4ax; x \geq 0\}$ It is not a function because for each x > 0 there are two different rules of y.

$R^{-1} = \left\{(x,y)|\, x^2 = 4ay \Rightarrow y = \dfrac{1}{49}x^2, x \geq 0\right\}$ It is a function.

v. $\{(x,y)|\, x^2 + y^2 = 9, |x| \leq 3, |y| \leq 3\}$

Sol. $R = \{(x,y)|\, x^2 + y^2 = 9, |x| \leq 3, |y| \leq 3\}$

$$R^{-1} = \{y^2 + x^2 = 9, |x| \leq 3, |y| \leq 3\}$$

both R and R^{-1} represent same circular disc. As any vertical line will cut it more than one point. So R and R^{-1} are not function.

Unary operation:

A mathematical procedure that changes one number into an other. OR It is an operation which when applied on a single number to give an other number.

e.g. $\sqrt{4} = 2$, Here '$\sqrt{\ }$' is Unary operation.

Binary operation:

It is an operation which when applied on two numbers give 3rd number. Generally we use symbol " * " (Star) for a binary operation.

i.g. '+ ', '× ', '− ', *and* '÷ ' are used as Binary operation in different sets of numbers.

EXERCISE 2.7

1. Complete the table indicating by a tick mark those properties which are satisfied by the specified set of numbers.

Sol:

Property	Set of number	Natural "N"	Whole "W"	Integers "Z"	Rational "Q'	Real R
Closure	+	✓	✓	✓	✓	✓
	×	✓	✓	✓	✓	✓
Associative	+	✓	✓	✓	✓	✓
	×	✓	✓	✓	✓	✓
Identity	+	×	✓	✓	✓	✓
	×	✓	✓	✓	✓	✓
Inverse	+	×	×	✓	✓	✓
	×	×	×	×	×	×
Commutative	+	✓	✓	✓	✓	✓
	×	✓	✓	✓	✓	✓

2. What are the field axioms? In what respect does the field of real numbers differ from that of complex numbers?

Sol A non empty set F is called field if

i. It is abelian group under '+'

ii. Non zero elements of F from abelian group under '×'

iii. Distributive Laws held i.e.

$$a.(b+c) = a.b + a.c$$

& $$(b+c).c = a.c + b.c$$

Also set of real no's is subfield of Set of complex numbers.

3. Show that the adjoining table is that of ' X ' of elements of the set of residue classes of modulo 5.

*	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Sol The zero's in C_2 and R_2 are obtained by multiplication of 1,2,3,4 with '0'

$\Rightarrow$ It is a multiplication table.

$\because$ every element is less than 5. So the table is a multiplication table of the set of elements residue classes modulo 5.

4. Prepare a table of addition of the elements of the set of residue classes modulo 4.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Sol Clearly $\{0,1,2,3\}$ is the set of residues classes modulo.

We add the pair of elements as in ordinary ' + ' if answer is equal or greater then 4 then we subtract 4.

5. Which of the following binary operations shown in tables (I) and (II) is commutative:

(i)

*	a	b	c	D
a	a	c	b	D
b	b	c	b	A
c	c	d	b	C
d	a	a	b	B

(ii)

*	a	b	c	d
a	a	c	b	d
b	c	d	b	a
c	b	b	a	c
d	d	a	c	d

Sol In table – I $\because a*b=c$

$b*a=b$

$\Rightarrow a*b \neq b*a$

$\Rightarrow B.O.$ '*' is not commutative

in Table – II

$a*b=b*a=c$

$a*c=c*a=b$

$a*d=d*a=d$

$b*c=c*b=b$

$b*d=d*b=a$

$c*d=d*c=c$

6. Supply the missing elements of 3rd row of the give tables, so that the B.O '*' may be associative.

'*'	a	b	c	d
a	a	b	c	d
b	B	a	c	d
c	?	?	?	?
d	d	c	c	b

Sol. We want to find $c*a, c*b, c*c, c*d$ from table

$c = d*b$

$c*a = (d*b)*a$
$= d*(b*a)$ $\because$ *Associative*
$= d*b = c$
$\boxed{c*a=c}$

$c*c = (d*b)*c$
$= d*(b*c)$
$= d*c = c$
$\boxed{c*c=c}$

Again

$c = d*b$

$c*b = (d*b)*b$
$= d*(b*b)$ $\because$ *Associative*
$= d*a = d$
$\boxed{c*b=d}$

Again $c = d*b$

$c*d = (d*b)*d$
$= d*(b*d)$
$= d*d = b$
$\boxed{c*d=b}$

So third row will be completed as 3rd row.

c	c	d	c	d

7. **What operation is represented by adjoining table? Name the identity elements of the relevant set, if it exits. Is the operation associative? Find the inverse of 0,1,2,3, if they exit.**

'*'	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Sol. **(i)** **The operation used the set of residue class mod 4 is ' + '**

(ii) The identity element is zero.
$\because 0+0=0, 0+1=1, 0+2=2, 0+3=3$

(iii) The operation is associative
e.g. (1 + 2) +3=1 +(2 + 3)
3+3=1+1 $\Rightarrow$ 2=2
Similarly it can be verified for any other choice of elements.

(iv) $\because 1+3=3+1=0$ 1 and 3 are inverse of each other.
2+2=0 also 0+0=0

Groupoid:

A non empty set which is closed under given Binary Operation '*' is called groupoid it is denoted as (S'*')

Example:

The $\{E.O\}$ is closed under addition, since

$E + E = E$; $O + E = O$

$E + O = O$; $O + O = E$

$\therefore$ $\{E, O\}$ is groupoid

Semi group: **Multan 2008, Faisalabad 2008, Sargodha 2010**

A non empty set is called Semi group if

i. It is closed under given Binary Operation

ii. The Binary Operation is associative.

Example:

The set of Natural nos 'N' under Binary Operation ' + ' is semi group.

i. i.e B.O ' + ' is defined in N.

ii. for any three elements $a, b, c, \in N$

$(a + b) + c = a + (b + c)$

i.e. associative Law holds.

Monoid:

A non empty set is called Monoid.

i. It is closed w.r.t given Binary Operation '*'

ii. Binary Operation '*' is associative

iii. The set has identity element w.r.t Binary Operation '*'

Example:

If $Z' = \{0, 1, 2, 3, ...\}$

i. Z' *is closed w.r.t* '+'

ii. Binary Operation ' + ' is associative.

iii. 'O' is identity element w.r.t to Binary Operation ' + "

$\therefore$ Given set is Monoid.

Group: A non empty set G is called a group w.r.t Binary Operation '*'

i. It is closed under Binary Operation '*' if
i.e. $\forall\ a, b, \in G;\ a * b \in G$

ii. Binary Operation is associative
$\forall\ a, b, c \in G\ ; (a * b) * c = a * (b * c)$

iii. G has Identity elements w.r.t
Binary Operation
'*' $i.e\ \forall\ a \in G\quad \exists e \in G\quad s.t\ a * e = e * a = a$ then 'e' is identity element w.r.t Binary Operation.

iv. Every element of G has an inverse in G .W.r.t Binary Operation. i.e.
a *a'=a' *a=e

where $a' \in G$ is called inverse of $a \in G$ w.r.t Binary Operation *

Ableian group:

A group G under Binary Operation '*' is called Abelian group if Binary Operation is commutative i.e. $\forall a, b \in G;\quad a*b=b*a$

Finite Infinite group:

A group G is said to be finite if it contains finite no. of elements. Otherwise G is an infinite group.

Reversal Law of Inverse:

Theorem. If $a,b,$ are elements of G then show that $(ab)^{-1}=b^{-1}a^{-1}$

Faisalabad 2007, 08 Sargodha 2008,11

Sol. $abb^{-1}a^{-1}=a(bb^{-1})a^{-1}$ Associative Law

$=aea^{-1}$ (Inverse Law)

$=aa^{-1}$ (Identity Law)

$=e$

$b^{-1}a^{-1}.ab=b^{-1}(a^{-1}a)b$

Also $=b^{-1}eb$

$=b^{-1}b$

$=e$

$\Rightarrow ab \;\; and \;\; b^{-1}a^{-1}$ are inverse of each other.

Thus inverse of $ab \;\; is \;\; b^{-1}a^{-1}$

$i.e \;\; (ab)^{-1}=b^{-1}a^{-1}$

Theorem .

Federal

If $(G,*)$ is a group. Then there is a unique inverse for each element of G.

Sol. Let $(G,*)$ be a group and $\forall\, a \in G$ Let a' and a'' are the inverse of a.

Then $a'*a=e$ ________ I if a' is inverse of a.

Also $a''*a=e$ ________ II if a'' is inverse of a

By Association Law in G.

$(a'*a)*a''=a'*(a*a'')$

$\Rightarrow (e)*a''=a'*(e) \qquad use\ (I,II)$

$a''=a' \qquad (\because e \text{ is identity})$

Hence a', a'' are same inverse of each element of a in G .

Theorem: If $(G,*)$ is a group with e its identity then e is unique.

Proof. Suppose e and e' are two identities.

Then $e'*e=e*e'=e' \rightarrow I$ (e is identity)

$e'*e=e*e'=e \rightarrow II$ (e' is identity)

Compare (I) & (II) $\Rightarrow$ e = e'

EXERCISE 2.8

1. Operation $\oplus$ performed on the two member set G = $\{0,1\}$ is shown in the adjoining table. Answer the questions.

+	0	1
0	0	1
1	1	0

i. Name the identity element if it exists?

Sol. 'O' is identity element.

ii. What is the Inverse of 1?

Sol. $\because 1+1=0$ (i.e identity element)

$\Rightarrow$Inverse of 1 is 1.

iii. Is the set G, under the given operation a group?

Sol.

i. Since all the elements of table $\in$ G so B.O " + " is closed.

ii. Clearly B.O is associative.

iii. Identity element w.r.t ' + ' is ' O ' $\in$ G.

iv. Additive inverse of each element of G belongs to G.

$\Rightarrow$(G, +) is a group.

iv. Ablelian or non-Abelian

Sol.

$\forall\, 1, 0 \in G$

$\Rightarrow \quad 1+0=0+1 \quad \Rightarrow \quad 1=1$

$\Rightarrow$ G is commutative w.r,t '+'

$\Rightarrow$ Group G is abelian (i.e commutative) w.r.t. '+'

2. The Operation $\oplus$ as performed on the set $\{0,1,2,3\}$ is shown in the adjoining table show that set is an Abelian group? (Multan 2010, Faisalabad 2008, Sargodha 2009)

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Sol.

i. Since all the element of table belongs to the set $\{0,1,2,3\}$. So $\Rightarrow$is closed w.r.t '+'

ii. It is clear the set is associative w.r.t '+'

iii. 'O' is the additive identity.

iv. Each element has inverse because $1+3=3+1=0$ and 0 + 0 = 0 and 2 + 2 =0

v. $\forall\ 1,0 \in G=\{0,1,2,3\}, 1+0=0+1=1$

$\Rightarrow$ **G is abelian.**

3. For each of the following sets, determine, whether or not the set forms a group with respect to the indicated operation.

	Set	Operation
i.	The set of rational numbers	×
ii.	The set of rational numbers	+
iii.	The set of positive rational numbers	×
iv.	The set of integers.	+
v.	The set of integers	×

Sol. i.

Q = Set of rational no's.
is not a group w.r.t "×"
Since inverse of 0 w.r.t '×' does not exit

ii.
Sol $(Q,+)$ is a group.

iii. **Set of +ve rational nos.** ×
Sol. it is a group w.r.t. '×'

iv. **The set of integers** +
Sol. it is a group w.r.t. '+'

v. **The set of integers** ×
Sol. If is not a group. Since multiplicative inverse of zero does not exist.

4. Show that the adjoining table represent the sum of the elements of the set $\{E,O\}$

+	E	O
E	E	O
O	O	E

What is the identity element of this set? Show that this set is an abelian group.

Sol.

Answer I
E + E = E (even)
E + O = O (odd);
O + O = E (even)
Here 'E' is the identity element.

Answer I I
i. Table shows that set satisfies the closure law w.r.t. '+'. Because all elements of table $\in \{E,O\}$

ii. The set is associative under '+' 'E' (O+E)+O=O+(O+E)
$\Rightarrow$ O+O=O+O $\Rightarrow$ E=E

iii. E is identity $\in \{E,O\}$

iv. Each element has inverse (O+E=E+O=O and E+E=O and O+O=O)

v. Commutative Law holds. (O+E=E+O)

So set $\{E,O\}$ is abelian group.

5. Show that the set S= $\{1,\omega,\omega^2\}$ when $\omega^3=1$ is an Abelian group w.r.t. ordinary multiplication. **Multan 2009, Faisalabad 2008, Lahore 2009, Sargodha 2007,08**

Sol. From multiplication table.

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Sol. i. Table shows that set shows closure law w.r.t "$\times$"

ii. $1,\omega,\omega^2 \in S$

$(1.\omega).\omega^2 = 1.(\omega.\omega^2)$

$\omega.\omega^2 = 1.\omega^3$

$\omega^3 = \omega^3$

$1 = 1$

$\Rightarrow$associative law of '$\times$' is satisfied.

iii. '1' is identity element w.r.t. '$\times$'

iv. Multiplicative inverse of 1 is 1

$\omega.\omega^2 = \omega^2.\omega = 1$

$\Rightarrow \omega$ and ω^2 are inverse of each other.

v. Commutative law holds in the given set. ($1\times\omega = \omega\times 1$)

S is an abelian group w.r.t "$\times$"

6. If G is a group under $*$ and $a,b\in G$, find the solution of the equations:

(i) $a*x=b$ (ii) $x*a=b$ **Faisalabad 2009**

Sol. (i) $a*x=b \longrightarrow (i)$

$\because a\in G,\ so\ a^{-1}\in G\ \ pre\,multiply\ (i)\ by\ a^{-1}$

$a^{-1}*(a*x)=a^{-1}*b$

$(a^{-1}*a)*x=a^{-1}*b$ $(Associative)$

$$e * x = a^{-1} * b$$

$$x = a^{-1} * b$$

ii. $x * a = b \longrightarrow (i)$ **Multan 2009, 10**

Sol. Post Multiplying by a^{-1}

$$(x * a) * a^{-1} = b * a^{-1}$$

$$x * (a * a^{-1}) = b * a^{-1} \quad (Associative)$$

$$x * e = b * a^{-1}$$

$$x = b * a^{-1}$$

7. **Show that the set consisting of elements of the form $a + \sqrt{3}b$ $(a, b,$ being rational) Is an abelian group w.r.t addition.**

Sol. **Let S in a set which contains the elements of the form $a + \sqrt{3}b$ where a, b are rational.**

i. For $a + \sqrt{3}\,b, c + \sqrt{3}\,d \in S$ a, b, c, d are rational

$(a + \sqrt{3}\,b) + (c + \sqrt{3}\,d) = (a + b) + (b + d)\sqrt{3} \in S$ so closed

ii. Association Law of '+' for

$a + \sqrt{3}b$ $c + \sqrt{3}d$, $e + \sqrt{3}f \in s$ $s.t\ a, b, c, d, e, f \in Q$

then:

Sol. L.H.S $= \left[(a + \sqrt{3}b) + (c + \sqrt{3}d)\right] + (e + \sqrt{3}f)$

$$= \left[a + c + \sqrt{3}\,(b + d)\right] + (e + \sqrt{3}\,f)$$

$$= (a + c + e) + \sqrt{3}(b + d + f) \qquad \rightarrow (I)$$

R.H.S

$$= (a + \sqrt{3}b) + \left[(c + \sqrt{3}d + e + \sqrt{3}f)\right] = (a + \sqrt{3}b) + \left[(c + e) + \sqrt{3}(d + f)\right]$$

$$= (a + c + e) + \sqrt{3}(b + d + f) \qquad \rightarrow (II)$$

$I = II$ Hence Addition is Associative

iii. $\forall\ a + \sqrt{3}b \in S$ $\exists$ $(0 + \sqrt{3}(0))$ as identity element w.r.t "+"

iv. For each $(a + \sqrt{3}b) \in S \exists (-a - \sqrt{3}\,b) \in S$

$s.t$ $(a + \sqrt{3}b) + (-a - \sqrt{3}b) = a + (-a) + \sqrt{3}(b - b) = 0 + \sqrt{3}\,0$

So inverse of each element of S is in S:

v. For each $a + \sqrt{3}b, c + \sqrt{3}d \in S$

$a+\sqrt{3}b+c+\sqrt{3d}=(a+c)+\sqrt{3}(d+b) \quad \rightarrow (I)$

$=(c+a)+\sqrt{3}(d+b)=(c+\sqrt{3}d)+(a+\sqrt{3}b)$

$\Rightarrow \quad (a+\sqrt{3}b)+(c+\sqrt{3}d)=(c+\sqrt{3}d)+(a+\sqrt{3})$

Sol. Hence S is Abelian group under addition.

8. Determine whether $(P(S),*)$, where *stands for intersection is a semi group, a monoid or neither. If it is a monoid, specify its identity.

Sol. **Let $P(S)$ power set of S i.e. consisting of all subsets of s and $*=\cap$ then for** $A,B\in P(s)$

i. **Since $A*B = A\cap B\in P(s) \Rightarrow P(s)$ is close under $\cap$**

ii. **Since $\cap$ is always associative.**

iii. $A\in S \quad A\cap S=A$ **so every set is identity element it self in P(s) UNDER * P(S)** is monoid.

9. Complete the following table to obtain a semi-group under *

*	a	b	c
a	c	a	b
b	a	b	c
c	-	-	a

Sol. **We want to find c*a, and c*b.**

$a*a=c \quad \longrightarrow (i)$

Now

$c*a=(a*a)*a$

$=a*(a*a) \quad \because Associative$

$=a*c$

$c*a=b$

$c*b=(a*a)*b$

$=a*(a*b) \quad \because Associative$

$=a*a$

$c*b=c$

10. Prove that all 2×2 non singular matrices over the real field form a non abelian group under multiplication.

Sol. **Let S be the set of all 2×2 Non singular matrices over R. so let**

i. **AB is also matrices of same order so $\in S$**
Multiplication is closed in S.

ii. **Since Matrix multiplication is Association.**

$\because (A.B).C=A(B.C)\forall \ A,B,C\in S$

iii. **The unit Matrix** $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\in S$ I **is the identity element is S.**

iv. **The inverse of each element of S exists in S.**

Sol. $\because \forall\ A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

$$A^{-1}=\frac{adiA}{|A|}=\frac{\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}}{ad-bc}\ which \in S$$

All four condition are satisfied. Hence S is a group under the multiplication since $AB\neq BA$

$\because$ S is not abelian group under multiplication.

TEST YOUR SKILLS Marks: 50

Q # 1. Select the Correct Option **(10)**

i. The statement written as p *iff* q is denoted by

a) b) $p \leftrightarrow q$

c) $q \rightarrow p$ d) $p \wedge q$

ii. The Tabular form of the set $\{x | x \in p \wedge x < 12\}$ is:

a) $\{3,5,7,11\}$ b) $\{1,2,3,5,7,11\}$

c) $\{2,3,5,7,11\}$ d) $\{3,5,7,9,11\}$

iii. Let $p \rightarrow q$ be given conditional then $\sim q \rightarrow \sim p$ is called

a) Converse b) Inverse

c) Reverse d) None of these

iv. A and B disjoint set then $A \cap B$ is equal to

a) A b) B

c) φ d) U

v. If a, b are elements of a group G then $(ab)^{-1}$ is equal to

a) $a^{-1}b^{-1}$ b) $\frac{1}{ab}$

c) ab d) $b^{-1}a^{-1}$

vi. If A' is the complement of the set A then $(A \cap A')$ equala

a) A b) A'

c) U d) φ

vii. The Set $\{(a,b)\}$ is called:

a) Infinite Set b) Singleton Set

c) Set with two elements d) Empty Set

viii. The numbers of all subsets of a set having three elements is :

a) 4 b) 6

c) 8 d) 10

ix. If $A \subseteq B$ then $A - B$ is:

a) A' b) B'

c) φ d) A

x. If A and B are two sets then $A \cap (A \cup B) =$

a) A b) B

c) φ d) $A \cup B$

Q # 2. Short Questions: **(2 X 20 = 40)**

i. Convert De Morgan's Laws to logical form:

ii. For $S=\{1,-1,i,-i\}$ write its multiplication table:

iii. Define Semi Group:

iv. Show $A \cap B$ by Venn Diagram where A and B are over lapping:

v. What is Proposition

vi. If $(G,*)$ is a group with e its identity then show that e is unique:

vii. Construct truth table of $(p \wedge \sim p) \rightarrow q$:

viii. What is Tautology?

ix. Find the inverse of relation $v=\{(1,3),(2,5),(3,7),(4,9),(5,11)\}$

x. Write power set of $\{+, -, \times, \div\}$

Q # 3. **(a)** Convert $(A \cap B)' = A' \cup B'$ into logical and prove by constructing the truth table:

(b) If a,b are elements of a group G then show that $(ab)^{-1} = b^{-1}a^{-1}$

Q # 4. **(a)** Prove that $pv(\sim p \wedge \sim q) \vee (p \wedge q) = pv(\sim p \wedge \sim q)$

(b) Prove logically $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Matrices and Determinants 3

Matrix:

An arrangement of different elements in the form of rows and columns, within square brackets is called Matrix. It is always denoted by capital Alphabets.

e.g $A=\begin{bmatrix}1 & 7\\ 3 & 2\end{bmatrix}$, $B=\begin{bmatrix}5 & 3 & 7\\ 1 & 6 & 8\end{bmatrix}$

Order:

Order of Matrix tells us about no of rows and no of columns.

Order of Matrix = no of rows × no of columns.

If $A=\begin{bmatrix}1 & 1 & 3\\ 2 & 1 & -7\end{bmatrix}$ then order of $A=2\times 3$

Row Matrix:

A matrix having single row is called row matrix.

e.g $A=\begin{bmatrix}1 & 3 & 7\end{bmatrix}$, $B=\begin{bmatrix}1 & 6 & 3\end{bmatrix}$

Column Matrix:

A matrix having single column is called Column Matrix.

e.g $A=\begin{bmatrix}5\\ 1\\ 4\end{bmatrix}$

Square Matrix:

A matrix in which no of rows equal to the no of columns is called square matrix.

e.g $A=\begin{bmatrix}2 & 3\\ 1 & 7\end{bmatrix}$, $B=\begin{bmatrix}1 & 1 & 7\\ -1 & 5 & 1\\ 7 & 4 & 6\end{bmatrix}$

Rectangular Matrix:

The matrix in which no of rows is not equal to the no of columns is called Rectangular Matrix.

e.g $A=\begin{bmatrix}1 & 5 & 2\\ 3 & 7 & 1\end{bmatrix}$

Diagonal Matrix: **Multan 2008**

A Square matrix having each of its elements equal to zero except at least one element in its diagonal is called diagonal matrix.

$$\text{e.g } A=\begin{bmatrix}3 & 0\\0 & 5\end{bmatrix},\quad B=\begin{bmatrix}0 & 0\\0 & 9\end{bmatrix},\quad C=\begin{bmatrix}1 & 0 & 0\\0 & 5 & 0\\0 & 0 & -4\end{bmatrix}$$

Scalar Matrix:

A diagonal matrix having same elements in its diagonal is called a Scalar matrix.

$$\text{e.g } A=\begin{bmatrix}5 & 0\\0 & 5\end{bmatrix},\quad B=\begin{bmatrix}3 & 0 & 0\\0 & 3 & 0\\0 & 0 & 3\end{bmatrix}$$

Identity Matrix:

A scalar matrix having 1 as its elements in the diagonal is called an identity matrix.

$$\text{e.g } A=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\quad B=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Null Matrix:

A matrix in which all elements are equal to zero is called Null matrix or zero matrix.

$$\text{e.g } A=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix},\quad B=\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Equal Matrixes:

Two matrixes are said to be equal if they are of same order with the same correspondence elements.

$$\text{e.g } A=\begin{bmatrix}a & b\\c & d\end{bmatrix},\quad B=\begin{bmatrix}3 & 1\\4 & 7\end{bmatrix}\ \textit{If } A=B \textit{ then } a=3,\ b=1,\ c=4,\ d=7$$

Upper Triangular Matrix: **Multan 2008, 2009**

If all elements below the main diagonal of a square matrix are zero then it is called upper triangular matrix.

$$\text{e.g } \begin{bmatrix}2 & 5 & 6\\0 & 4 & 3\\0 & 0 & 1\end{bmatrix},\quad \begin{bmatrix}1 & 0 & 7\\0 & 2 & 3\\0 & 0 & 4\end{bmatrix}$$

Lower Triangular Matrix: **Multan 2009**

If all elements above the main diagonal of a square matrix are zero then it is called Lower triangular matrix.

e.g $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 8 & 0 \\ 3 & 5 & 7 \end{bmatrix}$$

Triangular Matrix:

A matrix which is either upper triangular or lower triangular is called a triangular matrix.

Symmetric Matrix:

Let "A" be a square matrix if $A' = A$ then "A" is called symmetric matrix.

Skew Symmetric Matrix:

Let "A" be a square matrix if $A' = -A$ then "A" is called skew symmetric matrix or Anti symmetric matrix.

Hermitian Matrix: **Sargodha 2008, 2009, Multan 2010**

Let "A" be a square matrix if $(\overline{A})' = A$ then "A" is called Hermitian Matrix.

Skew Hermitian Matrix: **Multan 2009**

Let "A" be a square matrix if $(\overline{A})' = -A$ then "A" is called Skew Hermitian Matrix or Anti Hermitian Matrix.

Leading Entry (L.E):

The first non zero entry in any non zero row of an matrix is called leading entry.

Echelon Form: **Multan 2008**

(i) First non zero element of each row should be 1.

(ii) All elements under this 1 should be zero

e.g $$\begin{bmatrix} 0 & 1 & 7 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 5 & 6 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

Reduce Echelon Form:

First two conditions are same of echelon form

All elements above leading entry (1) should be zero.

$$\begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} etc$$

Rank: **Federal, Faisalabad 2008**

Number of non zero rows (not all elements zero) in the echelon form of a matrix is called Rank of the matrix.

Example:

Solve the following system of linear equations.

$3x_1 - x_2 = 1$ **Faisalabad 2008**

$x_1 + x_2 = 3$

Sol. The matrix form of system is

$$\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

A X =B

$\Rightarrow$ X =A^{-1}B

Now $|A| = \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} = 3-(-1) = 3+1 = 4$

adjA $= \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} adj\, A = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

$$X = A^{-1}B$$

$$X = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1+3 \\ -1+9 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4/4 \\ 8/4 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 1 \\ y_1 = 2 \end{matrix}$$

EXERCISE: 3.1

1. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 7 \\ 6 & 4 \end{bmatrix}$, then show that

i. $\mathbf{4A - 3A = A}$

Sol. L.H.S $= 4A - 3A = 4\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 8 & 12 \\ 4 & 20 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ 3 & 15 \end{bmatrix} = \begin{bmatrix} 8-6 & 12-9 \\ 4-3 & 20-5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} = A = \text{R.H.S.}$$

II. $\mathbf{3B - 3A = 3(B - A)}$

Sol. L.H.S $= 3B - 3A = 3\begin{bmatrix}1 & 7\\ 6 & 4\end{bmatrix} - 3\begin{bmatrix}2 & 3\\ 1 & 5\end{bmatrix}$

$$= \begin{bmatrix}3 & 21\\ 18 & 12\end{bmatrix} - \begin{bmatrix}6 & 9\\ 3 & 15\end{bmatrix} = \begin{bmatrix}3-6 & 21-9\\ 18-3 & 12-15\end{bmatrix} = \begin{bmatrix}-3 & 12\\ 15 & -3\end{bmatrix}$$

R.H.S. $= 3(B - A) = 3\left(\begin{bmatrix}1 & 7\\ 6 & 4\end{bmatrix} - \begin{bmatrix}2 & 3\\ 1 & 5\end{bmatrix}\right)$

$$= 3\begin{bmatrix}1-2 & 7-3\\ 6-1 & 4-5\end{bmatrix} = 3\begin{bmatrix}-1 & 4\\ 5 & -1\end{bmatrix} = \begin{bmatrix}-3 & 12\\ 15 & -3\end{bmatrix} \Rightarrow \text{L.H.S = R.H.S}$$

2. **If** $A = \begin{bmatrix}i & 0\\ 1 & -i\end{bmatrix}$**, Show that** $A^4 = I_2$ **Note** i **read as iota** $i = \sqrt{-1} \Rightarrow i^2 = -1$

Multan 2008, 2009, Sargodha 2009, Faisalabad 2008, Lahore 2009

Sol. $A^2 = A \times A = \begin{bmatrix}i & 0\\ 1 & -i\end{bmatrix}\begin{bmatrix}i & 0\\ 1 & -i\end{bmatrix} = \begin{bmatrix}i^2+0 & 0-0\\ i-i & 0+i^2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$

$$A^4 = A^2 \times A^2 = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} \times \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

$$= \begin{bmatrix}1+0 & 0+0\\ 0+0 & 0+1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = A^4 = I_2.$$ Hence proved

3. **Find x and y if.**

i. $\begin{bmatrix}x+3 & 1\\ -3 & 3y-4\end{bmatrix} = \begin{bmatrix}2 & 1\\ -3 & 2\end{bmatrix}$ **Sargodha 2011**

Sol. $\Rightarrow \quad x+3=2 \quad \Rightarrow x = 2-3 \quad \Rightarrow \boxed{x=-1}$

and $\quad 3y-4=2 \quad \Rightarrow 3y = 2+4 = 6 \quad \Rightarrow \boxed{y=2}$

ii. $\begin{bmatrix}x+3 & 1\\ -3 & 3y-4\end{bmatrix} = \begin{bmatrix}y & 1\\ -3 & 2x\end{bmatrix}$ **Sargodha 2008**

Sol. $\Rightarrow \quad x+3=y \longrightarrow I$

$3y-4=2x$

$\Rightarrow 2x-3y+4=0 \longrightarrow II$

Put I in II

$2x-3(x+3)+4=0 \Rightarrow 2x-3x-9+4=0 \Rightarrow -x-5=0 \Rightarrow \boxed{x=-5}$

Put value of x in I.

$$x+3=y \Rightarrow -5+3=y \Rightarrow -2=y \Rightarrow \boxed{y=-2}$$

4. If $A=\begin{bmatrix}-1 & 2 & 3\\ 1 & 0 & 2\end{bmatrix}$ and $B=\begin{bmatrix}0 & 3 & 2\\ 1 & -1 & 2\end{bmatrix}$, **find the following matrixes.**

i. $4A-3B$ **Multan 2007**

$$4A-3B=4\begin{bmatrix}-1 & 2 & 3\\ 1 & 0 & 2\end{bmatrix}-3\begin{bmatrix}0 & 3 & 2\\ 1 & -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}-4 & 8 & 12\\ 4 & 0 & 8\end{bmatrix}-\begin{bmatrix}0 & 9 & 6\\ 3 & -3 & 6\end{bmatrix}=\begin{bmatrix}-4-0 & 8-9 & 12-6\\ 4-3 & 0+3 & 8-6\end{bmatrix}$$

$$=\begin{bmatrix}-4 & -1 & 6\\ 1 & 3 & 2\end{bmatrix}$$

ii. $A+3(B-A)$

$$A+3(B-A)=\begin{bmatrix}-1 & 2 & 3\\ 1 & 0 & 2\end{bmatrix}+3\left(\begin{bmatrix}0 & 3 & 2\\ 1 & -1 & 2\end{bmatrix}-\begin{bmatrix}-1 & 2 & 3\\ 1 & 0 & 2\end{bmatrix}\right)$$

$$=\begin{bmatrix}-1 & 2 & 3\\ 1 & 0 & 2\end{bmatrix}+3\begin{bmatrix}0+1 & 3-2 & 2-3\\ 1-1 & -1-0 & 2-2\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 2 & 3\\ 1 & 0 & 2\end{bmatrix}+3\begin{bmatrix}1 & 1 & -1\\ 0 & -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 2 & 3\\ 1 & 0 & 2\end{bmatrix}+\begin{bmatrix}3 & 3 & -3\\ 0 & -3 & 0\end{bmatrix}=\begin{bmatrix}-1+3 & 2+3 & 3-3\\ 1+0 & 0-3 & 2+0\end{bmatrix}$$

$$=\begin{bmatrix}2 & 5 & 0\\ 1 & -3 & 2\end{bmatrix}$$

5. Find x and y if $\begin{bmatrix}2 & 0 & x\\ 1 & y & 3\end{bmatrix}+2\begin{bmatrix}1 & x & y\\ 0 & 2 & -1\end{bmatrix}=\begin{bmatrix}4 & -2 & 3\\ 1 & 6 & 1\end{bmatrix}$

Sol. $$\begin{bmatrix}2 & 0 & x\\ 1 & y & 3\end{bmatrix}+2\begin{bmatrix}1 & x & y\\ 0 & 2 & -1\end{bmatrix}=\begin{bmatrix}4 & -2 & 3\\ 1 & 6 & 1\end{bmatrix}$$

Or $$\begin{bmatrix}2 & 0 & x\\ 1 & y & 3\end{bmatrix}+\begin{bmatrix}2 & 2x & 2y\\ 0 & 4 & -2\end{bmatrix}=\begin{bmatrix}4 & -2 & 3\\ 1 & 6 & 1\end{bmatrix}$$

$$\begin{bmatrix}2+2 & 0+2x & x+2y\\ 1+0 & y+4 & 3-2\end{bmatrix}=\begin{bmatrix}4 & -2 & 3\\ 1 & 6 & 1\end{bmatrix}$$

Or $\begin{bmatrix} 4 & 2x & x+2y \\ x & y+4 & 3-2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$

$\Rightarrow 2x = -2 \Rightarrow \boxed{x=-1}$ & $y+4=6 \Rightarrow y=6-4 \Rightarrow \boxed{y=2}$

6. $A=\left[a_{ij}\right]_{3\times 3}$ **Show that:**

i. $\lambda(\mu A) = (\lambda\mu)A$

Sol *Let* $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$L.H.S = \lambda(\mu A) = \lambda\left(\mu\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)$$

$$= \lambda\begin{bmatrix} \mu a_{11} & \mu a_{12} & \mu a_{13} \\ \mu a_{21} & \mu a_{22} & \mu a_{23} \\ \mu a_{31} & \mu a_{32} & \mu a_{33} \end{bmatrix} = \begin{bmatrix} \lambda\mu a_{11} & \lambda\mu a_{12} & \lambda\mu a_{13} \\ \lambda\mu a_{21} & \lambda\mu a_{22} & \lambda\mu a_{23} \\ \lambda\mu a_{31} & \lambda\mu a_{32} & \lambda\mu a_{33} \end{bmatrix}$$

$$= (\lambda\mu)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = (\lambda\mu)A = R.H.S$$

ii. $(\lambda+\mu)A = \lambda A + \mu A$

Sol $L.H.S = (\lambda+\mu)A = (\lambda+\mu)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$= \begin{bmatrix} (\lambda+\mu)a_{11} & (\lambda+\mu)a_{12} & (\lambda+\mu)a_{13} \\ (\lambda+\mu)a_{21} & (\lambda+\mu)a_{22} & (\lambda+\mu)a_{23} \\ (\lambda+\mu)a_{31} & (\lambda+\mu)a_{32} & (\lambda+\mu)a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11}+\mu a_{11} & \lambda a_{12}+\mu a_{12} & \lambda a_{13}+\mu a_{13} \\ \lambda a_{21}+\mu a_{21} & \lambda a_{22}+\mu a_{22} & \lambda a_{23}+\mu a_{23} \\ \lambda a_{31}+\mu a_{31} & \lambda a_{32}+\mu a_{32} & \lambda a_{33}+\mu a_{33} \end{bmatrix}$$

$$=\begin{bmatrix}\lambda a_{11} & \lambda a_{12} & \lambda a_{13}\\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23}\\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33}\end{bmatrix}+\begin{bmatrix}\mu a_{11} & \mu a_{12} & \mu a_{13}\\ \mu a_{21} & \mu a_{22} & \mu a_{23}\\ \mu a_{31} & \mu a_{32} & \mu a_{33}\end{bmatrix}$$

$$=\lambda\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}+\mu\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\lambda A+\mu A=R.H.S$$

iii. $\lambda A-A=(\lambda-1)A$

Sol L.H.S $=\lambda A-A$

$$=\lambda\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}-\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

$$=\begin{bmatrix}\lambda a_{11} & \lambda a_{12} & \lambda a_{13}\\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23}\\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33}\end{bmatrix}-\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

$$=\begin{bmatrix}\lambda a_{11}-a_{11} & \lambda a_{12}-a_{12} & \lambda a_{13}-a_{13}\\ \lambda a_{21}-a_{21} & \lambda a_{22}-a_{22} & \lambda a_{23}-a_{23}\\ \lambda a_{31}-a_{31} & \lambda a_{32}-a_{32} & \lambda a_{33}-a_{33}\end{bmatrix}$$

$$=\begin{bmatrix}(\lambda-1)a_{11} & (\lambda-1)a_{12} & (\lambda-1)a_{13}\\ (\lambda-1)a_{21} & (\lambda-1)a_{22} & (\lambda-1)a_{23}\\ (\lambda-1)a_{31} & (\lambda-1)a_{32} & (\lambda-1)a_{33}\end{bmatrix}$$

$$=(\lambda-1)\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=(\lambda-1)A=R.H.S$$

7. $A=\left[a_{ij}\right]_{2\times3}$ **and** $B=\left[b_{ij}\right]_{2\times3}$ **show that** $\lambda(A+B)=\lambda A+\lambda B$.

Sol. Let $A=\left[a_{ij}\right]_{2\times3}$ & $B=\left[b_{ij}\right]_{2\times3}$

$$A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\end{bmatrix},\quad B=\begin{bmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\end{bmatrix}$$

L.H.S = $\lambda(A+B)=\lambda\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}\right)$

$=\lambda\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$

$=\begin{bmatrix} \lambda a_{11}+\lambda b_{11} & \lambda a_{12}+\lambda b_{12} & \lambda a_{13}+\lambda b_{13} \\ \lambda a_{21}+\lambda b_{21} & \lambda a_{22}+\lambda b_{22} & \lambda a_{23}+\lambda b_{23} \end{bmatrix}=\begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \end{bmatrix}+\begin{bmatrix} \lambda b_{11} & \lambda b_{12} & \lambda b_{13} \\ \lambda b_{21} & \lambda b_{22} & \lambda b_{23} \end{bmatrix}$

$=\lambda\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}+\lambda\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}=\lambda A+\lambda B=R.H.S$

8. **If $A=\begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}$ and $A^2=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ find the values of a and b.**

Sol. $A=\begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}$ & $A^2=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\longrightarrow I$ **Faisalabad 2007, Lahore 2009**

Now $A^2=A\times A=\begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}\begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}=\begin{bmatrix} 1+2a & 2+2b \\ a+ab & 2a+b^2 \end{bmatrix}$ $\longrightarrow II$

(Compare I and II) $\begin{bmatrix} 1+2a & 2+2b \\ a+ab & 2a+b^2 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow 1+2a=0$ & $2+2b=0$

$\Rightarrow \boxed{a=-1/2}$ $\Rightarrow 2b=-2\Rightarrow \boxed{b=-1}$

9. **If $A=\begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix}$ and $A^2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ find the values of a and b.**

Sol. $A=\begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix}$ & $A^2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\longrightarrow I$ **Multan 2008**

$A^2=A\times A=\begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix}\times\begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix}=\begin{bmatrix} 1-a & -1-b \\ a+ab & -a+b^2 \end{bmatrix}$ $\rightarrow II$

(Compare I and II) $\begin{bmatrix} 1-a & -1-b \\ a+ab & -a+b^2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow 1-a=1\Rightarrow \boxed{a=0}$ & $-1-b=0\Rightarrow \boxed{b=-1}$

10. **If $A=\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ then show that $(A+B)'=A'+B'$**

Sol. $A=\begin{bmatrix}1 & -1 & 2\\0 & 3 & 1\end{bmatrix}$ $B=\begin{bmatrix}2 & 3 & 0\\1 & 2 & -1\end{bmatrix}$ **Multan 2010, Sargodha 2008**

L.H.S $(A+B)^t=\left(\begin{bmatrix}1 & -1 & 2\\0 & 3 & 1\end{bmatrix}+\begin{bmatrix}2 & 3 & 0\\1 & 2 & -1\end{bmatrix}\right)^t$

$$=\begin{bmatrix}1+2 & -1+3 & 2+0\\0+1 & 3+2 & 1-1\end{bmatrix}^t=\begin{bmatrix}3 & 2 & 2\\1 & 5 & 0\end{bmatrix}^t=\begin{bmatrix}3 & 1\\2 & 5\\2 & 0\end{bmatrix}$$

R.H.S= $A^t+B^t=\begin{bmatrix}1 & -1 & 2\\0 & 3 & 1\end{bmatrix}^t+\begin{bmatrix}2 & 3 & 0\\1 & 2 & -1\end{bmatrix}^t$

$$=\begin{bmatrix}1 & 0\\-1 & 3\\2 & 1\end{bmatrix}+\begin{bmatrix}2 & 1\\3 & 2\\0 & -1\end{bmatrix}=\begin{bmatrix}1+2 & 0+1\\-1+3 & 3+2\\2+0 & 1-1\end{bmatrix}=\begin{bmatrix}3 & 1\\2 & 5\\2 & 0\end{bmatrix}\Rightarrow \text{L.H.S = R.H.S}$$

11. Find A^3 if $A=\begin{bmatrix}1 & 1 & 3\\5 & 2 & 6\\-2 & -1 & -3\end{bmatrix}$

Sol. $A=\begin{bmatrix}1 & 1 & 3\\5 & 2 & 6\\-2 & -1 & -3\end{bmatrix}$

$$A^2=A\times A=\begin{bmatrix}1 & 1 & 3\\5 & 2 & 6\\-2 & -1 & -3\end{bmatrix}\begin{bmatrix}1 & 1 & 3\\5 & 2 & 6\\-2 & -1 & -3\end{bmatrix}$$

$$=\begin{bmatrix}1+5-6 & 1+2-3 & 3+6-9\\5+10-12 & 5+4-6 & 15+12-18\\-2-5+6 & -2-2+3 & -6-6+9\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\3 & 3 & 9\\-1 & -1 & -3\end{bmatrix}$$

$$A^3=A^2\times A=\begin{bmatrix}0 & 0 & 0\\3 & 3 & 9\\-1 & -1 & -3\end{bmatrix}\begin{bmatrix}1 & 1 & 3\\5 & 2 & 6\\-2 & -1 & -3\end{bmatrix}$$

$$=\begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1+5+6 & -1-2+3 & -3-6+9 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=0_3$$

12. Find the matrix X if;

i. $X=\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}=\begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}$ **Multan 2008**

Sol. $X \quad A=B$

$\Rightarrow X=BA^{-1} \longrightarrow I$

Now $|A|=\begin{vmatrix} 5 & 2 \\ -2 & 1 \end{vmatrix}=5-(-4)=5+4=9\neq 0$

adj $A=\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}, A^{-1}=\dfrac{adjA}{|A|}=\dfrac{1}{9}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$

(I become) $= X=BA^{-1}=\begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}\dfrac{1}{9}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$

$$X=\frac{1}{9}\begin{bmatrix} -1+10 & 2+25 \\ 12+6 & -24+15 \end{bmatrix}=\frac{1}{9}\begin{bmatrix} 9 & 27 \\ 18 & -9 \end{bmatrix}=\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

ii. $\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}X=\begin{bmatrix} 2 & 1 \\ 5 & 10 \end{bmatrix}$

Sol. $A \quad X = B$

$\Rightarrow X = A^{-1}B \longrightarrow I$

$|A|=\begin{vmatrix} 5 & 2 \\ -2 & 1 \end{vmatrix}=5-(-4)=5+4=9\neq 0$

adj $A=\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix},$

$A^{-1}=\dfrac{adjA}{|A|}=\dfrac{1}{9}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$

(I become) $\Rightarrow X=\dfrac{1}{9}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 10 \end{bmatrix}$

$$X=\frac{1}{9}\begin{bmatrix} 2-10 & 1-20 \\ 4+25 & 2+50 \end{bmatrix}=\frac{1}{9}\begin{bmatrix} -8 & -19 \\ 29 & 52 \end{bmatrix}=\begin{bmatrix} -8/9 & -19/9 \\ 29/9 & 52/9 \end{bmatrix}$$

13. Find the matrix A if;

i. $$\begin{bmatrix} 5 & -1 \\ 0 & 0 \\ 3 & 1 \end{bmatrix} A = \begin{bmatrix} 3 & -7 \\ 0 & 0 \\ 7 & 2 \end{bmatrix}$$

Sol. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$\begin{bmatrix} 5 & -1 \\ 0 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ 0 & 0 \\ 7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 5a-c & 5b-d \\ 0+0 & 0+0 \\ 3a+c & 3b+d \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ 0 & 0 \\ 7 & 2 \end{bmatrix}$$

$$\Rightarrow 5a-c=3 \ \ \& \ \ 5b-d=-7$$

$$\Rightarrow 3a+c=7 \ \ \& \ \ 3b+d=2$$

Solve above equations

$$5a-\cancel{c} = 3 \quad \rightarrow I \qquad 5b-\cancel{d}=7 \quad \rightarrow III$$

$$3a+\cancel{c} = 7 \quad \rightarrow II \qquad 3b+\cancel{d}=2 \quad \rightarrow IV$$

$$8a = 10 \qquad\qquad 8b = -5$$

$$\boxed{a=5/4} \qquad \Rightarrow \boxed{b=-5/8}$$

I become $5\left(\frac{5}{4}\right)-c=3 \Rightarrow c=\frac{25}{4}-3 \Rightarrow \boxed{c=\frac{13}{4}}$

III become $5\left(\frac{-5}{8}\right)-d=-7 \Rightarrow d=\frac{-25}{8}+7 \Rightarrow \boxed{d=\frac{31}{8}}$

Hence $A = \begin{bmatrix} 5/4 & -5/8 \\ 13/4 & 31/8 \end{bmatrix}$

ii. $$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} A = \begin{bmatrix} 0 & -3 & 8 \\ 3 & 3 & -7 \end{bmatrix}$$

Sol. $B \ . \ A = C \Rightarrow A = B^{-1}C \longrightarrow I$

$$|B| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = (2)(2) - (-1)(-1) = 4 - 1 = 3 \neq 0$$

$$\text{adj}' B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, B^{-1} = \frac{adjB}{|B|} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(I\,become)\; A = B^{-1}C = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & -3 & 8 \\ 3 & 3 & -7 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 0+3 & -6+3 & 16-7 \\ 0+6 & -3+6 & 8-14 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & -3 & 9 \\ 6 & 3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix}$$

14. **Show that** $\begin{bmatrix} r\cos\phi & 0 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -r\sin\phi & 0 & r\cos\phi \end{bmatrix} = rI_3$ **Faisalabad 2008**

Sol. L.H.S= $\begin{bmatrix} r\cos\phi & 0 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -r\sin\phi & 0 & r\cos\phi \end{bmatrix}$

$$= \begin{bmatrix} r\cos^2\phi + 0 + r\sin^2\phi & 0+0+0 & r\cos\phi\sin\phi + 0 - r\cos\phi\sin\phi \\ 0+0+0 & 0+r+0 & 0+0+0 \\ r\sin\phi\cos\phi + 0 - r\sin\phi\cos\phi & 0+0+0 & r\sin^2\phi + 0 + r\cos^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} r(\cos^2\phi + r\sin^2\phi) & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r(\sin^2\phi + r\cos^2\phi) \end{bmatrix}$$

$$\begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} = r\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = rI_3$$

EXERCISE: 3.2

1. If $A=\left[a_{ij}\right]_{3\times4}$, then show that (i) $I_3A=A$ (ii) $AI_4=A$

Sol. then $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$ & $I_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

i. $I_3A=A$

Sol. $L.H.S=I_3A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

$$=\begin{bmatrix} a_{11}+0+0 & a_{12}+0+0 & a_{13}+0+0 & a_{14}+0+0 \\ 0+a_{21}+0 & 0+a_{22}+0 & 0+a_{23}+0 & 0+a_{24}+0 \\ 0+0+a_{31} & 0+0+a_{32} & 0+0+a_{33} & 0+0+a_{34} \end{bmatrix}$$

$$=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}=A \text{ Hence Proved } \boxed{I_3A=A}$$

ii. $AI_4=A$

Sol. $AI_4=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$=\begin{bmatrix} a_{11}+0+0+0 & 0+a_{12}+0+0 & 0+0+a_{13}+0+0 & 0+0+0+a_{14} \\ a_{21}+0+0+0 & 0+a_{22}+0+0 & 0+0+a_{23}+0 & 0+0+0+a_{24} \\ a_{31}+0+0+0 & 0+a_{32}+0+0 & 0+0+a_{33}+0 & 0+0+0+a_{34} \end{bmatrix}$$

$$=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}=A \text{ Hence } \boxed{AI_4=A}$$

2. **Find the inverse of the following matrices:**

i. $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ **Multan 2010**

Sol. $A=\begin{bmatrix}3 & -1\\2 & 1\end{bmatrix}$ *then* $|A|=\begin{vmatrix}3 & -1\\2 & 1\end{vmatrix}=3-(-2)=3+2=5\neq 0$

$adj\ A=\begin{bmatrix}1 & 1\\-2 & 3\end{bmatrix}$ So $A^{-1}=\dfrac{adjA}{|A|}=\dfrac{1}{5}\begin{bmatrix}1 & 1\\-2 & 3\end{bmatrix}=\begin{bmatrix}1/5 & 1/5\\-2/5 & 3/5\end{bmatrix}$

ii. $\begin{bmatrix}-2 & 3\\-4 & 5\end{bmatrix}$

Sol. Let $A=\begin{bmatrix}-2 & 3\\-4 & 5\end{bmatrix}$ *then* $|A|=\begin{vmatrix}-2 & 3\\-4 & 5\end{vmatrix}=-10-(-12)=10+12=2\neq 0$

$adj\ A=\begin{bmatrix}5 & -3\\4 & -2\end{bmatrix}$ So $A^{-1}=\dfrac{adjA}{|A|}=\dfrac{1}{2}\begin{bmatrix}5 & -3\\4 & -2\end{bmatrix}=\begin{bmatrix}5/2 & -3/2\\2 & -1\end{bmatrix}$

iii. $\begin{bmatrix}2i & i\\i & -i\end{bmatrix}$ **Multan 2009**

Sol. Let $A=\begin{bmatrix}2i & -i\\i & -i\end{bmatrix}$ *then* $|A|=\begin{vmatrix}2i & i\\i & -i\end{vmatrix}=-2i^2-i^2=-3i^2=-3(-1)=3$

$adj\ A=\begin{bmatrix}-i & +i\\-i & 2i\end{bmatrix}$ *then* $A^{-1}=\dfrac{adjA}{|A|}=\dfrac{1}{3}\begin{bmatrix}-i & i\\-i & 2i\end{bmatrix}=\begin{bmatrix}-i/3 & i/3\\-i/3 & 2i/3\end{bmatrix}$

3. **Solve the following system of linear equations.**

i. $2x_1-3x_2=5$
$5x_1+x_2=4$

Sol. In matrix form

$\begin{bmatrix}2 & -3\\5 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}$

$AX-B\Rightarrow X=A^{-1}B$———$I$

$|A|=\begin{vmatrix}2 & -3\\5 & 1\end{vmatrix}$

$=2-(-15)=2+15=17\neq 0$

$adj\ A=\begin{bmatrix}1 & 3\\-5 & 2\end{bmatrix}$

$A^{-1}=\dfrac{adjA}{|A|}=\dfrac{1}{17}\begin{bmatrix}1 & 3\\-5 & 2\end{bmatrix}$

$X=A^{-1}B=\dfrac{1}{17}\begin{bmatrix}1 & 3\\-5 & 2\end{bmatrix}\begin{bmatrix}5\\4\end{bmatrix}$

$X=\dfrac{1}{17}\begin{bmatrix}5+12\\-25+8\end{bmatrix}=\dfrac{1}{17}\begin{bmatrix}17\\-17\end{bmatrix}$

$X=\begin{bmatrix}1\\-1\end{bmatrix}\Rightarrow\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$

$\Rightarrow x_1=1\ \&\ x_2=-1$

ii. $4x_1 + 3x_2 = 5$
$3x_1 - x_2 = 7$

Sol. In matrix form

$$\begin{bmatrix} 4 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$A\ \ X = B \Rightarrow X = A^{-1}B$$

$$|A| = \begin{vmatrix} 4 & 3 \\ 3 & -1 \end{vmatrix} = -4-9 = -13 \neq 0$$

$$adj\ A = \begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{-13}\begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}$$

Now $X = A^{-1}B$

$$X = \frac{1}{-13}\begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$X = \frac{-1}{13}\begin{bmatrix} -5-21 \\ -15+28 \end{bmatrix}$$

$$X = \frac{-1}{13}\begin{bmatrix} -26 \\ 13 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, x_1 = 2, x_2 = -1$$

iii. $3x - 5y = 1$
$-2x + y = -3$

Sol. In matrix form

$$\begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$A\ \ X = B \Rightarrow X = A^{-1}B$$

$$|A| = \begin{vmatrix} 3 & -5 \\ -2 & 1 \end{vmatrix} = 3-10 = -7 \neq 0$$

$$adj\ A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{-7}\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$$

I become

$$x = \frac{1}{-7}\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \frac{1}{-7}\begin{bmatrix} 1-15 \\ 2-9 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-7}\begin{bmatrix} -14 \\ -7 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$x = 2$ & $y = 1$

4. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$ *and* $C = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$ then find

i. $A - B$

Sol.

$$A - B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & -1-1 & 2+1 \\ 3-1 & 2-3 & 5-4 \\ -1+1 & 0-2 & 4-1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 3 \\ 2 & -1 & 1 \\ 0 & -2 & 3 \end{bmatrix}$$

ii. $B-A$

Sol. $$B-A=\begin{bmatrix}2&1&-1\\1&3&4\\-1&2&1\end{bmatrix}-\begin{bmatrix}1&-1&2\\3&2&5\\-1&0&4\end{bmatrix}$$

$$=\begin{bmatrix}2-1&1+1&-1-2\\1-3&3-2&4-5\\-1+1&2-0&1-4\end{bmatrix}=\begin{bmatrix}1&2&-3\\-2&1&-1\\0&2&-3\end{bmatrix}$$

iii. $(A-B)-C$

Sol. $$(A-B)-C=\left(\begin{bmatrix}1&-1&2\\3&2&5\\-1&0&4\end{bmatrix}-\begin{bmatrix}2&1&-1\\1&3&4\\-1&2&1\end{bmatrix}\right)-\begin{bmatrix}1&3&-2\\-1&2&0\\3&4&-1\end{bmatrix}$$

$$=\begin{bmatrix}1-2&-1-1&2+1\\3-1&2-3&5-4\\-1+1&0-2&4-1\end{bmatrix}-\begin{bmatrix}1&3&-2\\-1&2&0\\3&4&-1\end{bmatrix}$$

$$=\begin{bmatrix}-1&-2&3\\2&-1&1\\0&-2&3\end{bmatrix}-\begin{bmatrix}1&3&-2\\-1&2&0\\3&4&-1\end{bmatrix}$$

$$=\begin{bmatrix}-1-1&-2-3&3+2\\2+1&-1-2&1-2\\0-3&-2-4&3+1\end{bmatrix}-\begin{bmatrix}-2&-5&5\\3&-3&1\\-3&-6&4\end{bmatrix}$$

iv. $A-(B-C)$

Sol. $$A-(B-C)=\begin{bmatrix}1&-1&2\\3&2&5\\-1&0&4\end{bmatrix}-\left(\begin{bmatrix}2&1&-1\\1&3&4\\-1&2&1\end{bmatrix}-\begin{bmatrix}1&3&-2\\-1&2&0\\3&4&-1\end{bmatrix}\right)$$

$$=\begin{bmatrix}1&-1&2\\3&2&5\\-1&0&4\end{bmatrix}-\begin{bmatrix}2-1&1-3&-1+2\\1+1&3-2&4-0\\-1-3&2-4&1+1\end{bmatrix}$$

$$=\begin{bmatrix}1&-1&2\\3&2&5\\-1&0&4\end{bmatrix}-\begin{bmatrix}1&-2&1\\2&1&4\\-4&-2&2\end{bmatrix}$$

$$=\begin{bmatrix} 1-1 & -1+2 & 2-1 \\ 3-2 & 2-1 & 5-4 \\ -1+4 & 0+2 & 4-2 \end{bmatrix}=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}$$

5. If $A=\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}, B=\begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix} and\ C=\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$ then find

I. $(AB)C=A(BC)$ **Multan 2007**

Sol. L.H.S $=(AB)C=\left(\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}\right)\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$

$$=\begin{bmatrix} -i^2+4i^2 & i+2i^2 \\ -i-2i^2 & 1-i^2 \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$=\begin{bmatrix} -(-1)+4(-1) & i+2(-1) \\ -i-2(-1) & 1-(-1) \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}=\begin{bmatrix} -3 & i-2 \\ -i+2 & 2 \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$=\begin{bmatrix} -6i-i^2+2i & 3+i^2-2i \\ -2i^2+4i-2i & i-2+2i \end{bmatrix}=\begin{bmatrix} -6i-(-1)+2i & 3+(-1)-2i \\ -2(-1)+4i-2i & 3i-2 \end{bmatrix}$$

$$=\begin{bmatrix} 1-4i & 2-2i \\ 2+2i & 3i-2 \end{bmatrix}$$

R.H.S $=A(BC)=\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\left(\begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}\right)$

$$=\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\begin{bmatrix} -2i^2-i & i+i \\ 4i^2-i^2 & -2i+i^2 \end{bmatrix}=\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\begin{bmatrix} -2(-1)-i & 2i \\ 4(-1)-(-1) & -2i+(-1) \end{bmatrix}$$

$$=\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\begin{bmatrix} 2-i & 2i \\ -3 & -2i-1 \end{bmatrix}=\begin{bmatrix} 2i-i^2-6i & 2i^2-4i^2-2i \\ 2-i+3i & 2i+2i^2+i \end{bmatrix}$$

$$=\begin{bmatrix} -4i-(-1) & -2i^2-2i \\ 2+2i & 3i+2(-1) \end{bmatrix}=\begin{bmatrix} 1-4i & -2(-1)-2i \\ 2+2i & 3i-2 \end{bmatrix}$$

$$=\begin{bmatrix} 1-4i & 2-2i \\ 2+2i & 3i-2 \end{bmatrix} Hence\ L.H.S=R.H.S$$

II. $(A+B)C=AC+BC$

Sol. L.H.S $=(A+B)C=\left(\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}+\begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}\right)\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$

$$=\begin{bmatrix} i-i & 2i+1 \\ 1+2i & -i+i \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}=\begin{bmatrix} 0 & 1+2i \\ 1+2i & 0 \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$=\begin{bmatrix} 0-i-2i^2 & 0+i+2i^2 \\ 2i+4i^2-0 & -1-2i+0 \end{bmatrix}=\begin{bmatrix} -i-2(-1) & i+2(-i) \\ 2i+4(-1) & -1-2i \end{bmatrix}$$

$$=\begin{bmatrix} 2-i & -2+i \\ 2i-4 & -1-2i \end{bmatrix}$$

$$\text{R.H.S} = AC+BC=\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}+\begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$=\begin{bmatrix} 2i^2-2i^2 & -i+2i^2 \\ 2i+i^2 & -1-i^2 \end{bmatrix}+\begin{bmatrix} -2i^2-i & i+i \\ 4i^2-i^2 & -2i+i^2 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & -i+2(-1) \\ 2i-1 & -1+1 \end{bmatrix}+\begin{bmatrix} -2(-1)-i & 2i \\ 4(-1)-(-1) & -2i+(-1) \end{bmatrix}$$

$$=\begin{bmatrix} 0 & -i-2 \\ 2i-1 & 0 \end{bmatrix}+\begin{bmatrix} 2-i & 2i \\ -3 & -2i-1 \end{bmatrix}=\begin{bmatrix} 0+2-i & -i-2+2i \\ 2i-1-3 & 0-2i-1 \end{bmatrix}$$

$$=\begin{bmatrix} 2-i & i-2 \\ 2i-4 & -1-2i \end{bmatrix}$$ Hence L.H.S = R.H.S

6. **If A and B are square matrices of the same order, then explain why in general;**

i. $(A+B)^2 \neq A^2+2AB+B^2$ **Faisalabad 2007**

sol.

$L.H.S=(A+B)^2=(A+B)(A+B)$

$=A^2+AB+BA+B^2$

Since $AB \neq BA$ *in general so* $AB+BA \neq 2AB$

Hence $(A+B)^2 \neq A^2-2AB+B^2$

ii. $(A-B)^2 \neq A^2-2AB+B^2$

$L.H.S=(A-B)^2=(A-B)(A-B)=A^2-AB-BA+B^2$

Since $AB \neq BA$ *in general so* $-AB-BA \neq -2AB$

Hence $(A-B)^2 \neq A^2-2AB+B^2$

iii. $(A+B)(A-B) \neq A^2-B^2$ **Rawalpindi 2009**

$L.H.S=(A+B)(A-B)=A^2-AB-BA+B^2$

Since $AB \neq BA$ *in general so*

$(A+B)(A-B) \neq A^2-B^2$

7. If $A=\begin{bmatrix}2 & -1 & 3 & 0\\ 1 & 0 & 4 & -2\\ -3 & 5 & 2 & -1\end{bmatrix}$ then find AA^t and $A^t A$.

Sol. $A=\begin{bmatrix}2 & -1 & 3 & 0\\ 1 & 0 & 4 & -2\\ -3 & 5 & 2 & -1\end{bmatrix}$, $A'=\begin{bmatrix}2 & 1 & -3\\ -1 & 0 & 5\\ 3 & 4 & 2\\ 0 & -2 & -1\end{bmatrix}$

$$AA'=\begin{bmatrix}2 & -1 & 3 & 0\\ 1 & 0 & 4 & -2\\ -3 & 5 & 2 & -1\end{bmatrix}\begin{bmatrix}2 & 1 & -3\\ -1 & 0 & 5\\ 3 & 4 & 2\\ 0 & -2 & -1\end{bmatrix}$$

$$=\begin{bmatrix}4+1+9+0 & 2-0+12-0 & -6-5+6-0\\ 2-0+12-0 & 1+0+16+4 & -3+0+8+2\\ -6-5+6-0 & -3+0+8+2 & 9+25+4+1\end{bmatrix}=\begin{bmatrix}14 & 14 & -5\\ 14 & 21 & 7\\ -5 & 7 & 39\end{bmatrix}$$

Now $A'A=\begin{bmatrix}2 & 1 & -3\\ -1 & 0 & 5\\ 3 & 4 & 2\\ 0 & -2 & -1\end{bmatrix}\begin{bmatrix}2 & -1 & 3 & 0\\ 1 & 0 & 4 & -2\\ -3 & 5 & 2 & -1\end{bmatrix}$

$$=\begin{bmatrix}4+1+9 & -2+0-15 & 6+4-6 & 0-2+3\\ -2+0-15 & 1+0+25 & -3+0+10 & 0+0-5\\ 6+4-6 & -3+0+10 & 9+16+4 & 0-8-2\\ 0-2+3 & 0-0-5 & 0-8-2 & 0+4+1\end{bmatrix}=\begin{bmatrix}14 & -17 & 4 & 1\\ -17 & 26 & 7 & -5\\ 4 & 7 & 29 & -10\\ 1 & -5 & -10 & 5\end{bmatrix}$$

8. **Solve the following matrix equations for X:**

i. $3X-2A=B$ if $A=\begin{bmatrix}2 & 3 & -2\\ -1 & 1 & 5\end{bmatrix}$ and $B=\begin{bmatrix}2 & -3 & 1\\ 5 & 4 & -1\end{bmatrix}$

Sol. $3X-2A=B \Rightarrow 3X=2A+B$

$$X=\frac{1}{3}(2A+B)=\frac{1}{3}\left(2\begin{bmatrix}2 & 3 & -2\\ -1 & 1 & 5\end{bmatrix}+\begin{bmatrix}2 & -3 & 1\\ 5 & 4 & -1\end{bmatrix}\right)$$

$$X=\frac{1}{3}\left(\begin{bmatrix}4 & 6 & -4\\ -2 & 2 & 10\end{bmatrix}+\begin{bmatrix}2 & -3 & 1\\ 5 & 4 & -1\end{bmatrix}\right)=\frac{1}{3}\begin{bmatrix}4+2 & 6-3 & -4+1\\ -2+5 & 2+4 & 10-1\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}6 & 3 & -3\\ 3 & 6 & 9\end{bmatrix}=\begin{bmatrix}3 & 1 & -1\\ 1 & 2 & 3\end{bmatrix}$$

ii. $\mathbf{2X-3A=B}$ **if** $\mathbf{A=\begin{bmatrix}1 & -1 & 2\\ -2 & 4 & 5\end{bmatrix}}$ **and** $\mathbf{B=\begin{bmatrix}3 & -1 & 0\\ 4 & 2 & 1\end{bmatrix}}$

$$2X-3A=B \quad\Rightarrow 2X=3A+B\Rightarrow X=\frac{1}{2}(3A+B)$$

$$X=\frac{1}{2}\left(3\begin{bmatrix}1 & -1 & 2\\ -2 & 4 & 5\end{bmatrix}+\begin{bmatrix}3 & -1 & 0\\ 4 & 2 & 1\end{bmatrix}\right)$$

$$=\frac{1}{2}\left(\begin{bmatrix}3 & -3 & 6\\ -6 & 12 & 15\end{bmatrix}+\begin{bmatrix}3 & -1 & 0\\ 4 & 2 & 1\end{bmatrix}\right)$$

$$=\frac{1}{2}\begin{bmatrix}3+3 & -3-1 & 6+0\\ -6+4 & 12+2 & 15+1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}6 & -4 & 6\\ -2 & 14 & 16\end{bmatrix}$$

$$X=\begin{bmatrix}3 & -2 & 3\\ -1 & 7 & 8\end{bmatrix}$$

9. Solve the following matrix equations for A:

i. $\begin{bmatrix}4 & 3\\ 2 & 2\end{bmatrix}\mathbf{A}-\begin{bmatrix}2 & 3\\ -1 & -2\end{bmatrix}=\begin{bmatrix}-1 & -4\\ 3 & 6\end{bmatrix}$

Sol. $\begin{bmatrix}4 & 3\\ 2 & 2\end{bmatrix}A=\begin{bmatrix}2 & 3\\ -1 & -2\end{bmatrix}+\begin{bmatrix}-1 & -4\\ 3 & 6\end{bmatrix}$

$$\begin{bmatrix}4 & 3\\ 2 & 2\end{bmatrix}A=\begin{bmatrix}2-1 & 3-4\\ -1+3 & -2+6\end{bmatrix}\Rightarrow\begin{bmatrix}4 & 3\\ 2 & 2\end{bmatrix}A=\begin{bmatrix}1 & -1\\ 2 & 4\end{bmatrix}$$

$$B \quad A=C$$

$$\Rightarrow \qquad A=B^{-1}C \text{———} I \qquad\qquad |B|=\begin{vmatrix}4 & 3\\ 2 & 2\end{vmatrix}=8-6=2\neq 0$$

$$\Rightarrow \qquad A=\frac{1}{2}\begin{bmatrix}2 & -3\\ -2 & 4\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & 4\end{bmatrix} \qquad\qquad adj\,B=\begin{bmatrix}2 & -3\\ -2 & 4\end{bmatrix}$$

$$A=\frac{1}{2}\begin{bmatrix}2-6 & -2-12\\ -2+8 & 2+16\end{bmatrix} \qquad\qquad B^{-1}=\frac{adj\,B}{|B|}=\frac{1}{2}\begin{bmatrix}2 & -3\\ -2 & 4\end{bmatrix}$$

$$A=\frac{1}{2}\begin{bmatrix}-4 & -14\\ 6 & 18\end{bmatrix} \qquad \Rightarrow \qquad A=\begin{bmatrix}-2 & -7\\ 3 & 9\end{bmatrix}$$

ii. $$A\begin{bmatrix}3 & 1\\4 & 2\end{bmatrix}-\begin{bmatrix}-1 & 2\\3 & 1\end{bmatrix}=\begin{bmatrix}2 & 0\\-1 & 5\end{bmatrix}$$

Sol. $$A\begin{bmatrix}3 & 1\\4 & 2\end{bmatrix}=\begin{bmatrix}-1 & 2\\3 & 1\end{bmatrix}+\begin{bmatrix}2 & 0\\-1 & 5\end{bmatrix}$$

$$A\begin{bmatrix}3 & 1\\4 & 2\end{bmatrix}=\begin{bmatrix}-1+2 & 2+0\\3-1 & 1+5\end{bmatrix}\Rightarrow A\begin{bmatrix}3 & 1\\4 & 2\end{bmatrix}=\begin{bmatrix}1 & 2\\2 & 6\end{bmatrix}$$

$$AB=C$$

Now $|B|=\begin{vmatrix}3 & 1\\4 & 2\end{vmatrix}=6-4=2\neq 0 \quad\Rightarrow\quad A=CB^{-1}$

$$adj\ B=\begin{bmatrix}2 & -1\\-4 & 3\end{bmatrix}\quad\Rightarrow\quad B^{-1}=\frac{adj\ B}{|B|}=\frac{1}{2}\begin{bmatrix}2 & -1\\-4 & 3\end{bmatrix}$$

$$A=CB^{-1}$$

$$=\begin{bmatrix}1 & 2\\2 & 6\end{bmatrix}\frac{1}{2}\begin{bmatrix}2 & -1\\-4 & 3\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2-8 & -1+6\\4-24 & -2+18\end{bmatrix}$$

$$A=\frac{1}{2}\begin{bmatrix}-6 & 5\\-20 & 16\end{bmatrix}\quad\Rightarrow\quad A=\begin{bmatrix}-3 & 5/2\\-10 & 8\end{bmatrix}$$

Example: **Find the cofactors A_{12}, A_{22}, & A_{32} if**

$$A=\begin{bmatrix}1 & -2 & 3\\-2 & 3 & 1\\4 & -3 & 2\end{bmatrix}\text{ also find } |A|$$

Sol. $$A_{12}=(-1)^{1+2}\begin{vmatrix}-2 & 1\\4 & 2\end{vmatrix}=(-1)^3[-4-4]=-1(-8)=\boxed{8}\ Ans$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}1 & 3\\4 & 2\end{vmatrix}=(-1)^4[2-12]=1(-10)=\boxed{-10}\ Ans$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1 & 3\\-2 & 1\end{vmatrix}=(-1)^5[1+6]=(-1)(7)=\boxed{-7}\ Ans$$

and $$|A|=a_{12}A_{12}+a_{22}A_{22}+a_{32}A_{32}$$
$$=(-2)(8)+(3)(-10)+(-3)(-7)$$
$$=-16-30+21=-25\ Ans$$

Important Question: Write any two properties of determinants.

Sol. Property 1. If a square matrix A has two identical rows or two identical columns then $|A| = 0$

Property 2: If all the entries of a row (or a column) of a square matrix. A are zero, then $|A| = 0$

Proof 2: Prove that If all entries of any row (column) of a square matrix are zero; then value of determinants is zero.

Sol. $Let\ A = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$|A| = \begin{vmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - 0\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + 0\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= 0 - 0 + 0 = 0$$

Proof 1: Prove that If any two rows (column) of a determinant are identical then value of determinant is zero. Multan 2009, Lahore 2009

Sol. $Let\ A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & y & z \end{bmatrix}$

$$|A| = \begin{vmatrix} a & b & c \\ a & b & c \\ a & y & z \end{vmatrix} = a\begin{vmatrix} b & c \\ y & z \end{vmatrix} - b\begin{vmatrix} a & c \\ a & z \end{vmatrix} + c\begin{vmatrix} a & b \\ a & y \end{vmatrix}$$

$$= a(bz - yc) - b(az - ac) + c(ay - ab)$$

$$= abz - acy - abz + abc + acy - abc$$

$$= \boxed{0}$$ proved.

EXERCISE: 3.3

1. Evaluate the following determinants.

i. $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$

Sol. $=5\begin{vmatrix} -1 & -3 \\ 1 & 2 \end{vmatrix}-(-2)\begin{vmatrix} 3 & -3 \\ -2 & 2 \end{vmatrix}+(-4)\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$

$=5(-2+3)+2(6-6)-4(3-2)$

$=5(1)+2(0)-4(1)=5-0-4=1$

ii. $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$ **Faisalabad 2009**

Sol. $=5\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix}-2\begin{vmatrix} 3 & 1 \\ -2 & -2 \end{vmatrix}+(-3)\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$

$=5(2-1)-2(-6+2)-3(3-2)$

$=5(1)-2(-4)-3(1)=5+8-3=10$

iii. $\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$ **Multan 2008**

Sol. $=1\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}-2\begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix}+(-3)\begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix}$

$=1(18-20)-2(-6+8)-3(-5+6)$

$=-2-4-3=-9$

iv. $\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}=(a+l)\begin{vmatrix} a+l & a-l \\ a & a+l \end{vmatrix}-(a-l)\begin{vmatrix} a & a-l \\ a-l & a+l \end{vmatrix}+a\begin{vmatrix} a & a+l \\ a-l & a \end{vmatrix}$

Sol. $=(a+l)\left[(a+l)^2-a(a-l)\right]-(a-l)\left[a(a+l)-(a-l)^2\right]+a\left[a^2-(a-l)(a+l)\right]$

$=(a+l)\left[a^2+l^2+2al-a^2+al\right]-(a-l)\left[a^2+al-a^2-l^2+2al\right]+a\left[a^2-a^2+l^2\right]$

$=(a+l)(l^2+3al)-(a-l)(3al-l^2)+al^2$

$=al^2+3a^2l+l^3+3al^2-3a^2l+al^2+3al^2-l^3+al^2$

$=9al^2$

v. $\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix}$

Sol. $= 1\begin{vmatrix} 1 & -3 \\ 4 & -1 \end{vmatrix} - 2\begin{vmatrix} -1 & -3 \\ 2 & -1 \end{vmatrix} + (-2)\begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix}$

$= 1(-1+12) - 2(1+6) - 2(-4-2)$

$= 1(11) - 2(7) - 2(-6) = 11 - 14 + 12 = 9$

vi. $\begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix}$

Sol. $= 2a\begin{vmatrix} 2b & b \\ c & 2c \end{vmatrix} - a\begin{vmatrix} b & b \\ c & 2c \end{vmatrix} + a\begin{vmatrix} b & 2b \\ c & c \end{vmatrix}$

$= 2a(4bc - bc) - a(2bc - bc) + a(bc - 2bc)$

$= 2a(3bc) - a(bc) + a(-bc)$

$= 6abc - abc - abc = 4abc$

2. **Without expansion show that.**

i. L.H.S $= \begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 0$ **Sargodha 2011, Faisalabad 2007**

Sol. $C_2 - C_1$ *and* $C_3 - C_2$

$= \begin{vmatrix} 6 & 1 & 1 \\ 3 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0 =$ R.H.S (Because C_2 and C_3 are identical)

ii. $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix} = 0$ **Sargodha 2009, 2010 Faisalabad 2008, Multan 2009**

Sol. L.H.S $= \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$ *Add* C_2 *in* C_3

$$= \begin{vmatrix} 2 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & -3 & 2 \end{vmatrix} = 0 = \text{R.H.S} \qquad \text{(Because } C_1 \text{ and } C_3 \text{ are identical)}$$

iii. L.H.S = $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$ **Sargodha 2006, Multan 2010**

Sol. $C_2 - C_1$ *and* $C_3 - C_2$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 1 \\ 7 & 1 & 1 \end{vmatrix} = 0 \qquad \text{(Because } C_2 \text{ and } C_3 \text{ are identical)}$$

3. Show that

i. $$\begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$$

Sol. L.H.S = $\begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{33}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix}$ opening from C_3

$$= (a_{13}+\alpha_{13})\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - (a_{23}+\alpha_{23})\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + (a_{33}+\alpha_{33})\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} + \alpha_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} - \alpha_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \alpha_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \alpha_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - \alpha_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + \alpha_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$$

ii. $\begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{vmatrix} = 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix}$ **Multan 2007, Lahore 2009, Faisalabad 2008**

Sol. $\text{L.H.S} = \begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{vmatrix}$

$= 3\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 2 \\ 2 & 15 & 1 \end{vmatrix}$ Take 3 common from R_2

$= 3.3\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix}$ $(\text{Take 3 Common from } C_2) = 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix} = \text{R.H.S}$

iii. $\begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix} = l^2(3a+l)$ **Multan 2010, Faisalabad 2008**

Sol. $\text{L.H.S} = \begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix}$ *Add* R_2, R_3 *in* R_1

$= \begin{vmatrix} 3a+l & 3a+l & 3a+l \\ a & a+l & a \\ a & a & a+l \end{vmatrix}$

$= (3a+l)\begin{vmatrix} 1 & 1 & 1 \\ a & a+l & a \\ a & a & a+l \end{vmatrix}$ (Take Common from $(3a+l)$)

$C_2 - C_1,\ C_3 - C_1$

$= (3a+l)\begin{vmatrix} 1 & 0 & 0 \\ a & l & 0 \\ a & 0 & l \end{vmatrix} = (3a+l)\left[1\begin{vmatrix} l & 0 \\ 0 & l \end{vmatrix} - 0 + 0\right] = l^2(3a+l) = R.H.S$

iv. $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$ **Faisalabad 2009**

Sol. L.H.S $= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix}$

$$= \frac{1}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix} \quad xC_1, yC_2, zC_3$$

$$= \frac{xyz}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$ Take common xyz from R_3

$$= -\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2 \end{vmatrix}$$ Interchange R_2 and R_3

$$= (-)(-)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$ Again Interchange R_2 and R_1

$$= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \text{R.H.S}$$

v. $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

Sol. L.H.S $= \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ Expand by R_1

$$=(b+c)\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - a\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + a\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$$=(b+c)\left[(c+a)(a+b)-bc\right]-a\left[b(a+b)-bc\right]+a\left[bc-c(c+a)\right]$$

$$=(b+c)(ac+bc+a^2+ab)-a(ab+b^2-bc)+a(bc-c^2-ac)$$

$$=abc+b^2c+a^2b+ab^2+ac^2+bc^2+a^2c+abc-a^2b-ab^2+abc+abc-ac^2-a^2c$$

$=4abc=$ R.H.S

vi. $$\begin{vmatrix} b & -1 & a \\ a & b & o \\ 1 & a & b \end{vmatrix} = a^3+b^3$$

Sol. L.H.S = $\begin{vmatrix} b & -1 & a \\ a & b & o \\ 1 & a & b \end{vmatrix}$ Expand by R_1

$$=b\begin{vmatrix} b & 0 \\ a & b \end{vmatrix} - (-1)\begin{vmatrix} a & 0 \\ 1 & b \end{vmatrix} + a\begin{vmatrix} a & b \\ 1 & a \end{vmatrix}$$

$$=b(b^2-0)+1(ab-0)+a(a^2-b)$$

$=b^3+ab+a^3-ab=a^3+b^3=$ R.H.S

vii. $$\begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & 1 & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix} = r$$

Sol. L.H.S = $\begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & 1 & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix} = -0+1\begin{vmatrix} r\cos\phi & -\sin\phi \\ r\sin\phi & \cos\phi \end{vmatrix} - 0$ Expand by R_2

$$=r\cos^2\phi+r\sin^2\phi$$

$=r(\cos^2\phi+\sin^2\phi)=r(1)=r$ R.H.S

viii. $$\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix} = a^3+b^3+c^3-3abc$$ **Falsalabad 2008**

Sol. $\text{L.H.S} = \begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix}$ *Add* C_2 *in* C_1

$$= \begin{vmatrix} a+b+c & b+c & a+b \\ a+b+c & c+a & b+c \\ a+b+c & a+b & c+a \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & b+c & a+b \\ 1 & c+a & b+c \\ 1 & a+b & c+a \end{vmatrix} \quad \text{Take } (a+b+c) \text{ common from } C_1$$

$$= (a+b+c)\begin{vmatrix} 1 & b+c & a+b \\ 0 & a-b & c-a \\ 0 & a-c & c-b \end{vmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= (a+b+c)\left[1\begin{vmatrix} a-b & c-a \\ a-c & c-b \end{vmatrix} - 0 + 0\right]$$

$$= (a+b+c)\left[((a-b)(c-b)-(a-c)(c-a))\right]$$

$$= (a+b+c)(ac-ab-bc+b^2-ac+a^2+c^2-ac)$$

$$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$= a^3+b^3+c^3-3abc \quad = \text{R.H.S}$$

xi. $\begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix} = \lambda^2(a+b+c+\lambda)$

Sol. $\text{L.H.S} = \begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix}$

$$= \begin{vmatrix} a+b+c+\lambda & b & c \\ a+b+c+\lambda & b+\lambda & c \\ a+b+c+\lambda & b & c+\lambda \end{vmatrix} \quad \textit{Add } C_2,\ C_3 \textit{ in } C_1$$

$$=(a+b+c+\lambda)\begin{vmatrix}1 & b & c\\ 1 & b+\lambda & c\\ 1 & b & c+\lambda\end{vmatrix}$$ Take Common $(a+b+c+\lambda)$ from C_1

$$=(a+b+c+\lambda)\begin{vmatrix}1 & b & c\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{vmatrix} \quad R_2-R_1, R_3-R_1$$

$$=(a+b+c+\lambda)\left[\begin{vmatrix}\lambda & 0\\ 0 & \lambda\end{vmatrix} -0+0\right]$$ Expand by C_1

$$=(a+b+c+\lambda)\left[\lambda^2-0-0+0\right]$$

$$=\lambda^2(a+b+c+\lambda)=\text{R.H.S}$$

x. $$\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}=(a-b)(b-c)(c-a)$$

Sol. $$\text{L.H.S}=\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}$$

$$=\begin{vmatrix}1 & 0 & 0\\ a & b-a & c-a\\ a^2 & b^2-a^2 & c^2-a^2\end{vmatrix} \quad C_2-C_1, C_3-C_1$$

(Take common $b-a$ from C_2, $c-a$ from C_3)

$$=\begin{vmatrix}1 & 0 & 0\\ a & b-a & c-a\\ a^2 & (b-a)(b+a) & (c-a)(c+a)\end{vmatrix}=(b-a)(c-a)\begin{vmatrix}1 & 0 & 0\\ a & 1 & 1\\ a^2 & b+a & c+a\end{vmatrix}$$

$$=(b-a)(c-a)\left[1\begin{vmatrix}1 & 1\\ b+a & c+a\end{vmatrix}-0+0\right]$$

$$=(b-a)(c-a)(c+a-b-a)$$

$$=(b-a)(c-a)(c-b)$$

$$=\left[-(a-b)\right](c-a)\left[-(b-c)\right]$$

$$=(a-b)(b+c)(c-a) \;=\; \text{R.H.S}$$

xi. $$\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$$ **Sargodha 2009**

Sol. L.H.S $= \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix}$

$$= \begin{vmatrix} a+b+c & a & a^2 \\ a+b+c & b & b^2 \\ a+b+c & c & c^2 \end{vmatrix} \quad add\ C_2\ in\ C_1$$

$$= (a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$ Take Common $(a+b+c)$ from C_1

$$= (a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \quad R_2 - R_1,\ R_3 - R_1$$

$$= (a+b+c)\left[1\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} - 0 + 0\right]$$

Expand by C_1

$$= (a+b+c)\begin{vmatrix} b-a & (b-a)(b+a) \\ c-a & (c-a)(c+a) \end{vmatrix}$$

$$= (a+b+c)(b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$ Take Common $(b-a)$, $(c-a)$ from R_1, R_2

$= (a+b+c)(b-a)(c-a)(c+a-b-a)$

$= (a+b+c)(b-a)(c-a)(c-b)$

$= (a+b+c)[-(a-b)](c-a)[-(b-c)]$

$= (a+b+c)(a-b)(b-c)(c-a)$ = R.H.S

4. If $A=\begin{bmatrix}1 & 2 & -3\\ 0 & -2 & 0\\ -2 & -2 & 1\end{bmatrix}$ *and* $B=\begin{bmatrix}5 & -2 & 5\\ 3 & -1 & 4\\ -2 & 1 & -2\end{bmatrix}$, then find;

i. A_{12}, A_{22}, A_{32} and $|A|$ **Faisalabad 2007,08,09 Sargodha 2007,08, Multan 2007**

Sol. $$|A|=\begin{vmatrix}1 & 2 & -3\\ 0 & -2 & 0\\ -2 & -2 & 1\end{vmatrix}=1\begin{vmatrix}-2 & 0\\ -2 & 1\end{vmatrix}-2\begin{vmatrix}0 & 0\\ -2 & 1\end{vmatrix}+(-3)\begin{vmatrix}0 & -2\\ -2 & -2\end{vmatrix}$$

$$=1(-2+0)-2(0-0)-3(0-4)$$

$$=-2-0+12=10$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix}0 & 0\\ -2 & 1\end{vmatrix}=-(0+0)=0$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}1 & -3\\ -2 & 1\end{vmatrix}=(1-6)=-5$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1 & -3\\ 0 & 0\end{vmatrix}=(-1)(0-0)=0$$

ii. B_{21}, B_{22}, B'_{23} and $|B|$ **Lahore 2009, Gujranwala 2009**

Sol. $$B=\begin{bmatrix}5 & -2 & 5\\ 3 & -1 & 4\\ -2 & 1 & -2\end{bmatrix}$$

$$|B|=\begin{vmatrix}5 & -2 & 5\\ 3 & -1 & 4\\ -2 & 1 & -2\end{vmatrix}=5\begin{vmatrix}-1 & 4\\ 1 & -2\end{vmatrix}-(-2)\begin{vmatrix}3 & 4\\ -2 & -2\end{vmatrix}+5\begin{vmatrix}3 & -1\\ -2 & -1\end{vmatrix}$$

$$=5(2-4)+2(-6+8)+5(3-2)$$

$$=-10+4+5=-1$$

$$B_{21}=(-1)^{2+1}\begin{vmatrix}-2 & 5\\ 1 & -2\end{vmatrix}=-(4-5)=1$$

$$B_{22}=(-1)^{2+2}\begin{vmatrix}5 & 5\\ -2 & -2\end{vmatrix}=(-10+10)=0$$

$$B_{23}=(-1)^{2+3}\begin{vmatrix}5 & -2\\ -2 & 1\end{vmatrix}=-(5-4)=-1$$

5. Without Expansion verify that:

i. $\begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix} = 0$ **Sargodha 2009, Multan 2010, Fsd 2008, Gujranwala 2009**

Sol. $\text{L.H.S} = \begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix}$

$= \begin{vmatrix} \alpha+\beta+\gamma & \beta+\gamma & 1 \\ \alpha+\beta+\gamma & \gamma+\alpha & 1 \\ \alpha+\beta+\gamma & \alpha+\beta & 1 \end{vmatrix}$ *Add C_2 in C_1*

$= (\alpha+\beta+\gamma)\begin{vmatrix} 1 & \beta+\gamma & 1 \\ 1 & \gamma+\alpha & 1 \\ 1 & \alpha+\beta & 1 \end{vmatrix}$ Take $(\alpha+\beta+\gamma)$ Common from C_1

$= (\alpha+\beta+\gamma)(0) = 0$ (Because C_1 and C_3 are identical)

ii. $\begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix} = 0$ **Multan 2007, 2008, 2009**

Sol. $\text{L.H.S} = \begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix}$

$= 3x\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 5 & 3 \end{vmatrix}$ Take Common $3x$ from C_3

$= 3x(0) = 0$ (Because C_1 and C_3 are identical)

iii. $\begin{vmatrix} 1 & a^2 & a/bc \\ 1 & b^2 & b/bc \\ 1 & c^2 & c/ab \end{vmatrix} = 0$

Sol. $\text{L.H.S} = \begin{vmatrix} 1 & a^2 & a/bc \\ 1 & b^2 & b/bc \\ 1 & c^2 & c/ab \end{vmatrix}$

Multiplying C_3 by abc and $\div$ outside

$$=\frac{1}{abc}\begin{vmatrix}1 & a^2 & (abc)a/bc \\ 1 & b^2 & (abc)b/bc \\ 1 & c^2 & (abc)c/ab\end{vmatrix}$$

$$=\frac{1}{abc}\begin{vmatrix}1 & a^2 & a^2 \\ 1 & b^2 & b^2 \\ 1 & c^2 & c^2\end{vmatrix}$$

$=\frac{1}{abc}(0)=0$ (Because C_2 and C_3 are identical)

iv. $\begin{vmatrix}a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c\end{vmatrix}=0$ **Federal**

Sol. L.H.S $=\begin{vmatrix}a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c\end{vmatrix}$

Add C_2, C_3 in C_1

$$=\begin{vmatrix}a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c\end{vmatrix}=\begin{vmatrix}0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c\end{vmatrix}$$

$=0$ (Because C_1 is zero)

v. $\begin{vmatrix}bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c\end{vmatrix}=0$

Sol. L.H.S $=\begin{vmatrix}bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c\end{vmatrix}$

Multiplying R_2 by abc and divide outside

$$= \frac{1}{abc}\begin{vmatrix} bc & ca & ab \\ \frac{abc}{a} & \frac{abc}{b} & \frac{abc}{c} \\ a & b & c \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} bc & ca & ab \\ bc & ca & ab \\ a & b & c \end{vmatrix}$$

$= \frac{1}{abc}(0) = 0$ Because R_1 and R_2 are identical.

vi. $$\begin{vmatrix} mn & l & l^2 \\ ln & m & m^2 \\ lm & n & n^2 \end{vmatrix} = \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$$

Sol. $$\text{L.H.S} = \begin{vmatrix} mn & l & l^2 \\ ln & m & m^2 \\ lm & n & n^2 \end{vmatrix}$$

$$= \frac{l}{lmn}\begin{vmatrix} lmn & l^2 & l^3 \\ lmn & m^2 & m^3 \\ lmn & n^2 & n^3 \end{vmatrix} \quad l \times R_1, m \times R_2, n \times R_3$$

$$= \frac{\cancel{lmn}}{\cancel{lmn}}\begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$$ = Taking common from C_1

$$= \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix} = \text{R.H.S}$$

vii. $$\begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix} = 0$$

Sol. $$\text{L.H.S} = \begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$$

$$=2\begin{vmatrix} a & b & c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix} \quad \text{Take 2 common from } R_1$$

$$=2\begin{vmatrix} a & b & c \\ b & b & b \\ c & c & c \end{vmatrix} \quad R_2-R_1, R_3-R_1$$

$$=2bc\begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \quad \text{Take common b from } R_2, \text{ c from } R_3$$

$=2bc(0)=0$ Because R_2, and R_3 are identical

viii. $\begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$

Sol. R.H.S $= \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$

Add C_3 of Both.

$$= \begin{vmatrix} 7 & 2 & 7-1 \\ 6 & 3 & 5-3 \\ -3 & 5 & -3+4 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = \text{L.H.S}$$

ix. $\begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix}$ **Rawalpindi 2009**

Sol. $= \dfrac{1}{abc}\begin{vmatrix} -ab & 0 & bc \\ 0 & ac & -bc \\ ab & -ac & 0 \end{vmatrix}, R_1\times b, R_2\times c, R_3\times a$

$$= \frac{1}{abc}\begin{vmatrix} -ab+ab & ac-ac & bc-bc \\ 0 & ac & -bc \\ ab & -ac & 0 \end{vmatrix} \; Add\ R_2, R_3\ in\ R_1$$

$$=\frac{1}{abc}\begin{vmatrix}0 & 0 & 0\\ 0 & ac & -bc\\ ab & -ac & 0\end{vmatrix}$$

$$=\frac{1}{abc}(0)=0 \text{ (Because } R_1 \text{ is zero)}$$

6. **Find values of x if**

i. $$\begin{vmatrix}3 & 1 & x\\ -1 & 3 & 4\\ x & 1 & 0\end{vmatrix}=-30$$ **Sargodha 2008, Multan 2009**

Sol.

$$3\begin{vmatrix}3 & 4\\ 1 & 0\end{vmatrix}-1\begin{vmatrix}-1 & 4\\ x & 0\end{vmatrix}+x\begin{vmatrix}-1 & 3\\ x & 1\end{vmatrix}=-30$$

$$3(0-4)-1(0-4x)+x(-1-3x)=-30$$

$$-12+4x-x-3x^2=-30$$

$$-3x^2+3x-12+30=0$$

$$-3x^2+3x+18=0$$

$\div$ by -3

$$x^2-x-6=0 \Rightarrow x^2-3x+2x-6=0$$

$$x(x-3)+2(x-3)=0$$

$$x(x-3)+2(x-3)=0$$

$$x-3=0 \quad or \quad x+2=0$$

$$x=3 \quad or \quad x=-2$$

ii. $$\begin{vmatrix}1 & x-1 & 3\\ -1 & x+1 & 2\\ 2 & -2 & x\end{vmatrix}=0$$ **Federal**

Sol. $$1\begin{vmatrix}x+1 & 2\\ -2 & x\end{vmatrix}-(x-1)\begin{vmatrix}-1 & 2\\ 2 & x\end{vmatrix}+3\begin{vmatrix}-1 & x+1\\ 2 & -2\end{vmatrix}=0$$

$$1(x^2+x+4)-(x-1)(-x-4)+3(2-2x-2)=0$$

$$x^2+x+4+x^2+4x-x-4-6x=0$$

$$2x^2-2x=0 \quad \Rightarrow \quad 2x(x-1)=0$$

$$2x=0 \quad or \quad x-1=0$$

$$x=0 \quad or \quad x=1$$

iii. $\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$ **Faisalabad 2009**

Sol. $1\begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & x \end{vmatrix} + 1\begin{vmatrix} 2 & x \\ 3 & 6 \end{vmatrix} = 0$

$$(x^2-12)-2(2x-6)+1(12-3x)=0$$
$$x^2-12-4x+12+12-3x=0$$
$$x^2-7x+12=0$$
$$x^2-3x-4x+12=0$$
$$x(x-3)-4(x-3)=0$$
$$(x-3)(x-4)=0$$
$$x-3=0 \quad or \quad x-4=0$$
$$x=3 \quad or \quad x=4$$

7. Evaluate the following determinants:

i. $\begin{vmatrix} 3 & 4 & 2 & 7 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix}$

Sol. $\begin{vmatrix} 3 & 4 & 2 & 7 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix}$

$$\begin{vmatrix} 1 & -1 & 2 & 4 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix} R_1 - R_2$$

$$= \begin{vmatrix} 1 & -1 & 2 & 4 \\ 0 & 7 & -4 & -5 \\ 0 & 3 & -5 & 1 \\ 0 & 5 & -10 & -10 \end{vmatrix} \quad R_2-2R_1,\ R_3-R_1,\ R_4-4R_1$$

$$=\begin{vmatrix}7 & -4 & -5\\ 3 & -5 & 1\\ 5 & -10 & -10\end{vmatrix}-0+0-0 \text{ Expand by } C_1$$

$$=7\begin{vmatrix}-5 & 1\\ -10 & -10\end{vmatrix}-(-4)\begin{vmatrix}3 & 1\\ 5 & -10\end{vmatrix}+(-5)\begin{vmatrix}3 & -5\\ 5 & -10\end{vmatrix}$$

$$=7(50+10)+4(-30-5)-5(-30+25)$$

$$=420-140+25$$

$$=305$$

ii. $\begin{vmatrix}2 & 3 & 1 & -1\\ 4 & 0 & 2 & 1\\ 5 & 2 & -1 & 6\\ 3 & -7 & 2 & -2\end{vmatrix}$

Sol.

$$=\begin{vmatrix}2 & 3 & 1 & -1\\ 6 & 3 & 3 & 0\\ 17 & 20 & 5 & 0\\ -1 & -13 & 0 & 0\end{vmatrix} R_2+R_1,\ R_3+6R_1,\ R_4-2R_1$$

$$=-(-1)\begin{vmatrix}6 & 3 & 3\\ 17 & 20 & 5\\ -1 & -13 & 0\end{vmatrix}+0-0+0 \text{ Expand from } C_4$$

$$=\begin{vmatrix}6 & 3 & 3\\ 17 & 20 & 5\\ -1 & -13 & 0\end{vmatrix}=3\begin{vmatrix}17 & 20\\ -1 & -13\end{vmatrix}-5\begin{vmatrix}6 & 3\\ -1 & -13\end{vmatrix}+0 \quad \text{Expand by } C_3$$

$$=3[(-221-(-20)]-5[(-78-(-3)]+0$$

$$=3(-201)-5(-75)$$

$$=-603+375=-228$$

iii. $\begin{vmatrix}-3 & 9 & 1 & 1\\ 0 & 3 & -1 & 2\\ 9 & 7 & -1 & 1\\ -2 & 0 & 1 & -1\end{vmatrix}$

Sol.

$$=\begin{vmatrix}-3 & 9 & 1 & 1\\ -3 & 12 & 0 & 3\\ 6 & 16 & 0 & 2\\ 1 & -9 & 0 & -2\end{vmatrix} \quad R_2+R_1,\ R_3+R_1,\ R_4-R_1$$

$$=1\begin{vmatrix}-3 & 12 & 3\\ 6 & 16 & 2\\ 1 & -9 & -2\end{vmatrix}-0+0-0 \qquad \text{Expand from } C_3$$

$$=-3\begin{vmatrix}16 & 2\\ -9 & -2\end{vmatrix}-12\begin{vmatrix}6 & 2\\ 1 & -2\end{vmatrix}+3\begin{vmatrix}6 & 16\\ 1 & -9\end{vmatrix}$$

$$=-3(-32+18)-12(-12-2)+3(-54-16)$$

$$=-3(-14)-12(-14)+3(-70)$$

$$=42+168-210=0$$

8. **Show that** $\begin{vmatrix}x & 1 & 1 & 1\\ 1 & x & 1 & 1\\ 1 & 1 & x & 1\\ 1 & x & 1 & x\end{vmatrix}=(x+3)(x-1)^3$ **Sgd 2008, Fsd 2007, Lahore 2009**

Sol L.H.S= $\begin{vmatrix}x & 1 & 1 & 1\\ 1 & x & 1 & 1\\ 1 & 1 & x & 1\\ 1 & x & 1 & x\end{vmatrix}$

$$=\begin{vmatrix}x+3 & 1 & 1 & 1\\ x+3 & x & 1 & 1\\ x+3 & 1 & x & 1\\ x+3 & 1 & 1 & x\end{vmatrix} \quad Add\ C_2, C_3, C_4\ in\ C_1$$

$$=(x+3)\begin{vmatrix}1 & 1 & 1 & 1\\ 1 & x & 1 & 1\\ 1 & 1 & x & 1\\ 1 & 1 & 1 & x\end{vmatrix} \qquad \text{Take } (x+3) \text{ Common from } C_1$$

$$R_2-R_1, R_3-R_1, R_4-R_1$$

$$= (x+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x-1 \end{vmatrix}$$

$$= (x+3)\left[1\begin{vmatrix} x-1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{vmatrix} - 0 + 0 - 0\right] \text{ Expand from } C_1$$

$$= (x+3)\begin{vmatrix} x-1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{vmatrix}$$

$$= (x+3)(x-1)^3\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \text{Take } (x-1) \text{ Common from } R_1, R_2, R_3$$

$$= (x+3)(x-1)^3\left[1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0 + 0\right]$$

$$= (x+3)(x-1)^3(1-0)$$

$$= (x+3)(x-1)^3$$

9. Find $|AA'|$ and $|A'A|$ if:

i. $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

Sol $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

$$A' = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$AA' = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 9+4+1 & 6+2-3 \\ 6+2-3 & 4+1+9 \end{bmatrix} = \begin{bmatrix} 14 & 5 \\ 5 & 14 \end{bmatrix}$$

$$|AA'| = \begin{vmatrix} 14 & 5 \\ 5 & 14 \end{vmatrix} = 196 - 25 = 171$$

$$A'A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9+4 & 6+2 & -3+6 \\ 6+2 & 4+1 & -2+3 \\ -3+6 & -2+3 & 1+9 \end{bmatrix} = \begin{bmatrix} 13 & 8 & 3 \\ 8 & 5 & 1 \\ 3 & 1 & 10 \end{bmatrix}$$

$$|A'A| = \begin{vmatrix} 13 & 8 & 3 \\ 8 & 5 & 1 \\ 3 & 1 & 10 \end{vmatrix} = 13\begin{vmatrix} 5 & 1 \\ 1 & 10 \end{vmatrix} - 8\begin{vmatrix} 8 & 1 \\ 3 & 10 \end{vmatrix} + 3\begin{vmatrix} 8 & 5 \\ 3 & 1 \end{vmatrix}$$

$$= 13(50-1) - 8(80-3) + 3(8-15)$$

$$= 637 - 616 - 21 = 0$$

ii. $$A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$$

Sol $$A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \textit{then } A' = \begin{bmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{bmatrix}$$

$$AA' = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9+16 & 6+4 & 3+4 & 6+12 \\ 6+4 & 4+1 & 2+1 & 4+3 \\ 3+4 & 2+1 & 1+1 & 2+3 \\ 6+12 & 4+3 & 2+3 & 4+9 \end{bmatrix} = \begin{bmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 18 & -7 & 5 & 13 \end{bmatrix}$$

$$|AA'| = \begin{vmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 18 & 7 & 5 & 13 \end{vmatrix}$$

$$R_4-(R_3+R_2) = \begin{vmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 1 & -1 & 0 & 1 \end{vmatrix}$$

$$R_1-25R_4, R_2-10R_4, R_3-7R_4 = \begin{vmatrix} 0 & 35 & 7 & -7 \\ 0 & 15 & 3 & -3 \\ 0 & 10 & 2 & -2 \\ 1 & 1 & 0 & 1 \end{vmatrix}$$

$$=0-0+0-1\begin{vmatrix} 35 & 7 & -7 \\ 15 & 3 & -3 \\ 10 & 2 & -2 \end{vmatrix} \quad \text{Expand from } C_1$$

$$=(-1)(-1)\begin{vmatrix} 35 & 7 & 7 \\ 15 & 3 & 3 \\ 10 & 2 & 2 \end{vmatrix} \quad \text{Take } -1 \text{ Common from } C_3$$

$=(-1)(-1)(0)=0$ Because C_2 and C_3 are same.

10. If A is a square matrix of roder 3, then show that $|kA| = k^3|A|$.

Sol. $$Let\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then}$$

$$KA = k\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} Ka_{11} & Ka_{12} & Ka_{13} \\ Ka_{21} & Ka_{22} & Ka_{23} \\ Ka_{31} & Ka_{32} & Ka_{33} \end{bmatrix}$$

$$|KA| = \begin{vmatrix} Ka_{11} & Ka_{12} & Ka_{13} \\ Ka_{21} & Ka_{22} & Ka_{23} \\ Ka_{31} & Ka_{32} & Ka_{33} \end{vmatrix}$$

Take k Common from $R_1, R_2, R_3 = K.K.K\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

$$= K^3|A| = \text{R.H.S}$$

11. Find the value of λ if A and B are singular.

$$A = \begin{bmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{bmatrix}$$

Sol. Given matrix is singular so

$$|A| = \begin{vmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$$4\begin{vmatrix} 3 & 6 \\ 3 & 1 \end{vmatrix} - \lambda\begin{vmatrix} 7 & 6 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 7 & 3 \\ 2 & 3 \end{vmatrix} = 0 \Rightarrow 4(3-18) - \lambda(7-12) + 3(21-6) = 0$$

$$-60 + 5\lambda + 45 = 0$$

$$5\lambda - 15 = 0 \Rightarrow 5\lambda = 15 \qquad \boxed{\lambda = 3}$$

ii. $$B = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{bmatrix}$$

Sol Given matrix is singular so $|B| = 0$

$$\begin{vmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 5 & 2 & 0 \\ 2 & 8 & 5 & 1 \\ 2 & 3 & 0 & 1 \\ \lambda & 2 & -1 & 3 \end{vmatrix} = 0 \qquad \text{'interchanged } C_1 \text{ and } C_2$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -2 & 1 & 1 \\ 2 & -7 & -4 & 1 \\ \lambda & 2-5\lambda & -1-2\lambda & 3 \end{vmatrix} = 0 \quad C_2-5C_1,\ C_3-2C_1$$

$$\begin{vmatrix} -2 & 1 & 1 \\ -7 & -4 & 1 \\ 2-5\lambda & -1-2\lambda & 3 \end{vmatrix} = 0 \quad \text{Expand from } R_1$$

$$\Rightarrow -2\begin{vmatrix} -4 & 1 \\ -1-2\lambda & 3 \end{vmatrix} - 1\begin{vmatrix} -7 & 1 \\ 2-5\lambda & 3 \end{vmatrix} + 1\begin{vmatrix} -7 & -4 \\ 2-5\lambda & -1-2\lambda \end{vmatrix} = 0$$

$$-2(-12+1+2\lambda)-1(-21-2+5\lambda)+1(7+14\lambda+8-20\lambda)=0$$

$$-2(-11+2\lambda)-1(-23+5\lambda)+1(15-6\lambda)=0$$

$$22-4\lambda+23-5\lambda+15-6\lambda=0 \Rightarrow -15\lambda+60=0$$

$$15\lambda=60$$

$$\boxed{\lambda=4}$$

12. Which of the following matrices are singular and which of them are non Singular?

i. $$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{3} \\ \mathbf{3} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{2} & \mathbf{4} \end{bmatrix}$$

Sol. $$Let\ A = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{vmatrix} = 1\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} - 0 + 3\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}$$

$$=1(4+2)+3(6-0)=6+18=24\neq 0$$

Non Singular

ii. $$B = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{-3} & \mathbf{5} \end{bmatrix}$$

Sol. $B=\begin{bmatrix}2&3&-1\\1&1&0\\2&-3&5\end{bmatrix}\Rightarrow|B|=\begin{vmatrix}2&3&-1\\1&1&0\\2&-3&5\end{vmatrix}$

$=2\begin{vmatrix}1&0\\-3&5\end{vmatrix}-3\begin{vmatrix}1&0\\2&5\end{vmatrix}+(-1)\begin{vmatrix}1&1\\+2&-3\end{vmatrix}$

$=2(5-0)-3(5-0)-1(-3-2)=10-15+5=0$

Singular

iii. $\begin{bmatrix}1&1&2&-1\\1&2&-1&-3\\2&3&1&2\\3&-1&3&4\end{bmatrix}$

Sol. $C=\begin{bmatrix}1&1&2&-1\\1&2&-1&-3\\2&3&1&2\\3&-1&3&4\end{bmatrix}\Rightarrow|C|=\begin{vmatrix}1&1&2&-1\\1&2&-1&-3\\2&3&1&2\\3&-1&3&4\end{vmatrix}$

$=\begin{vmatrix}1&1&2&-1\\0&1&-3&-2\\0&1&-3&4\\0&-4&-3&7\end{vmatrix}R_2-R_1,R_3-2R_1,R_4-3R_1$

$=1\begin{vmatrix}1&-3&-2\\1&-3&4\\-4&-3&7\end{vmatrix}-0+0-0$ Expand by C_1

$=1\begin{vmatrix}1&-3&-2\\0&0&6\\0&-15&-1\end{vmatrix}R_2-R_1,\ R_3+4R_1$

$=1\begin{vmatrix}0&6\\-15&-1\end{vmatrix}-0+0$ Expand by C_1

$=0+90=90$

No Singular

13. $A=\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$ **Find inverse and show that $A^{-1}A=I_3$**

$$|A|=\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}=2\begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix}-1\begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix}+0\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}$$

$$=2(5-0)-1(5-0)+0=10-5=5$$

$$A_{11}=(-1)^{1+1}\begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix}=(5-0)=5$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix}=-(5-0)=-5$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}=(-3-2)-5$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix}=-(5-0)=-5$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix} 2 & 0 \\ 2 & 5 \end{vmatrix}=(10-0)=10$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix}=-(-6-2)=8$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}=(0-0)=0$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix}=-(0-0)=0$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}=(2-1)=1$$

$$\text{Cofactor of } A=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}=\begin{bmatrix} 5 & -5 & -5 \\ -5 & 10 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Adj\ \ A=\begin{bmatrix}5 & -5 & 0\\ -5 & 10 & 0\\ -5 & 8 & 1\end{bmatrix}$$

$$A^{-1}=\frac{adj\ A}{|A|}=\frac{1}{5}\begin{bmatrix}5 & -5 & 0\\ -5 & 10 & 0\\ -5 & 8 & 1\end{bmatrix}$$

$$A^{-1}A=\frac{1}{5}\begin{bmatrix}5 & -5 & 0\\ -5 & 10 & 0\\ -5 & 8 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 1 & 1 & 0\\ 2 & -3 & 5\end{bmatrix}$$

$$=\frac{1}{5}\begin{bmatrix}10-5+0 & 5-5+0 & 0-0+0\\ -10+10+0 & -5+10-0 & 0+0+0\\ -10+8+2 & -5+8-3 & 0+0+5\end{bmatrix}=\frac{1}{5}\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I_3$$

14. Verify that $(AB)^{-1}=B^{-1}A^{-1}$ if:

i. $A=\begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix}, B=\begin{bmatrix}-3 & 1\\ 4 & -1\end{bmatrix}$

Sol. $AB=\begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix}\begin{bmatrix}-3 & 1\\ 4 & -1\end{bmatrix}=\begin{bmatrix}-3+8 & 1-2\\ 3+0 & -1-0\end{bmatrix}=\begin{bmatrix}5 & -1\\ 3 & -1\end{bmatrix}$

We know that $(AB)^{-1}=\dfrac{adj(AB)}{|AB|}$

$$|AB|=\begin{vmatrix}5 & -1\\ 3 & -1\end{vmatrix}=-5-(-3)=-5+3=-2\neq 0$$

$$adj\ AB=\begin{bmatrix}-1 & 1\\ -3 & 5\end{bmatrix}$$

$$\text{L.H.S}=(AB)^{-1}=\frac{ad(AB)}{|AB|}=\frac{1}{-2}\begin{bmatrix}-1 & 1\\ -3 & 5\end{bmatrix}=\begin{bmatrix}1/2 & -1/2\\ 3/2 & -5/2\end{bmatrix}$$

Now for B^{-1}

$$|B| = \begin{vmatrix} -3 & 1 \\ 4 & -1 \end{vmatrix} = 3-4 = -1 \neq 0$$

$$adj\ B = \begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix} \Rightarrow B^{-1} = \frac{adj\ B}{|B|} = \frac{\begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix}}{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}$$

For A^{-1}

$$|A| = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 0-(-1)(2) = 2 \neq 0$$

$$adj\ A, = \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}, A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{2}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

Now

$$\text{R.H.S} = B^{-1}A^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0+1 & -2+1 \\ 0+3 & -8+3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 3/2 & -5/2 \end{bmatrix}$$

Hence proved that $(AB)^{-1} = B^{-1}A^{-1}$ or L.H.S = R.H.S

ii. $A = \begin{bmatrix} 5 & 1 \\ 2 & 2 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

Sol. $AB = \begin{bmatrix} 5 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 20+2 & 15+1 \\ 8+4 & 6+2 \end{bmatrix} = \begin{bmatrix} 22 & 16 \\ 12 & 8 \end{bmatrix}$

$$(AB)^{-1} = \frac{adj(AB)}{|AB|}, adj(AB) = \begin{bmatrix} 8 & -16 \\ -12 & 22 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 22 & 16 \\ 12 & 8 \end{vmatrix} = (22)(8)-(16)(12) = 176-192 = -16 \neq 0$$

$$(AB)^{-1} = \frac{adj(AB)}{|AB|} = \frac{1}{-16}\begin{bmatrix} 8 & -16 \\ -12 & 22 \end{bmatrix} = \begin{bmatrix} \frac{8}{-16} & \frac{-16}{-16} \\ \frac{-12}{-16} & \frac{22}{-16} \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} & 1 \\ \frac{3}{4} & \frac{-11}{8} \end{bmatrix}$$

$$|B|=\begin{vmatrix}4 & 3\\ 2 & 1\end{vmatrix}=4-6=-2, \qquad adj\ B=\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix}$$

$$B^{-1}=\frac{adj\ B}{|B|}=\frac{1}{-2}\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix}$$

$$|A|=\begin{vmatrix}5 & 1\\ 2 & 2\end{vmatrix}=10-2=8\neq 0$$

$$adj\ A=\begin{bmatrix}2 & -1\\ -2 & 5\end{bmatrix},\ A^{-1}=\frac{adj\ A}{|A|}=\frac{1}{8}\begin{bmatrix}2 & -1\\ -2 & 5\end{bmatrix}$$

$$\text{R.H.S}==B^{-1}A^{-1}=\frac{1}{-2}\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix}\cdot\frac{1}{8}\begin{bmatrix}2 & -1\\ -2 & 5\end{bmatrix}=\frac{1}{-16}\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix}\begin{bmatrix}2 & -1\\ -2 & 5\end{bmatrix}$$

$$=\frac{1}{-16}\begin{bmatrix}2+6 & -1-15\\ -4-8 & 2+20\end{bmatrix}=\frac{1}{-16}\begin{bmatrix}8 & -16\\ -12 & 22\end{bmatrix}=\begin{bmatrix}\frac{8}{-16} & \frac{-16}{-16}\\ \frac{-12}{-16} & \frac{22}{-16}\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{1}{2} & 1\\ \frac{3}{4} & -\frac{11}{8}\end{bmatrix}$$ Hence proved that $(AB)^{-1}=B^{-1}A^{-1}$

15. **Verify that $(AB)^t=B^tA^t$ if:**

$$A=\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix}, B=\begin{bmatrix}1 & 1\\ 3 & 2\\ 0 & -1\end{bmatrix}$$ **Faisalabad 2009**

Sol. $$AB=\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 3 & 2\\ 0 & -1\end{bmatrix}$$

$$AB=\begin{bmatrix}1-3+0 & 1-2-2\\ 0+9+0 & 0+6-1\end{bmatrix}=\begin{bmatrix}-2 & -3\\ 9 & 5\end{bmatrix}$$

$$\text{L.H.S}=(AB)^t=\begin{bmatrix}-2 & 9\\ -3 & 5\end{bmatrix}$$

$$\text{Now } B' = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}, \qquad A' = \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$\text{R.H.S} = B'A' = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3+0 & 0+9+0 \\ 1-2-2 & 0+6-1 \end{bmatrix} = \begin{bmatrix} -2 & 9 \\ -3 & 5 \end{bmatrix}$$

Hence Prove that $(AB)' = B'A'$

L.H.S = R.H.S

16. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ verify that $(A^{-1})' = (A')^{-1}$

Sol. $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2(1) - (-1)(3) = 2 + 3 = 5$$

and Adj $A = \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}$

$$\text{Thus } A^{-1} = \frac{1}{|A|} Adj\, A = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} \Rightarrow (A^{-1})' = \frac{1}{5}\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{-3}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \longrightarrow (i)$$

$$\text{Now } A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$$

$$(A')^{-1} = \frac{1}{|A'|} Adj\,(A')$$

$$\text{So } |A'| = \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 2(1) - 3(-1) = 2 + 3 = 5$$

$$\text{Adj}(A')=\begin{bmatrix}1 & -3\\ 1 & 2\end{bmatrix}$$

$$\text{Thus } (A')^{-1}=\frac{1}{|A'|}Adj.\ A'=\frac{1}{5}\begin{bmatrix}1 & -3\\ 1 & 2\end{bmatrix}=\begin{bmatrix}\frac{1}{5} & \frac{-3}{5}\\ \frac{1}{5} & \frac{2}{5}\end{bmatrix} \longrightarrow (ii)$$

From Equations (i) and ii $\Rightarrow$ L.H.S = R.H.S

17. **If A and B are non-singular matrices, then show that**

i. $(AB)^{-1}=(B^{-1}A^{-1})$ **Federal, Multan 2008**

Sol We know that

$(AB).(AB)^{-1}=I$

Pre-multiplying by A^{-1}

$A^{-1}.AB(AB)^{-1}=A^{-1}I$

$IB(AB)^{-1}=A^{-1}\Rightarrow B(AB)^{-1}=A^{-1}$

Pre-multiplying by B^{-1}

$B^{-1}.B(AB)^{-1}=B^{-1}A^{-1}\Rightarrow I(AB)^{-1}=B^{-1}A^{-1}\Rightarrow(AB)^{-1}=B^{-1}A^{-1}$

ii. $(A^{-1})^{-1}=A$

Sol We know that $A^{-1}(A^{-1})^{-1}=I$

Pre-multiplying by A

$AA^{-1}(A^{-1})^{-1}=A.I$

$I.(A^{-1})^{-1}=A$

$\Rightarrow (A^{-1})^{-1}=A$

Hence Proved.

EXERCISE: 3.4

1. If $A=\begin{bmatrix}1 & -2 & 5\\ -2 & 3 & -1\\ 5 & -1 & 0\end{bmatrix}$ *and* $B=\begin{bmatrix}-3 & 1 & -2\\ 1 & 0 & -1\\ -2 & -1 & 2\end{bmatrix}$, then show that A + B is symmetric: Multan 2008

Sol. $A+B=\begin{bmatrix}1 & -2 & 5\\ -2 & 3 & -1\\ 5 & -1 & 0\end{bmatrix}+\begin{bmatrix}-3 & 1 & -2\\ 1 & 0 & -1\\ -2 & -1 & 2\end{bmatrix}$

$$=\begin{bmatrix}1-3 & -2+1 & 5-2\\ -2+1 & 3+0 & -1-1\\ 5-2 & -1-1 & 0+2\end{bmatrix}=\begin{bmatrix}-2 & -1 & 3\\ -1 & 3 & -2\\ 3 & -2 & 2\end{bmatrix}$$

Now $(A+B)'=\begin{bmatrix}-2 & -1 & 3\\ -1 & 3 & -2\\ 3 & -2 & 2\end{bmatrix}'$

$$=\begin{bmatrix}-2 & -1 & 3\\ -1 & 3 & -2\\ 3 & -2 & 2\end{bmatrix}=(A+B)$$

Hence $(A+B)$ is symmetric

2. If $A=\begin{bmatrix}1 & 2 & 0\\ 3 & 2 & -1\\ -1 & 3 & 2\end{bmatrix}$, then show that: Multan 2009, Faisalabad 2008

i. $A+A'$ is symmetric

Sol. then $A'=\begin{bmatrix}1 & 3 & -1\\ 2 & 2 & 3\\ 0 & -1 & 2\end{bmatrix}$

$$A+A'=\begin{bmatrix}1 & 2 & 0\\ 3 & 2 & -1\\ -1 & 3 & 2\end{bmatrix}+\begin{bmatrix}1 & 3 & -1\\ 2 & 2 & 3\\ 0 & -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix} 1+1 & 2+3 & 0-1 \\ 3+2 & 2+2 & -1+3 \\ -1+0 & 3-1 & 2+2 \end{bmatrix}=\begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

$$(A+A')'=\begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}'$$

$$(A+A')'=\begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}=(A+A')$$

Hence $(A+A')'$ is symmetric

ii. **$A-A'$ is skew symmetric** **Sargodha 2010**

Sol. then $A-A'=\begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}-\begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$

$$=\begin{bmatrix} 1-1 & 2-3 & 0+1 \\ 3-2 & 2-2 & -1-3 \\ -1-0 & 3+1 & 2-2 \end{bmatrix}=\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}$$

$$(A-A')'=\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}'=\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 4 \\ 1 & -4 & 0 \end{bmatrix}=-\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}$$

$(A+A')'=-(A+A')$

Hence skew symmetric.

3. **If A is any square matrix of order 3, show that:**

i. **$A+A'$ is symmetric**

Sol. $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$A+A'=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}+\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}'$$

$$=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}+\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}=\begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} & a_{13}+a_{31} \\ a_{21}+a_{12} & a_{22}+a_{22} & a_{23}+a_{32} \\ a_{31}+a_{13} & a_{32}+a_{23} & a_{33}+a_{33} \end{bmatrix}$$

$$(A+A')'=\begin{bmatrix} a_{11}+a_{11} & a_{21}+a_{12} & a_{31}+a_{13} \\ a_{21}+a_{12} & a_{22}+a_{22} & a_{32}+a_{23} \\ a_{13}+a_{31} & a_{23}+a_{32} & a_{33}+a_{23} \end{bmatrix}, A+A'$$

$(A+A')'=(A+A')$ Hence Symmetric.

ii. **$A-A^t$ is skew symmetric** **Rawalpindi 2009**

Sol. $$A-A'=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}-\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{12} & a_{23} & a_{33} \end{bmatrix}$$

$$=\begin{bmatrix} a_{11}-a_{11} & a_{12}-a_{21} & a_{13}-a_{31} \\ a_{21}-a_{12} & a_{22}-a_{22} & a_{23}-a_{32} \\ a_{31}-a_{13} & a_{32}-a_{23} & a_{33}-a_{33} \end{bmatrix}$$

$$A-A'=\begin{bmatrix} 0 & a_{12}-a_{21} & a_{13}-a_{31} \\ a_{21}-a_{22} & 0 & a_{23}-a_{32} \\ a_{31}-a_{13} & a_{32}-a_{23} & 0 \end{bmatrix}=-\begin{bmatrix} 0 & a_{21}-a_{12} & a_{31}-a_{13} \\ a_{22}-a_{21} & 0 & a_{32}-a_{23} \\ a_{13}-a_{31} & a_{23}-a_{32} & 0 \end{bmatrix}$$

$$=-(A-A')\Rightarrow(A-A')'=-(A-A')$$

Hence $A-A'$ is skew symmetric

4. **If the matrices A and B are symmetric and AB = BA, show that AB is symmetric.**

Sol. Now $(AB)'=B'A'$ $\quad$ *Given* $A'=A, B'=B, AB=BA$

$=BA$ $\quad$ by using $B'=B,\ A'=A$ (given A,B are symmetric)

$=AB$ $\quad$ by using $AB=BA$ (given)

$\Rightarrow(AB)' \quad =AB$

Hence AB is symmetric

5. **Show that AA^t and A^tA are symmetric for any matrix of order 2 x 3.**

i. AA'

Sol. **Let** $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, A'=\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$

Then

$$AA' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

$$AA' = \begin{bmatrix} a^2_{11} + a^2_{12} + a^2_{13} & a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} \\ a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} & a^2_{21} + a^2_{22} + a^2_{23} \end{bmatrix}$$

$$(AA')' = \begin{bmatrix} a^2_{11} + a^2_{12} + a^2_{13} & a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} \\ a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} & a^2_{21} + a^2_{22} + a^2_{23} \end{bmatrix}$$

$(AA')' = AA'$ $\quad AA'$ is symmetric.

ii. $A'A$

Sol. $$A'A = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a^2_{11} + a^2_{12} & a_{11}a_{12} + a_{21}a_{22} & a_{11}a_{13} + a_{21}a_{23} \\ a_{12}a_{11} + a_{22}a_{21} & a^2_{12} + a^2_{22} & a_{12}a_{13} + a_{22}a_{23} \\ a_{13}a_{11} + a_{23}a_{21} & a_{13}a_{12} + a_{23}a_{22} & a^2_{13} + a^2_{23} \end{bmatrix}$$

$$(A'A)' = \begin{bmatrix} a^2_{11} + a^2_{12} & a_{11}a_{12} + a_{21}a_{22} & a_{11}a_{13} + a_{21}a_{23} \\ a_{12}a_{11} + a_{22}a_{21} & a^2_{12} + a^2_{22} & a_{12}a_{13} + a_{22}a_{23} \\ a_{13}a_{11} + a_{23}a_{21} & a_{13}a_{12} + a_{23}a_{22} & a^2_{13} + a^2_{23} \end{bmatrix}$$

$\Rightarrow (A'A)' = A'A$ Hence $A'A$ is symmetric.

6. **If** $A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$**, Show that**

i. $A + (\bar{A})'$ **is hermitian** **(Federal)**

Sol. **Let** $A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} -i & 1-i \\ 1 & i \end{bmatrix}$$

$$(\bar{A})' = \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$$

$$A+(\bar{A})' = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix} + \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$$

$$= \begin{bmatrix} i-i & 1+i+1 \\ 1+1-i & -i+i \end{bmatrix} = \begin{bmatrix} 0 & 2+i \\ 2-i & 0 \end{bmatrix}$$

$$\overline{A+(\bar{A})'} = \begin{bmatrix} 0 & 2-i \\ 2+i & 0 \end{bmatrix}$$

$$\left(\overline{A+(\bar{A})'}\right)' = \begin{bmatrix} 0 & 2+i \\ 2-i & 0 \end{bmatrix}$$

$$\left(\overline{A+(\bar{A})'}\right)' = A+(\bar{A})'$$

So $A+(\bar{A})'$ is Hermitian

ii. **$A-(\bar{A})'$ is skew-hermitain** **Multan 2007**

Let $A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} -i & 1-i \\ 1 & i \end{bmatrix} \Rightarrow (\bar{A})' = \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$

Sol. **Now** $A-(\bar{A})' = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix} - \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix} = \begin{bmatrix} i+i & 1+i-1 \\ 1-1+i & -i-i \end{bmatrix}$

$$= \begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix} \Rightarrow \overline{(A-(\bar{A})'} = \begin{bmatrix} -2i & -i \\ -i & 2i \end{bmatrix} \Rightarrow \left(\overline{(A-(\bar{A})'}\right)' = \begin{bmatrix} -2i & -i \\ -i & 2i \end{bmatrix} = -\begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$$

$\left(\overline{(A-(\bar{A})'}\right)' = -\left(A-(\bar{A})'\right)$ Hence $A-(\bar{A})'$ is skew hermitaian.

7. **If A is symmetric or skew-symmetric, show that A^2 is symmetric.** **Lahore 2009**

Sol. Given $A' = A \longrightarrow (I)$ *or* $A' = -A \longrightarrow (II)$

$= (A^2)' = (A.A)'$ Now Also $(A^2)' = (A.A)'$

$= A'.A'$ or $= A'.A'$

$= A.A$ *use I* or $= (-A)(-A)(use\ II) = A^2$

$= A^2$ or So A^2 is symmetric

$= A^2$ is symmetric Hence In both cases A^2 is symmetric

8. If $A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$, **find** $A(\bar{A})'$

Sol. $A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} 1 \\ 1-i \\ -i \end{bmatrix}$,

$(\bar{A})' = [1 \quad 1-i \quad -i]$

$$A(\bar{A})' = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix} [1 \quad 1-i \quad -i]$$

$$A(\bar{A})' = \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 1-i^2 & -i-i^2 \\ i & i-i^2 & -i^2 \end{bmatrix} = \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 1-(-1) & -i-(-1) \\ i & i-(-1) & -(-1) \end{bmatrix}$$

$$A(\bar{A})' = \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 2 & 1-i \\ i & 1+i & 1 \end{bmatrix}$$

9. Find the inverse of the following matrices. Also find their inverse by using row and column operations.

i. $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{0} & \mathbf{-2} & \mathbf{0} \\ \mathbf{-2} & \mathbf{-2} & \mathbf{2} \end{bmatrix}$

Sol. $$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{vmatrix} = 1\begin{vmatrix} -2 & 0 \\ -2 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ -2 & 2 \end{vmatrix} + (-3)\begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix}$$

$$= 1(-4-0)-(0-0)-3(0-4) = -4+12 = 8 \neq 0$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -2 & 0 \\ -2 & 2 \end{vmatrix} = (-4-0) = -4$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 0 \\ -2 & 2 \end{vmatrix} = (0-0) = 0$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix} = (-0-4) = -4$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -3 \\ -2 & 2 \end{vmatrix} = -(4-6) = 2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -3 \\ -2 & 2 \end{vmatrix} = (2-6) = -4$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ -2 & -2 \end{vmatrix} = -(-2+4) = -2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -3 \\ -2 & 0 \end{vmatrix} = (0-6) = -6$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix} = -(0-0) = 0$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = (-2-0) = -2$$

Co-factor of $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} -4 & 0 & -4 \\ 2 & -4 & -2 \\ -6 & 0 & -2 \end{bmatrix}$

Adj A = (co-factor of A)^t = $\begin{bmatrix} -4 & 2 & -6 \\ 0 & -4 & 0 \\ -4 & -2 & -2 \end{bmatrix}$

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{8}\begin{bmatrix} -4 & 2 & -6 \\ 0 & -4 & 0 \\ -4 & -2 & -2 \end{bmatrix} = \begin{bmatrix} -4/8 & 2/8 & -6/8 \\ 0/8 & -4/8 & 0/8 \\ -4/8 & -2/8 & -2/8 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/4 & -3/4 \\ 0 & -1/2 & 0 \\ -1/2 & -1/4 & -1/4 \end{bmatrix}$$

A^{-1} By Row operation

$$\begin{bmatrix} 1 & 2 & -3 & : & 1 & 0 & 0 \\ 0 & -2 & 0 & : & 0 & 1 & 0 \\ -2 & -2 & 2 & : & 0 & 0 & 1 \end{bmatrix}$$

Add $2R_1$ in R_3

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -3 & : & 1 & 0 & 0 \\ 0 & -2 & 0 & : & 0 & 1 & 0 \\ 0 & 2 & -4 & : & 2 & 0 & 1 \end{bmatrix}$$

Add R_2 in R_1 & R_3

$$\underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 0 & -3 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 2 & 1 & 1 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 0 & -3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & -4 & 2 & 1 & 1 \end{array}\right]\frac{-1}{2}R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 0 & -3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{-1}{4} & \frac{-1}{4} \end{array}\right]\frac{-1}{4}R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -1/2 & 1/4 & -3/4 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1/2 & -1/4 & -1/4 \end{array}\right]3R_3+R_1$$

Hence $A^{-1}=\begin{bmatrix} \frac{-1}{2} & \frac{1}{4} & \frac{-3}{4} \\ 0 & \frac{-1}{2} & 0 \\ \frac{-1}{2} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix}$

A^{-1} Method of Column Operation.

$$A=\left[\begin{array}{ccc} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \\ \hdashline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \Rightarrow \underset{\sim}{C}\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -2 & 0 \\ -2 & 2 & -4 \\ \hdashline 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] C_2-2C_1 \ \& \ C_3+3C_1$$

$$\frac{-1}{2}C_2 \qquad\qquad \frac{-1}{4}C_3$$

$$\underset{\sim}{C}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-2 & -1 & -4\\ \cdots & \cdots & \cdots\\1 & 1 & 3\\0 & -1/2 & 0\\0 & 0 & 1\end{bmatrix}\Rightarrow\underset{\sim}{C}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-2 & -1 & 1\\ \cdots & \cdots & \cdots\\1 & 1 & -3/4\\0 & -1/2 & 0\\0 & 0 & -1/4\end{bmatrix}$$

$$\underset{\sim}{C}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\ \cdots & \cdots & \cdots\\-1/2 & 1/4 & -3/4\\0 & -1/2 & 0\\-1/2 & -1/4 & -1/4\end{bmatrix}\quad C_2+C_3\ \&\ C_1+2C_3$$

Hence $A^{-1}=\begin{bmatrix}-1/2 & 1/4 & -3/4\\0 & -1/2 & 0\\-1/2 & -1/4 & -1/4\end{bmatrix}$

ii. $A=\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{-1}\\\mathbf{0} & \mathbf{-1} & \mathbf{3}\\\mathbf{1} & \mathbf{0} & \mathbf{2}\end{bmatrix}$ **Multan 2009**

Sol. *Let* $A=\begin{bmatrix}1 & 2 & -1\\0 & -1 & 3\\1 & 0 & 2\end{bmatrix}$ then $|A|=\begin{vmatrix}1 & 2 & -1\\0 & -1 & 3\\1 & 0 & 2\end{vmatrix}$

$$|A|=1\begin{vmatrix}-1 & 3\\0 & 2\end{vmatrix}-2\begin{vmatrix}0 & 3\\1 & 2\end{vmatrix}+(-1)\begin{vmatrix}0 & -1\\1 & 0\end{vmatrix}$$

$$=1(-2-0)-2(0-3)-1(0+1)=-2+6-1=3\neq 0$$

$$adj\ A=\begin{bmatrix}A_{11} & A_{21} & A_{31}\\A_{12} & A_{22} & A_{32}\\A_{13} & A_{23} & A_{33}\end{bmatrix}$$

Now $A_{11}=(-1)^{1+1}\begin{vmatrix}-1 & 3\\ 0 & 2\end{vmatrix}=(-2-0)=-2$

$A_{12}=(-1)^{1+2}\begin{vmatrix}0 & 3\\ 1 & 2\end{vmatrix}=-(0-3)=3$

$A_{13}=(-1)^{1+3}\begin{vmatrix}0 & -1\\ 1 & 0\end{vmatrix}=(0+1)=1$

$A_{21}=(-1)^{2+1}\begin{vmatrix}2 & -1\\ 0 & 2\end{vmatrix}=-(4-0)=-4$

$A_{22}=(-1)^{2+2}\begin{vmatrix}1 & -1\\ 1 & 2\end{vmatrix}=(2+1)=3$

$A_{23}=(-1)^{2+3}\begin{vmatrix}1 & 2\\ 1 & 0\end{vmatrix}=-(0-2)=2$

$A_{31}=(-1)^{3+1}\begin{vmatrix}2 & -1\\ -1 & 3\end{vmatrix}=(6-1)=5$

$A_{32}=(-1)^{3+2}\begin{vmatrix}1 & -1\\ 0 & 3\end{vmatrix}=(3+0)=-3$

$A_{33}=(-1)^{3+3}\begin{vmatrix}1 & 2\\ 0 & -1\end{vmatrix}=(-1-0)=-1$

Adj $A=\begin{bmatrix}-2 & -4 & 5\\ 3 & 3 & -3\\ 1 & 2 & -1\end{bmatrix}$ then $A^{-1}=\dfrac{adjA}{|A|}=\dfrac{1}{3}\begin{bmatrix}-2 & -4 & 5\\ 3 & 3 & -3\\ 1 & 2 & -1\end{bmatrix}$

$$\Rightarrow \quad A^{-1}=\begin{bmatrix}-2/3 & -4/3 & 5/3\\ 1 & 1 & -1\\ 1/3 & 2/3 & -1/3\end{bmatrix}$$

For Row Operation

$$\left[\begin{array}{ccc:ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & -1 & 3 & 0 & 1 & 0\\ 1 & 0 & 2 & 0 & 0 & 1\end{array}\right]\Rightarrow \underset{\sim}{R}\left[\begin{array}{ccc:ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & -1 & 3 & 0 & 1 & 0\\ 0 & -2 & 3 & -1 & 0 & 1\end{array}\right] by\ R_3-R_1$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & -1 & : & 1 & 0 & 0\\ 0 & 1 & -3 & : & 0 & -1 & 0\\ 0 & -2 & 3 & : & -1 & 0 & 1\end{bmatrix} by\ (-1)\ R_2 \Rightarrow \underset{\sim}{R}\overset{R_1-2R_2,\, R_3+2R_2}{\begin{bmatrix}1 & 0 & 5 & : & 1 & 2 & 0\\ 0 & 1 & -3 & : & 0 & -1 & 0\\ 0 & 0 & -3 & : & -1 & -2 & 1\end{bmatrix}}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 5 & : & 1 & 2 & 0\\ 0 & 1 & -3 & : & 0 & -1 & 0\\ 0 & 0 & 1 & : & 1/3 & 2/3 & -1/3\end{bmatrix} by\ (-1/3)R_3$$

$$\Rightarrow \underset{\sim}{R}\begin{bmatrix}1 & 0 & 0 & : & -2/3 & -4/3 & 5/3\\ 0 & 1 & 0 & : & 1 & 1 & -1\\ 0 & 0 & 1 & : & 1/3 & 2/3 & -1/3\end{bmatrix} by\ R_1-5R_3,\ \ R_2+3R_3$$

Hence $A^{-1}=\begin{bmatrix}-2/3 & -4/3 & 5/3\\ 1 & 1 & 1\\ 1/3 & 2/3 & -1/3\end{bmatrix}$

For Column Operation $\begin{bmatrix}1 & 2 & -1\\ 0 & -1 & 3\\ 1 & 0 & 2\\ \hline 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \Rightarrow \underset{\sim}{C}\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 3\\ 1 & -2 & 3\\ \hline 1 & -2 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} by\ C_2-2C_1,\ C_3+C_2$

$$\Rightarrow \underset{\sim}{C}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 3\\ 1 & 2 & 3\\ \hline 1 & 2 & 1\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix} by(-1)\ C_2$$

$$C\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & -3 \\ \cdots & \cdots & \cdots \\ 1 & 2 & -5 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix} by\ C_3-3C_2 \Rightarrow C\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \\ \cdots & \cdots & \cdots \\ 1 & 2 & 5/3 \\ 0 & -1 & -1 \\ 0 & 0 & -1/3 \end{bmatrix} by\ (-\frac{1}{3})C_3$$

$$\Rightarrow C\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \cdots & \cdots & \cdots \\ -2/3 & -4/3 & 5/3 \\ 1 & 1 & -1 \\ 1/3 & 2/3 & -1/3 \end{bmatrix} by\ C_1-C_3, C_2-2C_3$$

Hence $A^{-1}=\begin{bmatrix} -2/3 & -4/3 & 5/3 \\ 1 & 1 & -1 \\ 1/3 & 2/3 & -1/3 \end{bmatrix}$

iii. $A=\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ **Sargodha 2007**

Sol. $|A|=1\begin{vmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix}=1\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}-(-3)\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}+2\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix}$

$$=1(1-0)+3(2-0)+2(-2-0)=1+6-4=3\neq 0$$

$$A_{11}=(-1)^{1+1}\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}=(1-0)=1$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}=-(2-0)=-2$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}2 & 1\\ 0 & -1\end{vmatrix}=(-2-0)=-2$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}-3 & 2\\ -1 & 1\end{vmatrix}=-(-3+2)=1$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix}=(1-0)=1$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}1 & -3\\ 0 & -1\end{vmatrix}=-(-1+0)=1$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}-3 & 2\\ 1 & 0\end{vmatrix}=(0-2)=-2$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1 & 2\\ 2 & 0\end{vmatrix}=-(0-4)=4$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1 & -3\\ 2 & 1\end{vmatrix}=(1+6)=7$$

Co-factor of $A=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}=\begin{bmatrix}1 & -2 & -2\\ 1 & 1 & 1\\ -2 & 4 & 7\end{bmatrix}$

adj A = (Co-factor of A)^t $=\begin{bmatrix}1 & 1 & -2\\ -2 & 1 & 4\\ -2 & 1 & 7\end{bmatrix}$

$$A^{-1}=\frac{adjA}{|A|}=\frac{1}{3}\begin{bmatrix}1 & 1 & -2\\ -2 & 1 & 4\\ -2 & 1 & 7\end{bmatrix}=\begin{bmatrix}1/3 & 1/3 & -2/3\\ -2/3 & 1/3 & 4/3\\ -2/3 & 1/3 & 7/3\end{bmatrix}$$

Method of Row Operation

$$\left[\begin{array}{ccc:ccc}1 & -3 & 2 & 1 & 0 & 0\\ 2 & 1 & 0 & 0 & 1 & 0\\ 0 & -1 & 1 & 0 & 0 & 1\end{array}\right]\underset{\sim}{R}\left[\begin{array}{ccc:ccc}1 & -3 & 2 & 1 & 0 & 0\\ 0 & 7 & -4 & -2 & 1 & 0\\ 0 & -1 & 1 & 0 & 0 & 1\end{array}\right]R_2-2R_1$$

R_2+6R_3 $\quad$ R_1+3R_2 & R_3+R_2

$$\underset{\sim}{R}\begin{bmatrix} 1 & -3 & 2 & : & 1 & 0 & 0 \\ 0 & 1 & 2 & : & -2 & 1 & 6 \\ 0 & -1 & 1 & : & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 8 & : & -5 & 3 & 18 \\ 0 & 1 & 2 & : & -2 & 1 & 6 \\ 0 & 0 & 3 & : & -2 & 1 & 7 \end{bmatrix}$$

$R_3/3$ $\qquad\qquad\qquad\qquad$ R_2+2R_3, R_1-8R_3

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 8 & : & -5 & 3 & 18 \\ 0 & 1 & 2 & : & -2 & 1 & 6 \\ 0 & 0 & 1 & : & -2/3 & 1/3 & 7/3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & : & 1/3 & 1/3 & -2/3 \\ 0 & 1 & 0 & : & -2/3 & 1/3 & 4/3 \\ 0 & 0 & 1 & : & -2/3 & 1/3 & 7/3 \end{bmatrix}$$

Hence $A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$

Method of Column Operation

$$\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \\ \cdots & \cdots & \cdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 7 & -4 \\ 0 & -1 & 1 \\ \cdots & \cdots & \cdots \\ 1 & 3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} C_2+3C_1, \quad C_3-2C_1$$

$\frac{1}{7}C_2$ $\qquad\qquad\qquad\qquad$ C_1-2C_2, C_3+4C_2

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -4 \\ 0 & -1/7 & 1 \\ \cdots & \cdots & \cdots \\ 1 & 3/7 & -2 \\ 0 & 1/7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2/7 & -1/7 & 3/7 \\ \cdots & \cdots & \cdots \\ 1/7 & 3/7 & -2/7 \\ -2/7 & 1/7 & 4/7 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{3}{7}C_3 \qquad\qquad C_2+\frac{1}{7}C_3, \& C_1-\frac{2}{7}C_3$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2/7 & -1/7 & 1 \\ \cdots & \cdots & \cdots \\ 1/7 & 3/7 & -2/3 \\ -2/7 & 1/7 & 4/3 \\ 0 & 0 & 7/3 \end{bmatrix} \Rightarrow \underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \cdots & \cdots & \cdots \\ 1/3 & 1/3 & -2/3 \\ -2/3 & 1/3 & 4/3 \\ -2/3 & 1/3 & 7/3 \end{bmatrix}$$

Hence $A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$

10. Find the rank of the following matrices:

i. $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -6 & 5 & 1 \\ 3 & 5 & 4 & -3 \end{bmatrix}$ **(Federal)**

Sol. $\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & 8 & -2 & -6 \end{bmatrix} \begin{matrix} by\ R_2-2R_1 \\ R_3-3R_1 \end{matrix} \Rightarrow \underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 8 & -2 & -6 \end{bmatrix} by\ (-1/4)$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 7/4 & 5/4 \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 0 & 0 & -8 \end{bmatrix} \begin{matrix} by\ R_1+R_2 \\ R_3-8R_2 \end{matrix} \Rightarrow \underset{\sim}{R}\begin{bmatrix} 1 & 0 & 7/4 & 5/4 \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix} by\ \frac{-1}{8}R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 7/4 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} by\ R_1-5/4R_3,\ R_2-\frac{1}{4}R_3$$

$\boxed{\textbf{Hence Rank } = 3}$

ii. $\begin{bmatrix} 1 & -4 & -7 \\ 2 & -5 & 1 \\ 1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix}$ **Faisalabad 2007**

Sol. $$\begin{bmatrix} 1 & -4 & -7 \\ 2 & -5 & 1 \\ 1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix} \Rightarrow \underset{\sim}{R} \begin{bmatrix} 1 & -4 & -7 \\ 0 & 3 & 15 \\ 0 & 2 & 10 \\ 0 & 5 & 25 \end{bmatrix} \begin{matrix} by \; R_2-2R_1 \\ R_3-R_1 \\ R_4-3R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -4 & -7 \\ 0 & 1 & 5 \\ 0 & 2 & 10 \\ 0 & 5 & 25 \end{bmatrix} by \; \frac{1}{3}R_2 \Rightarrow \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} by \; R_1+4R_2 \\ R_3-2R_2 \\ R_4-5R_2 \end{matrix}$$

$\boxed{\textbf{Rank} = \textbf{2}}$

iii. $\begin{bmatrix} 3 & -1 & 3 & 0 & -1 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -3 & -2 \end{bmatrix}$

Sol. $$\begin{bmatrix} 3 & -1 & 3 & 0 & -1 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix} \Rightarrow \underset{\sim}{R} \begin{bmatrix} 1 & -4 & -1 & -2 & -6 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix} by \; R_1-R_3$$

$$\Rightarrow \underset{\sim}{R} \begin{bmatrix} 1 & -4 & -1 & -2 & -6 \\ 0 & 6 & 0 & -1 & 4 \\ 0 & 11 & 6 & 6 & 17 \\ 0 & 13 & 0 & 1 & 15 \end{bmatrix} \begin{matrix} by \; R_2-R_1 \\ R_3-2R_1 \\ R_4-2R_1 \end{matrix}$$

$$\Rightarrow \underset{\sim}{R} \begin{bmatrix} 1 & -4 & -1 & -2 & -6 \\ 0 & 1 & 0 & -1/6 & 2/3 \\ 0 & 11 & 6 & 6 & 17 \\ 0 & 13 & 0 & 1 & 15 \end{bmatrix} by \; R_2/6$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 & -8/3 & -10/3 \\ 0 & 1 & 0 & -1/6 & 2/3 \\ 0 & 0 & 6 & 47/6 & 29/3 \\ 0 & 0 & 0 & 19/6 & 19/3 \end{bmatrix} \begin{matrix} by\ R_1+4R_2 \\ R_3-11R_2 \\ R_4-13R_2 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 & -8/3 & -10/3 \\ 0 & 1 & 0 & 1/6 & 2/3 \\ 0 & 0 & 1 & 47/36 & 29/8 \\ 0 & 0 & 0 & 19/6 & 19/3 \end{bmatrix} by\ \frac{1}{6}R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & -49/36 & -31/18 \\ 0 & 1 & 0 & -1/6 & 2/3 \\ 0 & 0 & 1 & 47/6 & 29/3 \\ 0 & 0 & 0 & 19/6 & 19/3 \end{bmatrix} by\ R_1+R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & -49/36 & -31/18 \\ 0 & 1 & 0 & -1/6 & 2/3 \\ 0 & 0 & 1 & 47/6 & 29/8 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} by\ \frac{6}{19}\times R_4$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} R_1+\frac{49}{36}R_4,\ R_2+\frac{1}{6}R_4,\ R_3-\frac{47}{36}$$

$\boxed{\textbf{Rank} = 4}$

EXERCISE: 3.5

1. **Solve the following systems of linear equations by Cramer's rule.**

i. $$\left.\begin{aligned} 2x+2y+z &= 3 \\ 3x-2y-2z &= 1 \\ 5x+y-3z &= 2 \end{aligned}\right\}$$ Sargodha 2009,2010, Multan 2009, Lahore 2009

Sol. $$A=\begin{bmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

$$\Rightarrow |A|=\begin{vmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{vmatrix}=2\begin{vmatrix} -2 & -2 \\ 1 & -3 \end{vmatrix}-2\begin{vmatrix} 3 & -2 \\ 5 & -3 \end{vmatrix}+1\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix}$$

$$=2(6+2)-2(-9+10)+1(3+10)$$

$$=16-2+13=27\neq 0$$

$$x=\frac{\begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & -3 \end{vmatrix}}{27}=\frac{3(6+2)-2(-3+4)+1(1+4)}{27}$$

$$x=\frac{24-2+5}{27}=\frac{27}{27}=1\Rightarrow x=1$$

$$y=\frac{\begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}}{27}=\frac{2(-3+4)-3(-9+10)+1(6-5)}{27}$$

$$y=\frac{2-3+1}{27}=\frac{0}{27}=0\Rightarrow y=0$$

$$Z=\frac{\begin{vmatrix} 2 & 2 & 3 \\ 3 & -2 & 1 \\ 5 & 1 & 2 \end{vmatrix}}{27}=\frac{2(-4-1)-2(6-5)+3(3+10)}{27}$$

$$Z=\frac{-10-2+39}{27}=\frac{27}{27}=1\Rightarrow Z=1$$

ii. $\left.\begin{aligned} 2x_1 - x_2 + x_3 &= 5 \\ 4x_1 + 2x_2 + 3x_3 &= 8 \\ 3x_1 - 4x_2 - x_3 &= 3 \end{aligned}\right\}$ **Multan 2007**

Sol. **Here**

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{bmatrix}$$

$$\Rightarrow |A| = \begin{vmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 3 \\ -4 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 3 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 3 & -4 \end{vmatrix}$$

$$= 2(-2+12) + 1(-4-9) + 1(-16-6)$$

$$= 2(10) + 1(-13) + 1(-22) = 20 - 13 - 22 = -15 \neq 0$$

$$x_1 = \frac{\begin{vmatrix} 5 & -1 & 1 \\ 8 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}}{-15} = \frac{5(-2+12)+1(-8-9)+1(-32-6)}{-15}$$

$$= \frac{5(10)+1(-17)+1(-38)}{-15} = \frac{50-17-38}{-15} = \frac{-5}{-15}$$

$$x_1 = \frac{1}{3}$$

$$x_2 = \frac{\begin{vmatrix} 2 & 5 & 1 \\ 4 & 8 & 3 \\ 3 & 3 & -1 \end{vmatrix}}{-15} = \frac{2(-8-9)-5(-4-9)+1(12-24)}{-15}$$

$$x_2 = \frac{2(-17)-5(-13)+1(-12)}{-15} = \frac{-34+65-12}{-15} = \frac{-19}{15}$$

$$x_3 = \frac{\begin{vmatrix} 2 & -1 & 5 \\ 4 & 2 & 8 \\ 3 & -4 & 3 \end{vmatrix}}{-15} = \frac{2(6+32)+1(12-24)+5(-16-6)}{-15}$$

$$=\frac{2(38)+1(-12)+5(-22)}{-15}=\frac{76-12-110}{-15}$$

$$x_3=\frac{-46}{-15}=\frac{46}{15}$$

iii. $\left.\begin{array}{l} 2x_1-x_2+x_3\ =8 \\ x_1+2x_2+2x_3=6 \\ x_1-2x_2-x_3\ =1 \end{array}\right\}$ **Sargodha 2006, Multan 2010**

Sol. **Here**

$$A=\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{bmatrix}$$

$$\Rightarrow |A|=\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}$$

$$=2\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix}-(-1)\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}+1\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix}$$

$$=2(-2+4)+1(-1-2)+1(-2-2)$$

$$=2(2)+1(-3)+1(-4)=4-3-4=-3\neq 0$$

$$x_1=\frac{\begin{vmatrix} 8 & -1 & 1 \\ 6 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}}{-3}=\frac{8(-2+4)+1(-6-2)+1(-12-2)}{-3}$$

$$x_1=\frac{8(2)+1(-8)+1(-14)}{-3}=\frac{16-8-14}{-3}=\frac{-6}{-3}=2$$

$$x_2=\frac{\begin{vmatrix} 2 & 8 & 1 \\ 1 & 6 & 2 \\ 1 & 1 & -1 \end{vmatrix}}{-3}=\frac{2(-6-2)-8(-1-2)+1(1-6)}{-3}$$

$$x_2=\frac{2(-8)-8(-3)+1(-5)}{-3}=\frac{-16+24-5}{-3}=\frac{3}{-3}=-1$$

$$x_3 = \frac{\begin{vmatrix} 2 & -1 & 8 \\ 1 & 2 & 6 \\ 1 & -2 & 1 \end{vmatrix}}{-3} = \frac{2(2+12)+1(1-6)+8(-2-2)}{-3}$$

$$x_3 = \frac{2(14)+1(-5)+8(-4)}{-3} = \frac{28-5-32}{-3} = \frac{-9}{-3} = 3$$

2. **Use matrices to solve the following systems:**

I. $\left.\begin{aligned} x-2y+z &= -1 \\ 3x+y-2z &= 4 \\ y-z &= 1 \end{aligned}\right\}$ **Multan 2009**

Sol. **In matrix form:**

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

Where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{vmatrix} = 1(-1+2)-(-2)(-3+0)+1(3-0)$$

$$= 1-6+3 = -2 \neq 0$$

Now

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = (-1+2) = 1$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} = -(-3+0) = 3$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} = (3-0) = 3$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = -(2-1) = -1$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}1 & 1\\0 & -1\end{vmatrix}=(-1-0)=-1$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}1 & -2\\0 & 1\end{vmatrix}=-(1+0)=-1$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}-2 & 1\\1 & -2\end{vmatrix}=(4-1)=3$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1 & 1\\3 & -2\end{vmatrix}=-(-2-3)=5$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1 & -2\\3 & 1\end{vmatrix}=(1+6)=7$$

$$\text{Co-factor of } A=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\A_{21} & A_{22} & A_{23}\\A_{31} & A_{32} & A_{33}\end{bmatrix}$$

$$=\begin{bmatrix}1 & 3 & 3\\-1 & -1 & -1\\3 & 5 & 7\end{bmatrix}$$

$$\text{adj A = (Co-factor of A)}^t=\begin{bmatrix}1 & -1 & 3\\3 & -1 & 5\\3 & -1 & 7\end{bmatrix}$$

$$A^{-1}=\frac{adjA}{|A|}=\frac{1}{-2}\begin{bmatrix}1 & -1 & 3\\3 & -1 & 5\\3 & -1 & 7\end{bmatrix}$$

$$X=A^{-1}B$$

$$X=\frac{1}{-2}\begin{bmatrix}1 & -1 & 3\\3 & -1 & 5\\3 & -1 & 7\end{bmatrix}\begin{bmatrix}-1\\4\\1\end{bmatrix}=\frac{1}{-2}\begin{bmatrix}-1-4+3\\-3-4+5\\-3-4+7\end{bmatrix}$$

$$X=\frac{1}{-2}\begin{bmatrix}-2\\-2\\0\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow x = 1, y = 1, z = 0$$

ii. $$\left.\begin{aligned} 2x_1 + x_2 + 3x_3 &= 3 \\ x_1 + x_2 - 2x_3 &= 0 \\ -3x_1 - x_2 + 2x_3 &= -4 \end{aligned}\right\}$$

Sol. **In matrix form:**

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -4 \end{bmatrix}$$

Where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{bmatrix}, X \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} -3 \\ 0 \\ -4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{vmatrix} = 2(2-2) - 1(2-6) + 3(-1+3)$$

$$= 2(0) - 1(-4) + 3(2)$$

$$= 0 + 4 + 6 = 10 \neq 0$$

Now

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} = (2-2) = 0$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} = -(2-6) = 4$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix} = (-1+3) = 2$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} = -(2+3) = -5$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 3 \\ -3 & 2 \end{vmatrix} = (4+9) = 13$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}2 & 1\\ -3 & -1\end{vmatrix}=-(-2+3)=-1$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}1 & 3\\ 1 & -2\end{vmatrix}=(-2-3)=-5$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}2 & 3\\ 1 & -2\end{vmatrix}=-(-4-3)=7$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}2 & 1\\ 1 & 1\end{vmatrix}=(2-1)=1$$

Co-factor of $A=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}$

$$=\begin{bmatrix}0 & 4 & 2\\ -5 & 13 & -1\\ -5 & 7 & 1\end{bmatrix}$$

adj A = (Co-factor of A)^t $=\begin{bmatrix}0 & -5 & -5\\ 4 & 13 & 7\\ 2 & -1 & 1\end{bmatrix}$

$$A^{-1}=\frac{adjA}{|A|}=\frac{1}{10}\begin{bmatrix}1 & -5 & -5\\ 4 & 13 & 7\\ 2 & -1 & 1\end{bmatrix}$$

$$X=A^{-1}B$$

$$X=\frac{1}{10}\begin{bmatrix}0 & -5 & -5\\ 4 & 13 & 7\\ 2 & -1 & 1\end{bmatrix}\begin{bmatrix}-3\\ 0\\ -4\end{bmatrix}$$

$$X=\frac{1}{10}\begin{bmatrix}0-0+20\\ -12+0-28\\ -6-0-4\end{bmatrix}=\frac{1}{10}\begin{bmatrix}20\\ -40\\ -10\end{bmatrix}=\begin{bmatrix}2\\ -4\\ -1\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}2\\ -4\\ -1\end{bmatrix}$$ So $x_1=2,\ x_2=-4,\ x_3=-1$

iii. $\left.\begin{aligned} x+y&=2 \\ 2x-z&=1 \\ 2y-3z&=-1 \end{aligned}\right\}$

Sol. **In matrix form:**

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{vmatrix} = 1(0+2)-1(-6+0)+0(4-0)$$

$$= 2+6+0 = 8 \neq 0$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 0 & -1 \\ 2 & -3 \end{vmatrix} = (0+2) = 2$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} = -(-6+0) = 6$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = (4-0) = 4$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = -(-3-0) = 3$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} = (-3-0) = -3$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -(2-0) = -2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = (-1-0) = -1$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -(-1-0) = 1$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1 & 1\\ 2 & 0\end{vmatrix}=(0-2)=-2$$

$$\text{Co-factor of } A=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}=\begin{bmatrix}2 & 6 & 4\\ 3 & -3 & -2\\ -1 & 1 & -2\end{bmatrix}$$

$$\text{adj A} = (\text{Co-factor of A})^t=\begin{bmatrix}2 & 3 & -1\\ 6 & -3 & 1\\ 4 & -2 & -2\end{bmatrix}$$

$$A^{-1}=\frac{adjA}{|A|}=\frac{1}{8}\begin{bmatrix}2 & 3 & -1\\ 6 & -3 & 1\\ 4 & -2 & -2\end{bmatrix}$$

$$X=A^{-1}B=\frac{1}{8}\begin{bmatrix}2 & 3 & -1\\ 6 & -3 & 1\\ 4 & -2 & -2\end{bmatrix}\begin{bmatrix}2\\ 1\\ -1\end{bmatrix}=\frac{1}{8}\begin{bmatrix}4+3+1\\ 12-3-1\\ 8-2+2\end{bmatrix}=\frac{1}{8}\begin{bmatrix}8\\ 8\\ 8\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$ **Hence** $x=1,\ y=1,\ z=1$

3. **Solve the following systems by reducing their augmented matrices to the echelon form and the reduced echelon forms:**

i. $$\left.\begin{aligned}x_1-2x_2-2x_3 &= -1\\ 2x_1+3x_2+x_3 &= 1\\ 5x_1-4x_2-3x_3 &= 1\end{aligned}\right\}$$

Sol. **The augmented matrix is:**

$$\begin{bmatrix}1 & -2 & -2 & : & -1\\ 2 & 3 & 1 & : & 1\\ 5 & -4 & -3 & : & 1\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & -2 & -2 & : & -1\\ 0 & 7 & 5 & : & 3\\ 0 & 6 & 7 & : & 6\end{bmatrix}\begin{matrix}\\ R_2-2R_1\\ R_3-5R_1\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & -2 & : & -3 \\ 0 & 6 & 7 & : & 6 \end{bmatrix} R_2 - R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & -2 & : & -3 \\ 0 & 0 & 19 & : & 24 \end{bmatrix} R_3 - 6R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & -2 & : & -3 \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix} \frac{1}{19}R_3 \quad \text{———} I$$

is squired Echelon from where

$x_1 - 2x_2 - 2x_3 = -1 \longrightarrow (i)$

$x_2 - 2x_3 = -3 \longrightarrow (ii)$

$\boxed{x_3 = \frac{24}{19}} \longrightarrow (iii)$

Put (iii) in (ii) $x_2 - 2(\frac{24}{19}) = -3$

$x_2 = -3 + \frac{48}{19}$

$x_2 = \frac{-57+48}{19} = \frac{-9}{19} \Rightarrow \boxed{x_2 = \frac{-9}{19}}$

Put in x_2 & x_3 in (i)

$x_1 - 2(\frac{-9}{19}) - 2(\frac{24}{19}) = -1$

$x_1 + \frac{18}{19} - \frac{48}{19} + 1 = 0$

$x_1 + \frac{18-48+19}{19} = 0 \Rightarrow x_1 - \frac{11}{19} = 0 \Rightarrow \boxed{x_1 = \frac{11}{19}}$

For Reduced Echelon form continue (I)

$$\underset{\sim}{R}\begin{bmatrix}1 & -2 & -2 & : & -1\\ 0 & 1 & -2 & : & -3\\ 0 & 0 & 1 & : & 24/19\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & -6 & : & -7\\ 0 & 1 & -2 & : & -3\\ 0 & 0 & 1 & : & 24/19\end{bmatrix} R_1+2R_2$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 0 & : & 11/19\\ 0 & 1 & 0 & : & -9/19\\ 0 & 0 & 1 & : & 24/19\end{bmatrix}\begin{matrix}R_1+6R_3\\ R_2+2R_3\end{matrix}$$

$so\ \ x_1 = 11/19,\ x_2 = -9/19,\ \ x_3 = 24/19$

ii. $\left.\begin{aligned} x+2y+z &= 2\\ 2x+y+2z &= -1\\ 2x+3y-z &= 9\end{aligned}\right\}$ **Federal**

Sol. **The augmented matrix is:**

$$\begin{bmatrix}1 & 2 & 1 & : & 2\\ 2 & 1 & 2 & : & -1\\ 2 & 3 & -1 & : & 9\end{bmatrix}$$

For Echelon form

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & 1 & : & 2\\ 0 & -3 & 0 & : & -5\\ 0 & -1 & -3 & : & 5\end{bmatrix}\begin{matrix}R_3-2R_1\\ R_2-2R_1\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & 1 & : & 2\\ 0 & 1 & 12 & : & -25\\ 0 & -1 & -3 & : & 5\end{bmatrix} R_2-4R_3$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & 1 & : & 2\\ 0 & 1 & 12 & : & -25\\ 0 & 0 & 9 & : & -20\end{bmatrix} R_3+R_2$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & 1 & : & 2\\ 0 & 1 & 12 & : & -25\\ 0 & 0 & 1 & : & -20/9\end{bmatrix}\frac{1}{9}R_3 \text{ ——— } (A)$$

Which is required echlon form

$$x+2y+z=2 \longrightarrow (i)$$
$$y+12z=-25 \longrightarrow (ii)$$
$$\boxed{z=-20/9}$$

Put in (ii) $y+12(\frac{-20}{9})=-25 \Rightarrow y-\frac{80}{3}=-25$

$$y=-25+\frac{80}{3}=\frac{-75+80}{3}=5/3 \Rightarrow \boxed{y=5/3}$$

Put values of Z & y in (i)

$$x+2(\frac{5}{3})-\frac{20}{9}=2$$
$$x+\frac{10}{3}-\frac{20}{9}-2=0$$
$$x+\frac{30-20-18}{9}=0 \Rightarrow x-\frac{8}{9}=0 \Rightarrow \boxed{x=8/9}$$

For Reduced Echelon form continue (A)

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 12 & : & -25 \\ 0 & 0 & 1 & : & -20/9 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -23 & : & 52 \\ 0 & 1 & 12 & : & -25 \\ 0 & 0 & 1 & : & -20/9 \end{bmatrix} R_1-2R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & : & 8/9 \\ 0 & 1 & 0 & : & 5/3 \\ 0 & 0 & 1 & : & -20/9 \end{bmatrix} \begin{matrix} R_2-12R_3 \\ R_1+23R_3 \end{matrix}$$

so $\boldsymbol{x=8/9,\ y=5/3,\ z=-20/9}$

iii. $$\left.\begin{aligned} x_1+4x_2+2x_3&=2 \\ 2x_1+x_2-2x_3&=9 \\ 3x_1+2x_2-2x_3&=12 \end{aligned}\right\}$$

Sol. **The augmented matrix is** $\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 2 & 1 & -2 & : & 9 \\ 3 & 2 & -2 & : & 12 \end{bmatrix}$

$$\underset{\sim}{R}\begin{bmatrix}1 & 4 & 2 & : & 2\\ 0 & -7 & -6 & : & 5\\ 0 & -10 & -8 & : & 6\end{bmatrix}\begin{matrix}R_2-2R_1\\ R_3-3R_1\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 4 & 2 & : & 2\\ 0 & 1 & 6/7 & : & -5/7\\ 0 & -10 & -8 & : & 6\end{bmatrix}\frac{-1}{7}R_2$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 4 & 2 & : & 2\\ 0 & 1 & 6/7 & : & -5/7\\ 0 & 0 & 4/7 & : & -8/7\end{bmatrix}R_3+10R_2$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 4 & 2 & : & 2\\ 0 & 1 & 6/7 & : & -5/7\\ 0 & 0 & 1 & : & -2\end{bmatrix}\frac{7}{4}R_3 \longrightarrow (A)$$

$$x_1+4x_2+2x_3=2 \longrightarrow (i)$$

$$x_2+\frac{6}{7}x_3=\frac{-5}{7} \longrightarrow (ii)$$

$$\boxed{x_3=-2} \longrightarrow (iii)$$

Put (iii) in (ii) $x_2+\frac{6}{7}(-2)=-\frac{5}{7}\Rightarrow x_2=\frac{-5}{7}+\frac{12}{7}=\frac{7}{7}=1$

Put x_3 and x_2 in (i) $\Rightarrow x_1+4(1)+2(-2)=2$

$x_1+4-4=2\Rightarrow \boxed{x_1=2}$

For Reduced Echenol form continue (A)

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & -10/7 & : & 34/2\\ 0 & 1 & 6/7 & : & -5/7\\ 0 & 0 & 1 & : & -2\end{bmatrix}R_1-4R_2$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 0 & : & 2\\ 0 & 1 & 0 & : & 1\\ 0 & 0 & 1 & : & -2\end{bmatrix}\begin{matrix}R_1-6/7R_3\\ R_1+10/7R_3\end{matrix}$$

so $x_1=2,\ x_2=1,\ x_3=-2$

4. Solve the following system of homogeneous linear equations.

i. $\left.\begin{aligned} x+2y-2z=0 \\ 2x+y+5z=0 \\ 5x+4y+8z=0 \end{aligned}\right\}$

Sol. $x+2y-2z=0 \longrightarrow I$

$2x+y+5z=0 \longrightarrow II$

$5x+4y+8z=0 \longrightarrow III$

$$A=\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{bmatrix}$$

$$|A|=\begin{vmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{vmatrix}$$

$=1(8-20)-2(16-25)+(-2)(8-5)$

$=-12+18-6=18-18=0$

So system has non trivial solution

$2\times I-II$

$2x+4y-4z=0$

$\underline{2x+y+5z=0}$

$3y-9z=0 \Rightarrow 3y=9z$

$y=3z$

$III-4\times II$

$5x+4y+8z=0$

$\underline{8x+4y+20z=0}$

$-3x-12z=0$

$3x=-12z$

$x=-4z$

Take $z=t$ then solution is $x=-4t,\ y=3t,\ z=t$

ii. $\left.\begin{aligned} x_1+4x_2+2x_3=0 \\ 2x_1+x_2-3x_3=0 \\ 3x_1+2x_2-4x_3=0 \end{aligned}\right\}$

Sol. $x_1+4x_2+2x_3=0 \longrightarrow I$

$$2x_1+x_2-3x_3=0 \longrightarrow II$$
$$3x_1+2x_2-4x_3=0 \longrightarrow III$$

$$A=\begin{bmatrix}1 & 4 & 2\\ 2 & 1 & -3\\ 3 & 2 & -4\end{bmatrix} \Rightarrow |A|=\begin{vmatrix}1 & 4 & 2\\ 2 & 1 & -3\\ 3 & 2 & -4\end{vmatrix}$$
$$=1(-4+6)-4(-8+9)+2(4-3)$$
$$=2-4+2=4-4=0$$

So system has non trivial solution

$II-2\times I \Rightarrow$

$$\begin{array}{r}2x_1+x_2-3x_3=0\\ \underline{2x_1\pm 8x_2\pm 4x_3=0}\\ -7x_2-7x_3=0\end{array}$$

$III-2II \Rightarrow$

$$\begin{array}{r}3x_1+2x_2-4x_3=0\\ \underline{4x_1\pm 2x_2\mp 6x_3=0}\\ -x_1+2x_3=0 \Rightarrow x_1=2x_3\end{array}$$

$$\Rightarrow 7x_2=-7x_3 \Rightarrow x_2=-x_3$$

Take $x_3=t$ then $x_1=2t$, $x_2=-t$, $x_3=t$ **is solution.**

iii. $$\left.\begin{array}{r}x_1-2x_2-x_3=0\\ x_1+x_2+5x_3=0\\ 2x_1-x_2-x_3=0\end{array}\right\}$$

Sol. $$x_1-2x_2-x_3=0 \longrightarrow I$$
$$x_1+x_2+5x_3=0 \longrightarrow II$$
$$2x_1-x_2+4x_3=0 \longrightarrow III$$

$$A=\begin{bmatrix}1 & -2 & -1\\ 1 & 1 & 5\\ 2 & -1 & 4\end{bmatrix}$$

$$|A|=\begin{vmatrix}1 & -2 & -1\\ 1 & 1 & 5\\ 2 & -1 & 4\end{vmatrix}=1((4+5)-(-2)(4-10)+(-1)(-1-2)=9-12+3=0$$

So system has non trivail satiation

$II-I$ $\quad$ $III+II$

$$\begin{array}{r} x_1+x_2+5x_3=0 \\ x_1-2x_2-x_3=0 \\ -\quad+\quad+ \\ \hline 3x_2+6x_3=0 \end{array} \qquad \begin{array}{r} 2x_1-x_2+4x_3=0 \\ x_1+x_2+5x_3=\ \ 0 \\ \hline 3x_1+9x_3=0 \end{array}$$

$\Rightarrow\ x_2=-2x_3$ $\qquad$ $3x_1=-9x_3$

Take $x_3=t$ then solution is $\qquad$ $\Rightarrow x_1=-3x_3$

$x_1=-3t,\quad x_2=-2t,\quad x_3=t$

5. **Find the value of λ for which the following systems have non-trivial solutions. Also solve the system for the value of λ.**

i. $$\left.\begin{array}{l} \boldsymbol{x+y+z=0} \\ \boldsymbol{2x+y-\lambda z=0} \\ \boldsymbol{x+2y-2z=0} \end{array}\right\}$$

Sol. $$A=\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{bmatrix}$$

Given system has non-trivial solution so $|A|=0$

$$|A|=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{vmatrix}=0$$

$\Rightarrow 1(-2+2\lambda)-1(-4+\lambda)+1(4-1)=0$

$\Rightarrow -2+2\lambda+4-\lambda+3=0$

$\Rightarrow \lambda+5=0 \Rightarrow \boxed{\lambda=-5}$

System becomes

$x+y+z=0 \longrightarrow I$

$2x+y+5z=0 \longrightarrow II$

$x+2y-2z=0 \longrightarrow III$

$II-2\times I$ $\qquad$ $III-2\times II$

$$\begin{array}{r} \Rightarrow 2x+\ \ y+5z=0 \\ 2x+2y+2z=0 \\ -\quad-\quad- \\ \hline -y+3z=0 \end{array} \qquad \begin{array}{r} x+2y-2z=0 \\ 4x+2y+10z=0 \\ -\quad-\quad- \\ \hline -3x-12z=0 \end{array}$$

$y = 3z \qquad \Rightarrow 3x = -12z$

$x = -4z$

Take $z = t$ then solution $x = -4t,\ y = 3t$

$z = t$

ii. $\left.\begin{aligned} x_1 + 4x_2 + \lambda x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 0 \\ 3x_1 + \lambda x_2 - 4x_3 &= 0 \end{aligned}\right\}$

Sol. $A = \begin{bmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{bmatrix}$

System has non trivial solution so $|A| = 0$ i.e. $\begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{vmatrix} = 0$

$\Rightarrow 1(-4+3\lambda) - 4(-8+9) + \lambda(2\lambda - 3) = 0$

$\Rightarrow -4 + 3\lambda - 4 + 2\lambda^2 - 3\lambda = 0$

$\Rightarrow 2\lambda^2 - 8 = 0 \Rightarrow \lambda^2 = 4 \Rightarrow \lambda = \pm 2$

When $\lambda = 2$ then system

$$x_1 + 4x_2 + 2x_3 = 0 \longrightarrow I$$
$$2x_1 + x_2 - 3x_3 = 0 \longrightarrow II$$
$$3x_1 + 2x_2 - 4x_3 = 0 \longrightarrow III$$

$II - 2 \times I$

$\cancel{2x_1} + x_2 - 3x_3 = 0$

$\cancel{2x_1} + 8x_2 + 4x_3 = 0$

$- \quad - \quad -$

$-7x_2 - 7x_3 = 0 \Rightarrow x_2 = -x_3$

$III - 2 \times II$ When $\lambda = 2$ then solution is if $x_3 = t, x_2 = -t, x_1 = 2t$

$3x_1 + \cancel{2x_2} - 4x_3 = 0$

$4x_1 + \cancel{2x_2} - 6x_3 = 0$

$- \quad - \quad +$

$-x_1 + 2x_3 = 0 \Rightarrow x_1 = 2x_3$

When $\lambda = -2$ then system is

$$x_1+4x_2-2x_3=0 \longrightarrow IV$$
$$2x_1+\ \ x_2-3x_3=0 \longrightarrow V$$
$$3x_1-2x_2-4x_3=0 \longrightarrow VI$$

$V-2\times IV$

$$\cancel{2x_1}+\ x_2-3x_3=0$$
$$\cancel{2x_1}+8x_2-4x_3=0$$
$$-\quad -\quad +$$
$$-7x_2+x_3=0$$
$$\Rightarrow 7x_2=x_3$$
$$\boxed{x_2=\frac{1}{7}x_3}$$

$V+2\times V$

$$3x_1-2x_2-4x_3=0$$
$$4x_1+2x_2-6x_3=0$$
$$7x_1-10x_3=0$$
$$\boxed{x_1=\frac{10}{7}x_3}$$

When $\lambda=-2$ & $x_3=t$ then $x_1=\frac{10}{7}t, x_2=-\frac{1}{7}t$

6. Find the value of λ for which the following systems does not possess a unique solution. Also solve the system for the value of λ.

$$\left.\begin{aligned} x_1+4x_2+\lambda x_3 &= 2 \\ 2x_1+x_2-2x_3 &= 11 \\ 3x_1+2x_2-2x_3 &= 16 \end{aligned}\right\} \qquad \textbf{Federal}$$

Sol. Augmented Matrix is:

$$\begin{bmatrix} 1 & 4 & \lambda & : & 2 \\ 2 & 1 & -2 & : & 11 \\ 3 & 2 & -2 & : & 16 \end{bmatrix} \Rightarrow \underset{\sim}{R} \begin{bmatrix} 1 & 4 & \lambda & : & 2 \\ 0 & -7 & -2-2\lambda & : & 7 \\ 0 & -10 & -2-3\lambda & : & 10 \end{bmatrix} R_2-2R_1 \ \& \ R_3-3R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & \lambda & : & 2 \\ 0 & 1 & \frac{-2-2\lambda}{-7} & : & -1 \\ 0 & -10 & -2-3\lambda & : & 10 \end{bmatrix} \frac{-1}{7}R_2 \Rightarrow \underset{\sim}{R} \begin{bmatrix} 1 & 4 & \lambda & : & 2 \\ 0 & 1 & \frac{2+2\lambda}{7} & : & -1 \\ 0 & 0 & \frac{6-\lambda}{7} & : & 0 \end{bmatrix} R_3+10R_2 \longrightarrow (A)$$

System does not possess unique solution for $\frac{6-\lambda}{7}=0 \Rightarrow 6-\lambda=0 \Rightarrow \boxed{\lambda=6}$

Put value of λ in (A)

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 6 & : & 2 \\ 0 & 1 & \frac{2+2(6)}{7} & : & -1 \\ 0 & 0 & \frac{6-6}{7} & : & 0 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 6 & : & 2 \\ 0 & 1 & 2 & : & -1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$x_1 + 4x_2 + 6x_3 = 2 \longrightarrow I$$

Here $$x_2 + 2x_3 = -1 \longrightarrow II$$

$$\Rightarrow x_2 = -2x_3 - 1 \longrightarrow III$$

Put III in I

$$x_1 + 4(-2x_3 - 1) + 6x_3 = 2$$

$$x_1 - 8x_3 - 4 + 6x_3 = 2$$

$$\Rightarrow x_1 - 2x_3 - 4 = 2$$

$$x_1 = 2x_3 + 6$$

Put $x_3 = t$

$$x_1 = 2t + 6$$

$$x_2 = -2t - 1$$

TEST YOUR SKILLS Marks: 50

Q # 1. Select the Correct Option **(10)**

i. A square matrix $A=\left[a_{ij}\right]$ with complex entries is skew hermetian if $\overline{(A)}'=?$

a) A b) $-A$ c) $|A|$ d) $|A|$

ii. The matrix $\begin{bmatrix} 5 & 1 \\ 15 & 3 \end{bmatrix}$ is:

a) Singular b) Non Singular c) Symmetric d) Skew symmetric

iii. For trivial solution $|A|$ is:

a) 1 b) -1 c) Zero d) Not defined

iv. (0,0,0) is ________ solution of homogeneous system of linear equation is

a) Trivial b) Non trivial c) unique d) Non

v. If $\begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$ then A_{22} is equal to:

a) 10 b) -10 c) -18 d) -11

vi. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is

a) Diagonal matrix b) Zero matrix c) Scalar matrix d) Identity matrix

vii. If $\begin{bmatrix} -1 & 3 \\ x & 1 \end{bmatrix}=0$ then value of x is

a) -3 b) $\frac{1}{3}$ c) $-\frac{1}{3}$ d) 3

viii. If A is a square matrix of order 2×2 then $|KA|$ equals:

a) $K|A|$ b) $\frac{1}{K}|A|$ c) $2K|A|$ d) $K^2|A|$

ix. If $A=\left[a_{ij}\right]$ is a square matrix of order n if $a_{ij}=0 \quad \forall i\neq j$ and $a_{ij}=1 \;,\forall \; i=j$ then A is matrix

a) Unit b) Null c) Symmetric d) Skew Symmetric

x. If A and B are confirmable for multiplication if $(AB)'=?$

a) AB b) BA c) $A'B'$ d) $B'A'$

Q # 2. Short Questions: **(2 X 20 = 40)**

i. Find x and y if $\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix}=\begin{bmatrix} y & 1 \\ -3 & 2x \end{bmatrix}$

ii. Without expansion show that $\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 0$

iii. If A is square matrix of order 3 then show that $A - A'$ is skew symmetric :

iv. Define Scalar Matrix

v. Without expansion prove that $\begin{vmatrix} \alpha & \beta+\alpha & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix} = 0$

vi. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$ Find A_{12} & A_{32}

vii. If $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ show that $A^4 = I_2$

viii. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ Show that $(A+B)' = A' + B'$

ix. Find x if $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = 0$

x. Define Hermetain matrix:

Q # 3. (a) Solve by Cramer's rule

$$\begin{aligned} 2x+2y+z &= 3 \\ 3x-2y-2z &= 1 \\ 5x+y-3z &= 2 \end{aligned}$$

(b) Show that $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

Q # 4. (a) Show that $\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = (x+3)(x-1)^3$

(b) Find the inverse of $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Quadratic Equations 4

Exercise 4.1

Solve the following equations by factorization:

1. $3x^2+4x+1=0$

Sol. $3x^2+3x+x+1=0$

$3x\ (x+1)+1(x+1)=0$

$(x+1)(3x+1)=0$

$x+1=0$ or $3x+1=0$

$x=-1$ or $x=-\frac{1}{3}$

$S.S\left\{-1,\frac{-1}{3}\right\}$

3. $9x^2-12x-5=0$

Sol. $9x^2-15x+3x-5=0$

$3x(3x-5)+1(3x-5)=0$

$(3x-5)(3x+1)=0$

$3x-5=0$ or $3x+1=0$

$x=\frac{5}{3}$ or $x=\frac{-1}{3}$

$S.S\left\{\frac{5}{3},\frac{-1}{3}\right\}$

2. $x^2+7x+12=0$

Sol. $x^2+3x+4x+12=0$

$x(x+3)+4(x+3)=0$

$(x+3)(x+4)=0$

$x+3=0$ or $x+4=0$

$x=-3$ or $x=-4$

$S.S\ \{-3,-4\}$

4. $x^2-x=2$ **Multan 2008, Sargodha 2006**

Sol. $x^2-x-2=0$

$x^2-2x+x-2=0$

$x(x-2)+1(x-2)=0$

$(x-2)(x+1)=0$

$x-2=0$ or $x+1=0$

$x=2$ or $x=-1$

$S.S=\{-1,2\}$

5. $x(x+7)=(2x-1)(x+4)$ **Multan 2007**

Sol. $x^2+7x=2x^2+8x-x-4$ **Faisalabad07,09**

or $2x^2+7x-4-x^2-7x=0$

or $x^2-4=0 \Rightarrow (x-2)(x+2)=0$

$x-2=0$ or $x+2=0$

$x=2$ or $x=-2$

$S.S=\{2,-2\}$

6. $\dfrac{x}{x+1}+\dfrac{x+1}{x}=\dfrac{5}{2}; x\neq -1,0$

Sol. $\dfrac{x^2+(x+1)^2}{x(x+1)}=\dfrac{5}{2}$

$\dfrac{x^2+x^2+2x+1}{x^2+x}=\dfrac{5}{2}$

$\dfrac{2x^2+2x+1}{x^2+x}=\dfrac{5}{2}$

$\Rightarrow \quad 2(2x^2+2x+1)=5(x^2+x)$

$4x^2+4x+2=5x^2+5x$

$5x^2+5x-4x^2-4x-2=0$

$x^2+x-2=0$

$\Rightarrow \quad x^2+2x-x-2=0$

$x(x+2)-1(x+2)=0$

$\Rightarrow \quad (x+2)(x-1)=0$

$x+2=0 \text{ or } x-1=0$

$x=-2 \text{ or } x=1$

$S.S=\{1,-2\}$

7. $\dfrac{1}{x+1}+\dfrac{2}{x+2}=\dfrac{7}{x+5}; x\neq -1,-2,-5$

Sol. $\dfrac{1(x+2)+2(x+1)}{(x+1)(x+2)}=\dfrac{7}{x+5}$

$\dfrac{x+2+2x+2}{x^2+3x+2}=\dfrac{7}{x+5}$

$\dfrac{3x+4}{x^2+3x+2}=\dfrac{7}{x+5}$

$\Rightarrow \quad (3x+4)(x+5)=7(x^2+3x+2))$

$3x^2+15x+4x+20=7x^2+21x+14$

$7x^2+21x+14-3x^2-19x-20=0$

$4x^2+2x-6=0$

$4x^2+6x-4x-6=0$

$2x(2x+3)-2(2x+3)=0$

$(2x+3)(2x-2)=0$

$2x+3=0 \text{ or } 2x-2=0$

$$x=\frac{-3}{2} \quad or \quad x=1$$

$$S.S=\left\{\frac{-3}{2},1\right\}$$

8. $$\frac{a}{ax-1}+\frac{b}{bx-1}=a+b;x\neq\frac{1}{a},\frac{1}{b}$$

Sol. $$or \quad \frac{a}{(ax-1)}-b+\frac{b}{bx-1}-a=0$$

$$\frac{a-b(ax-1)}{(ax-1)}+\frac{b-a(bx-1)}{bx-1}=0$$

$$\frac{a-abx+b}{(ax-1)}+\frac{b-abx+a}{bx-1}=0$$

$$(a-abx+b)\left[\frac{1}{ax-1}+\frac{1}{bx-1}\right]=0$$

$$(a-abx+b)\left(\frac{bx-1+ax-1}{(ax-1)(bx-1)}\right)=0$$

$$(a-abx+b)(ax+bx-2)=0$$

$$a-abx+b=0 \quad or \quad ax+bx-2=0$$

$$abx=a+b \quad or \quad x(a+b)=2$$

$$x=\frac{a+b}{ab} \quad or \quad x=\frac{2}{a+b}$$

$$S.S=\left\{\frac{a+b}{ab},\frac{2}{a+b}\right\}$$

Solve the following equations by completing the square:

9. $x^2-2x-899=0$

Sol. $x^2-2x-899=0 \Rightarrow x^2-2x=899$

Adding $(1)^2$ both sides.

$$x^2-2x+(1)^2=899+(1)^2 \Rightarrow (x-1)^2=900 \Rightarrow \sqrt{(x-1)^2}=\pm\sqrt{900}$$

$$x-1=\pm30 \Rightarrow x=1\pm30 \Rightarrow x=1+30 \ or \ x=1-30$$

$$x=31 \quad or \quad x=-29$$

$$S.S=\{-29,31\}$$

10. $\boldsymbol{x^2+4x-1085=0}$

Sol. $x^2+4x=1085$

Adding $(2)^2$ both sides.

$x^2+4x+(2)^2=1085+(2)^2$

$(x+2)^2=1089 \Rightarrow x+2=\pm 33$ (*By taking square root both side*)

$x=-2\pm 33$

$x=-2+33$ *or* $x=-2-33$

$x=31$ *or* $x=-35$

$S.S=\{31,-35\}$

11. $\boldsymbol{x^2+6x-567=0}$

Sol. $x^2+6x-567=0 \Rightarrow x^2+6x=567$

Adding $(3)^2$ *both sides*

$x^2+6x+(3)^2=567+(3)^2$

$(x+3)^2=576 \Rightarrow x+3=\pm 24$ (*By taking square root both side*)

$x=-3\pm 24$

$x=-3+24$ *or* $x=-3-24$

$x=21$ *or* $x=-27$

$S.S=\{21,-27\}$

12. $\boldsymbol{x^2-3x-648=0}$

Sol. $x^2-3x=648$

Adding $(\frac{3}{2})^2$ *both sides*

$x^2-3x+(\frac{3}{2})^2=648+(\frac{3}{2})^2$

$(x-\frac{3}{2})^2=648+\frac{9}{4} \Rightarrow (x-\frac{3}{2})^2=\frac{2592+9}{4}$

$(x-\frac{3}{2})^2=\frac{2601}{4} \Rightarrow x-\frac{3}{2}=\pm\frac{51}{2}$ (*By taking square root both side*)

$x=\frac{3}{2}\pm\frac{51}{2} \Rightarrow x=\frac{3}{2}\pm\frac{51}{2}$

$x=\frac{3}{2}+\frac{51}{2}$ *or* $x=\frac{3}{2}-\frac{51}{2}$

$x=\frac{54}{2}$ *or* $x=-\frac{48}{2} \Rightarrow x=27$ *or* $x=-24 \Rightarrow S.S=\{-24,27\}$

13. $\mathbf{x^2 - x - 1806 = 0}$

Sol. $x^2 - x = 1806$

Add both sides $\left(\frac{1}{2}\right)^2$ *we get*

$$x^2 - x + \left(\frac{1}{2}\right)^2 = 1806 + \left(\frac{1}{2}\right)^2$$

$$(x - \frac{1}{2})^2 = 1806 + \frac{1}{4}$$

$$\left(x - \frac{1}{2}\right)^2 = 1806 + \frac{1}{4}$$

$$\left(x - \frac{1}{2}\right)^2 = \frac{7224 + 1}{4} = \frac{7225}{4}$$

Taking square root both sides

$$\sqrt{\left(x - \left(\frac{1}{2}\right)\right)^2} = \pm\sqrt{\frac{7225}{4}}$$

$$x - \frac{1}{2} = \pm\frac{85}{2} \Rightarrow x = \frac{1}{2} \pm \frac{85}{2}$$

$$x = \frac{1 \pm 85}{2} \Rightarrow x = \frac{1 + 85}{2} \quad or \quad x = \frac{1 - 85}{2}$$

$$x = 43 \quad or \quad x = -42 \Rightarrow S.S = \{-42, 43\}$$

14. $\mathbf{2x^2 + 12x - 110 = 0}$

Sol. $2x^2 + 12x - 110 = 0(\div by 2)$

$$x^2 + 6x - 55 = 0$$

Adding both sides $(3)^2$ *we get*

$$x^2 + 2(3)x + (3)^2 = 55 + (3)^2$$

$$(x + 3)^2 = 55 + 9 = 64$$

$$or \ \sqrt{(x + 3)^2} = \pm\sqrt{64}$$

$$x + 3 = \pm 8$$

$$x = -3 \pm 8$$

$$x = -3 \pm 8$$

$$x = -3 + 8 \ or \ x = -3 - 8$$

$$x = 5 \ or \ x = -11 \Rightarrow S.S = \{5, -11\}$$

Find roots of the following equations by using quadratic formula:

15. $5x^2 - 13x + 6 = 0$

Sol. $a = 5,\ b = -13,\ c = 6$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-13)\sqrt{(-13)^2 - 4(5)(6)}}{2(5)} = \frac{13 \pm \sqrt{169 - 120}}{10}$$

$$x = \frac{13 \pm \sqrt{49}}{10} = \frac{13 \pm 7}{10}$$

$$x = \frac{13+7}{10} \quad or \quad \frac{13-7}{10}$$

$$x = \frac{2\cancel{0}}{1\cancel{0}} \quad or \quad x = \frac{6}{1\cancel{0}}$$

$$x = 2 \quad or \quad x = \frac{3}{5}$$

$$S.S = \left\{2, \frac{3}{5}\right\}$$

16. $4x^2 + 7x - 1 = 0$ **Multan 2008**

Sol. $a = 4,\ b = 7,\ c = -1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(4)(-1)}}{2(4)} = \frac{-7 \pm \sqrt{49 + 16}}{8} = \frac{-7 \pm \sqrt{65}}{8}$$

$$S.S = \left\{\frac{-7 \pm \sqrt{65}}{8}\right\}$$

17. $15x^2 + 2ax - a^2 = 0$ **Sargodha 2008**

Sol. $a = 15,\ b = 2a,\ c = -a^2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2a \pm \sqrt{(2a)^2 - 4(15)(-a^2)}}{2(15)} = \frac{-2a \pm \sqrt{4a^2 + 60a^2}}{30}$$

$$= \frac{-2a \pm \sqrt{64a^2}}{30} = \frac{-2a \pm 8a}{30}$$

$$x=\frac{-2a+8a}{30} \quad or \quad x=\frac{-2a-8a}{30}$$

$$x=\frac{6a}{30} \quad or \quad x=\frac{-10a}{30}$$

$$x=\frac{a}{5} \quad or \quad x=\frac{-a}{3} \Rightarrow S.S=\left\{\frac{-a}{3},\frac{a}{5}\right\}$$

18. $\mathbf{16x^2+8x+1=0}$

Sol. $a=16,\ b=8,\ c=1$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-8\pm\sqrt{(8)^2-4(16)(1)}}{2(16)}$$

$$x=\frac{-8\pm\sqrt{64-64}}{32}=\frac{-8\pm 0}{32}=\frac{-8}{32}=-\frac{1}{4} \Rightarrow S.S=\left\{-\frac{1}{4}\right\}$$

19. $\mathbf{(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0}$

Sol. $x^2-ax-bx+ab+x^2-bx-cx+bc+x^2-cx-ax+ac=0$

$3x^2-2ax-2bx-2cx+ab+bc+ca=0$

$3x^2-2(a+b+c)x+(ab+bc+ca)=0$

$A=3, B=-2(a+b+c), C=ab+bc+ca$

$$x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}=\frac{-(-2(a+b+c))\pm\sqrt{(-2(a+b+c))^2-4(3)(ab+bc+ca)}}{2(3)}$$

$$x=\frac{2(a+b+c)\pm\sqrt{4(a^2+b^2+c^2+2ab+2bc+2ca)-12(ab+bc+ca)}}{6}$$

$$x=\frac{2(a+b+c)\pm\sqrt{4a^2+4b^2+4c^2+8ab+8bc+8ca-12ab-12bc-12ca}}{6}$$

$$x=\frac{2(a+b+c)\pm\sqrt{4a^2+6b^2+4c^2-4ab-4bc-4ca}}{6}=\frac{2(a+b+c)\pm\sqrt{4(a^2+b^2+c^2-ab-bc-ca)}}{6}$$

$$\frac{2(a+b+c)\pm 2\sqrt{a^2+b^2+c^2-ab-bc-ca}}{6} \Rightarrow x=\frac{\cancel{2}\left[(a+b+c)\pm\sqrt{a^2+b^2+c^2-ab-bc-ca}\right]}{\cancel{6}_3}$$

$$x=\frac{(a+b+c)\pm\sqrt{a^2+b^2+c^2-ab-bc-ca}}{3}$$

$$S.S=\left\{\frac{(a+b+c)\pm\sqrt{a^2+b^2+c^2-ab-bc-ca}}{3}\right\}$$

20. $(a+b)x^2+(a+2b+c)x+b+c=0$

Sol. $A=a+b, B=a+2b+c, C=b+c$

$$x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}=\frac{-(a+2b+c))\pm\sqrt{(a+2b+c)^2-4(a+b)(b+c)}}{2(a+b)}$$

$$x=\frac{-(a+2b+c)\pm\sqrt{a^2+\cancel{4b^2}+c^2+\cancel{4ab}+\cancel{4bc}+2ca-\cancel{4ab}-4ac-\cancel{4bc}-4b^2}}{2(a+b)}$$

$$x=\frac{-(a+2b+c)\pm\sqrt{a^2+c^2-2ac}}{2(a+b)}=\frac{-(a+2b+c)\pm\sqrt{(a-c)^2}}{2(a+b)}$$

$$x=\frac{-a-2b-c\pm(a-c)}{2(a+b)}$$

$$x=\frac{-a-2b-c+(a-c)}{2(a+b)}\ or\ x=\frac{-a-2b-c-(a-c)}{2(a+b)}=\frac{-a-2b-c-a+c}{2(a+b)}$$

$$x=\frac{-2(b+c)}{2(a+b)}\ or\ x=\frac{-2a-2b}{2(a+b)}=\frac{-\cancel{2(a+b)}}{\cancel{2(a+b)}}\Rightarrow\ x=-1$$

$$S.S=\left\{-1,\frac{-(b+c)}{a+b}\right\}$$

Exponential Equation:

Equations in which variable occur in exponents.

Reciprocal Equations:

An equation which remains unchanged when x is replaced by $\frac{1}{x}$

Example 3: $\mathbf{2^{2x}-3.2^{x+2}+32=0}$

Sol.

$$2^{2x}-3.2^{x+2}+32=0$$
$$2^{2x}-3.2^2.2^x+32=0$$
$$2^{2x}-3.4.2^x+32=0\Rightarrow 2^{2x}-12.2^x+32=0$$
$$Put\ 2^x=y\Rightarrow 2^{2x}=y^2\Rightarrow y^2-12y+32=0$$

2008 – II Sargodha Just Covert to quadratic

Example* 5 :** ***Solve $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$

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Faisalabad 2008

Sol. $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$

'$\div$' *by* x^2

$$\Rightarrow x^2 - 3x + 4 - \frac{3}{x} + \frac{1}{x^2} = 0 \quad \Rightarrow \left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) + 4 = 0 \_\_\_\_ I$$

Put $x + \frac{1}{x} = y \Rightarrow x^2 + \frac{1}{x^2} + 2 = y^2$

or $x^2 + \frac{1}{x^2} = y^2 - 2$

(*I become*) $y^2 - 2 - 3y + 4 = 0 \Rightarrow y^2 - 3y + 2 = 0$

$y^2 - y - 2y + 2 = 0 \Rightarrow y(y-1) - 2(y-1) = 0$

$(y-1)(y-2) = 0 \Rightarrow (y-1) = 0$ *or* $(y-2) = 0$

$y = 1$ *or* $y = 2$

When $y = 1 \Rightarrow x + \frac{1}{x} = 1 \Rightarrow x^2 + 1 = x \Rightarrow x^2 - x + 1 = 0$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2}$$

When $y = 2 \Rightarrow x + \frac{1}{x} = 2 \Rightarrow x^2 + 1 = 2x \Rightarrow x^2 - 2x + 1 = 0$

$$(x-1)^2 = 0 \Rightarrow (x-1)(x-1) = 0 \Rightarrow x = 1, 1 \Rightarrow S.S = \left\{1, \frac{1 \pm \sqrt{-3}}{2}\right\}$$

Exercise 4.2

Solve the following equations

1. $x^4 - 6x^2 + 8 = 0$

Sol. *Put* $x^2 = y \Rightarrow x^4 = y^2$

$y^2 - 6y + 8 = 0$

or $y^2 - 2y - 4y + 8 = 0$

or $y(y-2) - 4(y-2) = 0$

or $(y-2)(y-4) = 0$

$y - 2 = 0$ *or* $y - 4 = 0$

$y = 2$ *or* $y = 4$

$$when\quad y=2\quad then\ x^2=2\Rightarrow x=\pm\sqrt{2}$$

$$when\quad y=4\quad then\ x^2=4\Rightarrow x=\pm 2$$

$$S.S=\left\{\pm 2,\pm\sqrt{2}\right\}$$

2. $\boldsymbol{x^{-2}-10=3x^{-1}}$ **Faisalabad 2008, Multan 2009**

Sol. $x^{-2}-10=3x^{-1}\quad or\quad \dfrac{1}{x^2}-10=\dfrac{3}{x}$

Multiplying both sides by x^2

$$1-10x^2=3x\quad or\quad 10x^2+3x-1=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\qquad a=10,\ b=3,\ c=-1$$

$$x=\frac{-3\pm\sqrt{(3)^2-4(10)(-1)}}{2(10)}$$

$$x=\frac{-3\pm\sqrt{9+40}}{20}=\frac{-3\pm\sqrt{49}}{20}=\frac{-3\pm 7}{20}$$

$$x=\frac{-3\pm 7}{20}\ or\ x=\frac{-3-7}{20}$$

$$x=\frac{4}{20}\quad or\quad x=\frac{-10}{20}$$

$$x=\frac{1}{5}\quad or\quad x=\frac{-1}{2}\ \Rightarrow\ S.S=\left\{-\frac{1}{2},\frac{1}{5}\right\}$$

3. $\boldsymbol{x^6-9x^3+8=0}$

Sol. $Put\quad x^3=y\Rightarrow x^6=y^2$

$$y^2-9y+8=0$$

$$or\qquad y^2-y-8y+8=0$$

$$or\qquad y(y-1)-8(y-1)=0\Rightarrow(y-1)(y-8)=0$$

$$\Rightarrow(y-1)=0\qquad or\quad (y-8)=0$$

$$y=1\quad or\quad y=8$$

$$when\qquad y=1\ then\ x^3=1\Rightarrow x^3-1=0$$

$$(x-1)(x^2+x+1)=0$$

$$x-1=0\quad or\quad x^2+x+1=0$$

$$x=1\quad or\quad x=\frac{-1\pm\sqrt{(1)^2-4(1)(1)}}{2(1)}$$

$$x = 1 \quad or \quad x = \frac{-1 \pm \sqrt{-3}}{2}$$

$$when \quad y = 8 \; then \quad x^3 = 8 \Rightarrow x^3 = (2)^3$$

$$x^3 - (2)^3 = 0 \Rightarrow (x-2)(x^2 + 2x + 4) = 0$$

$$x - 2 = 0 \qquad or \qquad x^2 + 2x + 4 = 0$$

$$x = 2 \quad or \quad x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{4-16}}{2} \quad \Rightarrow \quad x = \frac{-2 \pm \sqrt{-12}}{2}$$

$$x = \frac{-2 \pm 2\sqrt{-3}}{2} \Rightarrow x = \frac{2(-1 \pm \sqrt{-3})}{2} \quad \Rightarrow \quad x = -1 \pm \sqrt{-3}$$

$$S.S = \left\{1, 2, \frac{-1 \pm \sqrt{-3}}{2}, -1 \pm \sqrt{-3}\right\}$$

4. $\mathbf{8x^6 - 19x^3 - 27 = 0}$ **Multan 2008,**

Sol. $Put \quad x^3 = y \Rightarrow x^6 = y^2$

$$8y^2 - 19y - 27 = 0$$

$$or \quad 8y^2 + 8y - 27y - 27 = 0$$

$$8y(y+1) - 27(y+1) = 0$$

$$(y+1)(8y-27) = 0$$

$$y + 1 = 0 \quad or \quad 8y - 27 = 0$$

$$y = -1 \quad or \quad y = \frac{27}{8}$$

$$when \quad y = -1 \; then \; x^3 = -1$$

$$x^3 + 1 = 0 \qquad or \quad (x+1)(x^2 - x + 1) = 0$$

$$x + 1 = 0 \quad or \quad x^2 - x + 1 = 0$$

$$x = -1 \quad or \quad x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{1-4}}{2} \; x = \frac{1 \pm \sqrt{-3}}{2} \Rightarrow x = \frac{1 \pm \sqrt{3}i}{2}$$

$$when \quad y = \frac{27}{8} \; then \quad x^3 = \frac{27}{8}$$

$$x^3 = \left(\frac{3}{2}\right)^3 \Rightarrow x^3 - \left(\frac{3}{2}\right)^3 = 0$$

$$\left(x - \frac{3}{2}\right)\left(x^2 + \frac{3x}{2} + \frac{9}{4}\right) = 0$$

$$x - \frac{3}{2} = 0 \ \ or \ \left(x^2 + \frac{3x}{2} + \frac{9}{4} = 0\right)$$

$$x = \frac{3}{2} \ or \ 4x^2 + 6x + 9 = 0 \ \ ('\times' \ by \ 4)$$

$$x = \frac{-6 \pm \sqrt{(6)^2 - 4(4)(9)}}{2(4)} = \frac{-6 \pm \sqrt{-108}}{8}$$

$$x = \frac{-6 \pm 2\sqrt{-27}}{8} = \frac{\cancel{2}(-3 \pm 3\sqrt{-3})}{\cancel{8}}$$

$$x = \frac{3(-1 \pm \sqrt{-3})}{4} = \frac{3(-1 \pm \sqrt{3}i)}{4}$$

$$S.S = \left\{-1, \frac{3}{2}, \frac{1 \pm \sqrt{3}i}{2}, \frac{3(-1 \pm \sqrt{3}i)}{4}\right\}$$

5. $x^{2/5} + 8 = 6x^{1/5}$ ---------------------- **I**

Sol. $Put \ \ x^{1/5} = y \ \Rightarrow \ x^{2/5} = y^2$

$(I \ \ become) \ \ y^2 + 8 = 6y \ \ \Rightarrow \ \ y^2 - 6y + 8 = 0$

$$or \quad y^2 - 2y - 4y + 8 = 0$$

$$y(y-2) - 4(y-2) = 0$$

$$(y-2)(y-4) = 0$$

$$y - 2 = 0 \quad or \quad y - 4 = 0$$

$$y = 2 \quad or \quad y = 4$$

$when \quad y = 2 \ then \ x^{1/5} = 2$

$$\Rightarrow x = 2^5 \ \Rightarrow \ x = 32$$

$when \quad y = 4 \ then \ x^{1/5} = 4$

$$\Rightarrow x = 4^5 = 1024$$

$$S.S = \{32, 1024\}$$

6. $(x+1)(x+2)(x+3)(x+4) = 24$ **Multan 2009**

Sol. $or \ \ (x+1)(x+4)(x+2)(x+3) = 24$

$$(x^2 + 5x + 4)(x^2 + 5x + 6)$$

$$Put \ \ x^2 + 5x = y$$

$$(y+4)(y+6)=24$$

$$y^2+10y+24=24$$

$$y^2+10y+24-24 \Rightarrow y^2+10y=0$$

$$or \;\; y(y+10)=0$$

$$y=0 \quad or \quad y+10=0$$

$$y=0 \quad or \quad y=-10$$

$$when \qquad y=0 \; then \; x^2+5x=0$$

$$x(x+5)=0 \Rightarrow x=0 \; or \; x+5=0 \Rightarrow x=0 \; or \; x=-5$$

$$when \qquad y=-10 \; then \; x^2+5x=-10$$

$$x^2+5x+10=0$$

$$or \;\; x=\frac{-5\pm\sqrt{(5)^2-4(1)(10)}}{2(1)}$$

$$x=\frac{-5\pm\sqrt{25-40}}{2}=\frac{-5\pm\sqrt{-15}}{2}$$

$$S.S=\left\{0,-5,\frac{-5\pm\sqrt{-15}}{2}\right\}=\left\{0,-5,\frac{-5\pm\sqrt{15}i}{2}\right\}$$

7. $\mathbf{(x-1)(x+5)(x+8)(x+2)-880=0}$

Sol. $(x-1)(x+8)(x+5)(x+2)-880=0$

$$(x^2+7x-8)(x^2+7x+10)-880=0$$

$$Put \;\; x^2+7x=y$$

$$(y-8)(y+10)-880=0$$

$$y^2-8y+10y-80-880=0 \Rightarrow y^2+2y-960=0$$

$$or \;\; y^2+32y-30y-960=0$$

$$or \quad y(y+32)-30(y+32)=0$$

$$or \;\; (y+32)(y-30)=0$$

$$y+32=0 \quad or \quad y-30=0$$

$$y=-32 \quad or \quad y=30$$

$$when \qquad y=-32 \; then \; x^2+7x=-32$$

$$x^2+7x+32=0$$

$$or \;\; x=\frac{-7\pm\sqrt{(7)^2-4(1)(32)}}{2(1)}$$

$$x=\frac{-7\pm\sqrt{49-128}}{2}=\frac{-7\pm\sqrt{-79}}{2}$$

$when \quad y = 30 \Rightarrow x^2 + 7x = 30$

$or \quad x^2 + 7x - 30 = 0$

$or \quad x^2 + 10x - 3x - 30 = 0$

$x(x+10) - 3(x+10) = 0$

$(x+10)(x-3) = 0$

$x + 10 = 0 \;\; or \;\; x - 3 = 0$

$x = -10 \;\; or \;\; x = 3$

$$S.S = \left\{3, -10, \frac{-7 \pm \sqrt{-79}}{2}\right\} \quad = \left\{-10, 3, \frac{-7 \pm \sqrt{79}i}{2}\right\}$$

8. $(x-5)(x-7)(x+6)(x+4) - 504 = 0$

Sol. $(x-5)(x+4)(x-7)(x+6) - 504 = 0$

$(x^2 - x - 20)(x^2 - x - 42) - 504 = 0$

$Put \;\; x^2 - x = y$

$(y-20)(y-42) - 504 = 0$

$y^2 - 20y - 42y + 840 - 504 = 0$

$y^2 - 62 + 336 = 0$

$y^2 - 6y - 56y + 336 = 0$

$y(y-6) - 56(y-6) = 0$

$(y-6)(y-56) = 0$

$y - 6 = 0 \quad or \quad y - 56 = 0$

$y = 6 \quad or \quad y = 56$

$when \quad y = 6 \; then \; x^2 - x = 6$

$x^2 - x - 6 = 0$

$x^2 - 3x + 2x - 6 = 0$

$x(x-3) + 2(x-3) = 0$

$(x-3)(x+2) = 0 \quad \Rightarrow \quad x - 3 = 0 \; or \; x + 2 = 0 \;\Rightarrow\; x = 3 \; or \; x = -2$

$when \quad y = 56 \;\; then \;\; x^2 - x = 56$

$x^2 - x - 56 = 0$

$x^2 - 8x + 7x - 56 = 0$

$x(x-8) + 7(x-8) = 0$

$(x-8)(x+7) = 0$

$(x-8) = 0 \quad or \quad (x+7) = 0$

$x = 8 \quad or \quad x = -7$

9. $(x-1)(x-2)(x-8)(x+5)+360=0$

Sol. $(x^2-3x+2)(x^2-3x-40)+360=0$

$Put\ \ x^2-3x=y\ \ then$

$(y+2)(y-40)+360=0$

$y^2+2y-40y-80+360=0$

$y^2-38y+280=0$

$y^2-10y-28y+280=0$

$y(y-10)-28(y-10)=0$

$(y-10)(y-28)=0$

$y-10=0\quad or\quad y-28=0$

$y=10\quad or\quad y=28$

$when\qquad y=10\ then\ x^2-3x=10 \Rightarrow x^2-3x-10=0$

$$x=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(-10)}}{2(1)}$$

$$x=\frac{3\pm\sqrt{9+40}}{2}=\frac{3\pm\sqrt{49}}{2}=\frac{3\pm7}{2}$$

$$x=\frac{3+7}{2}\ and\ \frac{3-7}{2}$$

$$x=\frac{\cancel{10}}{\cancel{2}}=5\ and\ x=\frac{-4}{\cancel{2}}=-2$$

$when\qquad y=28\ \ then\ \ x^2-3x=28$

$x^2-3x-28=0$

$x^2-7x+4x-28=0$

$x(x-7)+4(x-7)=0$

$(x-7)(x+4)=0$

$x-7=0\quad or\quad x+4=0$

$x=7\quad or\quad x=-4$

$S.S=\{5,-2,7,-4\}$

10. $(x+1)(2x+3)(2x+5)(x+3)=945$

Sol. $(x+1)(x+3)(2x+3)(2x+5)=945$

$(x^2+x+3x+3)(4x^2+10x+6x+15)=945$

$(x^2+4x+3)(4x^2+16x+15)=945$

$(x^2+4x+3)\left[4(x^2+4x)+15)\right]=945$

$Put\ \ x^2+4x=y\ \ then$

$(y+3)(4y+15)=945$

$4y^2+15y+12y+45-945=0$

$4y^2+27y-900=0$

$4y^2+75y-48y-900=0$

$y(4y+75)-12(4y+75)=0$

$(4y+75)(y-12)=0$

$4y+75=0\quad or\quad y-12=0$

$y=\dfrac{-75}{4}\quad or\quad y=12$

$when\qquad y=\dfrac{-75}{4}\quad then\ x^2+4x=\dfrac{-75}{4}$

$or\quad 4x^2+16x=-75$

$4x^2+16x+75=0$

$$x=\frac{-16\pm\sqrt{(16)^2-4(4)(75)}}{2(4)}=\frac{-16\pm\sqrt{256-1200}}{8}$$

$$x=\frac{-16\pm\sqrt{-944}}{8}=\frac{-16\pm i\sqrt{944}}{8}=\frac{-16\pm i\sqrt{16\times 59}}{8}$$

$$x=\frac{-16\pm 4i\sqrt{59}}{8}=\frac{\cancel{4}(-4\pm i\sqrt{59})}{\cancel{8}}=\frac{(-4\pm i\sqrt{59})}{2}$$

$when\qquad y=12\ \ then\ \ x^2+4x=12$

$x^2+4x-12=0$

$x^2+6x-2x-12=0$

$x(x+6)-2(x+6)=0$

$(x+6)(x-\quad =0$

$x+6=0\quad or\quad x-2=0$

$x=-6\quad or\quad x=2$

$$S.S=\left\{2,-6,\frac{-4\pm i\sqrt{59}}{2}\right\}$$

11. $\mathbf{(2x-7)(x^2-9)(2x+5)-91=0}$

Sol. $(2x-7)(x+3)(x-3)(2x+5)-91=0$

$(2x^2+6x-7x-21)(2x^2+5x-6x-15)-91=0$

$(2x^2-x-21)(2x^2-x-15)-91=0$

$Put\ \ 2x^2 - x = y$

$(y-21)(y-15)-91=0$

$y^2-15y-21y+315-91=0$

$y^2-36y+224=0$

$y^2-8y-28y+224=0$

$y(y-8)-28(y-8)=0$

$(y-8)(y-28)=0$

$y-8=0 \quad or \quad y-28=0$

$y=8 \quad or \quad y=28$

$when \quad y=8 \quad then\ 2x^2-x=8$

$2x^2-x-8=0$

$$or \quad x=\frac{-(-1)\pm\sqrt{(-1)^2-4(2)(-8)}}{2(2)}$$

$$x=\frac{1\pm\sqrt{1+64}}{4}=\frac{1\pm\sqrt{65}}{4}$$

$when \quad y=28\ \ then\ \ 2x^2-x=28$

$2x^2-x-28=0$

$2x^2-8x+7x-28=0$

$2x(x-4)+7(x-4)=0$

$(x-4)(2x+7)=0$

$x-4=0 \quad or \quad 2x+7=0$

$$x=4 \quad or \quad x=\frac{-7}{2}$$

$$S.S=\left\{4,\frac{-7}{2},\frac{1\pm\sqrt{65}}{4}\right\}$$

12. $\mathbf{(x^2+6x+8)(x^2+14x+48)=105}$

Sol. $(x^2+2x+4x+8)(x^2+6x+8x+48)=105$

$[x(x+2)+4(x+2)][x(x+6)+8(x+6)]=105$

$(x+2)(x+4)(x+6)(x+8)=105$

$(x+2)(x+8)(x+4)(x+6)=105$

$(x^2+10x+16)(x^2+10x+24)=105$

$Put\ \ x^2+10x=y$

$(y+16)(y+24)=105$

$y^2+16y+24y+384-105=0$

$y^2+40y+279=0$

$y^2+9y+31y+279=0$

$y(y+9)+31(y+9)=0$

$(y+9)(y+31)=0$

$y=-9 \quad or \quad y=-31$

$when \quad y=-9 \quad then \; x^2+10x=-9$

$or \quad x^2+10x+9=0$

$x^2+x+9x+9=0$

$x(x+1)+9(x+1)=0 \Rightarrow (x+1)(x+9)=0$

$x+1=0 \quad or \quad x+9=0$

$x=-1 \quad or \quad x=-9$

$when \quad y=-31 \; then \; x^2+10x=-31$

$x^2+10x+31=0$

$$x=\frac{-10\pm\sqrt{(10)^2-4(1)(31)}}{2(1)}$$

$$x=\frac{-10\pm\sqrt{100-124}}{2}=\frac{-10\pm\sqrt{-24}}{2}=\frac{-10\pm\sqrt{4(-6)}}{2}$$

$$x=\frac{-10\pm2\sqrt{-6}}{2}=x=\frac{\cancel{2}(-5\pm\sqrt{6}i)}{\cancel{2}}\Rightarrow x=-5\pm\sqrt{6}$$

$S.S=\{-9,-1,-5\pm\sqrt{-6}\}$

13. $(x^2+6x-27)(x^2-2x-35)=385$

Sol. $or \quad (x^2+9x-3x-27)(x^2-7x+5x-35)=385$

$[x(x+9)-3(x+9)][x(x-7)+5(x-7)]=385$

$or \quad (x+9)(x-3)(x-7)(x+5)=385$

$or \quad (x-3)(x+5)(x+9)(x-7)=385$

$(x^2+2x-15)(x^2+2x-63)=385$

$Put \; x^2+2x=y \; then$

$(y-15)(y-63)=385$

$y^2-15y-63y+945-385=0$

$y^2-78y+560=0$

$y^2-8y-70y+560=0$

$y(y-8)-70(y-8)=0$

$(y-8)(y-70)=0$

$y=8 \quad or \quad y=70$

$when \quad y=8 \quad then\ x^2+2x=8$

$x^2+2x-8=0$

$x^2+4x-2x-8=0$

$x(x+4)+2(x+4)=0$

$(x+4)(x-2)=0$

$x+4=0 \quad or \quad x-2=0$

$x=-4 \quad or \quad x=2$

$when \quad y=70\ then\ x^2+2x=70$

$x^2+2x-70=0$

$$x=\frac{-2\pm\sqrt{(2)^2-4(1)(-70)}}{2(1)}$$

$$x=\frac{-2\pm\sqrt{4+280}}{2}=\frac{-2\pm\sqrt{284}}{2}=\frac{-2\pm\sqrt{4(71)}}{2}$$

$$x=\frac{-2\pm2\sqrt{71}}{2}=(-1\pm\sqrt{71})=-1\pm\sqrt{71}\Rightarrow S.S=\left\{2,-4,-1\pm\sqrt{71}\right\}$$

14. $\mathbf{4.2^{2x+1}-9.2^x+1=0}$ **Gujranwala 2009, Multan 2007** (just convert to quadratic)

Sol. $4.2.2^{2x}-9.2^x+1=0$

$8.2^{2x}-9.2^x+1=0$

$Put\ 2^x=y\Rightarrow 2^{2x}=y^2$

$8y^2-9y+1=0$

$8y^2-8y-y+1=0$

$8y(y-1)-1(y-1)=0$

$(y-1)(8y-1)=0$

$y-1=0 \quad or \quad 8y-1=0$

$y=1 \quad or \quad 8y=1$

$y=1 \quad or \quad y=\frac{1}{8}$

$when \quad y=1 \quad then\ 2^x=1\Rightarrow 2^x=2^0\Rightarrow x=0$

$when \quad y=\frac{1}{8}\ then\ 2^x=\frac{1}{8}=\frac{1}{2^3}$

$or \quad 2^x=2^{-3}\Rightarrow x=-3\Rightarrow S.S=\{0,-3\}$

15. $2^x + 2^{-x+6} - 20 = 0$ **Sargodha 2010, 11**

Sol. $2^x + 2^6 \times 2^{-x} - 20 = 0$

$2^x + 64 \times \dfrac{1}{2^x} - 20 = 0$

Put $2^x = y$ *then*

$y + \dfrac{64}{y} - 20 = 0$

'$\times$' *by* y *we get*

$y^2 + 64 - 20y = 0 \Rightarrow y^2 - 20y + 64 = 0$

$y^2 - 4y - 16y + 64 = 0$

$y(y-4) - 16(y-4) = 0$

$(y-4)(y-16) = 0$

$y - 4 = 0$ *or* $y - 16 = 0$

$y = 4$ *or* $y = 16$

when $y = 4$ *then* $2^x = 4 = 2^2$

$\Rightarrow \quad x = 2$

when $y = 16$ *then* $2^x = 16 = 2^4 \Rightarrow x = 4$

$S.S = \{2, 4\}$

16. $4^x - 3.2^{x+3} + 128 = 0$

Sol. $(2^2)^x - 3.2^3.2^x + 128 = 0$

$2^{2x} - 3.8.2^x + 128 = 0$

$2^{2x} - 24.2^x + 128 = 0$

Put $2^x = y \Rightarrow 2^{2x} = y^2$ *then*

$y^2 - 24y + 128 = 0$

$y^2 - 8y - 16y + 128 = 0$

$y(y-8) - 16(y-8) = 0$

$(y-8)(y-16) = 0$

$y - 8 = 0$ *or* $y - 16 = 0$

$y = 8$ *or* $y = 16$

when $y = 8$ *then* $2^x = 8 = 2^3 \Rightarrow \quad x = 3$

when $y = 16$ *then* $2^x = 16 = 2^4 \Rightarrow \quad x = 4$

$S.S = \{3, 4\}$

17. $3^{2x-1}-12.3^x+81=0$

Sol. $or\;\; 3^{2x}\cdot 3^{-1}-12.3^x+81=0$

$$\frac{3^{2x}}{3}-12.3^x+81=0$$

$$Put\;\; 3^x=y\Rightarrow 3^{2x}=y^2$$

$$\frac{y^2}{3}-12y+81=0$$

Multiplying by 3

$$y^2-36y+243=0$$

$$y^2-9y-27y+243=0$$

$$y(y-9)-27(y-9)=0$$

$$(y-9)(y-27)=0$$

$$y-9=0 \quad or \quad y-27=0$$

$$y=9 \quad or \quad y=27$$

$$when \quad y=9 \quad then\; 3^x=9\Rightarrow 3^x=3^2\Rightarrow x=2$$

$$when \quad y=27 \quad then\; 3^x=27=3^3\Rightarrow x=3$$

$$S.S=\{2,3\}$$

18. $\left(x+\frac{1}{x}\right)^2-3\left(x+\frac{1}{x}\right)-4=0$ **Federal**

Sol. $Put\;\; x+\frac{1}{x}=y\; then$

$$y^2-3y-4=0$$

$$y^2+y-4y-4=0$$

$$y(y+1)-4(y+1)=0$$

$$(y+1)(y-4)=0$$

$$y=-1 \quad or \quad y=4$$

$$when \quad y=-1 \quad then\; x+\frac{1}{x}=-1$$

$$or\;\; x^2+1=-x \quad or \quad x^2+x+1=0$$

$$x=\frac{-1\pm\sqrt{1-4}}{2(1)}=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm\sqrt{3}\,i}{2}$$

$$when \quad y=4 \quad then\; x+\frac{1}{x}=4$$

$$or \quad x^2+1=4x \quad or \quad x^2-4x+1=0$$

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)}$$

$$x=\frac{4\pm\sqrt{16-4}}{2}=\frac{4\pm\sqrt{12}}{2}$$

$$x=\frac{4\pm2\sqrt{3}}{2}=\frac{\cancel{2}(2\pm\sqrt{3})}{\cancel{2}}=2\pm\sqrt{3}$$

$$S.S=\left\{\frac{-1\pm\sqrt{3}\,i}{2},2\pm\sqrt{3}\right\}$$

19. $x^2+x-4+\frac{1}{x}+\frac{1}{x^2}=0$

Sol. $or \quad \left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-4=0$ ______ I

$$Put \quad x+\frac{1}{x}=y \quad then \Rightarrow x^2+\frac{1}{x^2}=+2=y^2$$

$$then \quad x^2+\frac{1}{x^2}=y^2-2$$

$$(y^2-2)+y-4=0 \quad (I \text{ become})$$

$$or \quad y^2+y-6=0$$

$$y^2+3y-2y-6=0$$

$$y(y+3)-2(y+3)=0$$

$$(y+3)(y-2)=0$$

$$y+3=0 \quad or \quad y-2=0$$

$$y=-3 \quad or \quad y=2$$

$$when \quad y=-3 \quad then\ x+\frac{1}{x}=-3$$

$$x^2+1=-3x \quad\Rightarrow\quad x^2+3x+1=0$$

$$x=\frac{-3\pm\sqrt{(3)^2-4(1)(1)}}{2(1)}$$

$$x=\frac{-3\pm\sqrt{9-4}}{2}=\frac{-3\pm\sqrt{5}}{2}$$

$$when \quad y=2 \quad then\ x+\frac{1}{x}=2.$$

$or \quad x^2+1=2x \quad \Rightarrow \quad x^2-2x+1=0$

$(x-1)^2=0 \quad \Rightarrow \quad x-1=0 \quad \Rightarrow \quad x=1 \quad \Rightarrow S.S=\left\{1, \dfrac{-3\pm\sqrt{5}}{2}\right\}$

20. $\left(x-\dfrac{1}{x}\right)^2+3\left(x+\dfrac{1}{x}\right)=0$ **Faisalabad 2008**

Sol. $or \quad (x^2+\dfrac{1}{x^2}-2)+3\left(x+\dfrac{1}{x}\right)=0 \_\_\_\_\_ I$

$Put \quad x+\dfrac{1}{x}=y \quad \Rightarrow \quad x^2+\dfrac{1}{x^2}=+2=y^2$

$then \quad x^2+\dfrac{1}{x^2}=y^2-2$

$y^2-2-2+3y=0 \; (I \; become)$

$\Rightarrow \quad y^2+3y-4=0$

$y^2+4y-y-4=0$

$y(y+4)-1(y+4)=0$

$(y+4)(y-1)=0$

$y+4=0 \quad or \quad y-1=0$

$y=-4 \quad or \quad y=1$

$when \quad y=-4 \quad then \; x+\dfrac{1}{x}=-4$

$or \quad x^2+1=-4x \quad or \quad x^2+4x+1=0$

$x=\dfrac{-4\pm\sqrt{(4)^2-4(1)(1)}}{2(1)}=\dfrac{-4\pm\sqrt{16-4}}{2}$

$x=\dfrac{-4\pm\sqrt{12}}{2}=\dfrac{-4\pm2\sqrt{3}}{2}=\dfrac{2(-2\pm\sqrt{3})}{2}=-2\pm\sqrt{3}$

$when \quad y=1 \quad then \; x+\dfrac{1}{x}=1$

$x^2+1=x \quad or \quad x^2-x+1=0$

$x=\dfrac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}$

$x=\dfrac{1\pm\sqrt{1-4}}{2}=\dfrac{1\pm\sqrt{-3}}{2}=\dfrac{1\pm\sqrt{3}i}{2} \Rightarrow S.S=\left\{-2\pm\sqrt{3}, \dfrac{1\pm\sqrt{3}i}{2}\right\}$

21. $2x^4-3x^3-x^2-3x+2=0$

Sol. *Divied by* x^2; $2x^2-3x-1-\frac{3}{x}+\frac{2}{x^2}=0 \Rightarrow 2x^2+\frac{2}{x^2}-3x-\frac{3}{x}-1=0$

or $2\left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)-1=0$ _____ I

Put $x+\frac{1}{x}=y$ *then*

$x^2+\frac{1}{x^2}+2=y^2 \quad \Rightarrow \quad x^2+\frac{1}{x^2}=y^2-2$

$2(y^2-2)-3y-1=0$ (I *become*)

or $2y^2-4-3y-1=0$

or $2y^2-3y-5=0$

$2y^2+2y-5y-5=0$

$2y(y+1)-5(y+1)=0$

$(y+1)(2y-5)=0$

$y+1=0 \quad$ *or* $\quad 2y-5=0$

$y=-1 \quad$ *or* $\quad y=5/2$

when $\quad y=-1$ *then* $x+\frac{1}{x}=-1$

or $x^2+1=-x$ *or* $x^2+x+1=0$

$$x=\frac{-1\pm\sqrt{(1)^2-4(1)(1)}}{2(1)}$$

$$x=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{-3}}{2}$$

when $\quad y=5/2$ *then* $x+\frac{1}{x}=\frac{5}{2} \Rightarrow \frac{x^2+1}{x}=\frac{5}{2}$

$\Rightarrow 2x^2+2=5x$ *or* $2x^2-5x+2=0$

$2x^2-4x-x+2=0$

$2x(x-2)-1(x-2)=0$

$(x-2)(2x-1)=0$

$x-2=0 \quad$ *or* $\quad 2x-1=0$

$x=2 \quad$ *or* $\quad x=1/2 \quad \Rightarrow \quad S.S=\left\{2,\frac{1}{2},\frac{-1\pm\sqrt{-3}}{2}\right\}$

22. $2x^4+3x^3-4x^2-3x+2=0$

Sol. *Divied by* x^2; $2x^2+3x-4-\frac{3}{x}+\frac{2}{x^2}=0$

or $2\left(x^2+\frac{1}{x^2}\right)+3\left(x-\frac{1}{x}\right)-4=0$ ______ I

Put $x-\frac{1}{x}=y \Rightarrow x^2+\frac{1}{x^2}-2=y^2$

then $x^2+\frac{1}{x^2}=y^2+2$

$2(y^2+2)+3y-4=0$ (*I become*)

$2y^2+4+3y-4=0$

or $2y^2+3y=0 \Rightarrow \quad y(2y+3)=0$

$y=0 \quad or \quad 2y+3=0$

$y=0 \quad or \quad y=-3/2$

when $y=0$ *then* $x-\frac{1}{x}=0$

or $x^2-1=0$

$(x-1)(x+1)=0$

$x-1=0 \quad or \quad x+1=0$

$x=1 \quad or \quad x=-1 \Rightarrow x=\pm 1$

when $y=-3/2$ *then* $x-\frac{1}{x}=-\frac{3}{2} \Rightarrow \frac{x^2-1}{x}=\frac{-3}{2}$

or $2x^2-2=-3x$

or $2x^2+3x-2=0$ *or* $2x^2+4x-x-2=0$

$2x(x+2)-1(x+2)=0$ *or* $(x+2)(2x-1)=0$

$x+2=0 \quad or \quad 2x-1=0$

$x=-2 \quad or \quad x=1/2$

$S.S=\left\{\pm 1,\frac{1}{2},-2\right\}$

23. $6x^4-35x^3+62x^2-35x+6=0$

Sol. *Divied by* x^2; $6x^2-35x+62-\dfrac{35}{x}+\dfrac{6}{x^2}=0$

or $6\left(x^2+\dfrac{1}{x^2}\right)-35\left(x+\dfrac{1}{x}\right)+62=0$ ------------I

Put $x+\dfrac{1}{x}=y \Rightarrow x^2+\dfrac{1}{x^2}+2=y^2$

$x^2+\dfrac{1}{x^2}=y^2-2$ *then* $6(y^2-2)-35y+62=0$ (*I become*)

or $6y^2-12-35y+62=0$

or $6y^2-35y+50=0$

$6y^2-15y-20y+50=0$

or $3y\ (2y-5)-10(2y-5)=0$

$(2y-5)(3y-10)=0$

$2y-5=0$ *or* $3y-10=0$

$y=5/2$ *or* $y=10/3$

when $y=5/2$ *then* $x+\dfrac{1}{x}=5/2 \Rightarrow \dfrac{x^2+1}{x}=\dfrac{5}{2}$

or $2x^2+2=5x$

$2x^2-5x+2=0$ *or* $2x^2-x-4x+2=0$

$x(2x-1)-2(2x-1)=0$

$(2x-1)(x-2) \Rightarrow 2x-1=0$ *or* $x-2=0$

$x=1/2$ *or* $x=2$

when $y=10/3$ *then* $x+\dfrac{1}{x}=10/3 \Rightarrow \dfrac{x^2+1}{x}=\dfrac{10}{3}$

$3x^2+3=10x$

or $3x^2-10x+3=0$

$3x^2-x-9x+3=0$

$x(3x-1)-3(3x-1)=0$

$(3x-1)(x-3)=0$

$3x-1=0$ *or* $x-3=0$

$x=1/3$ *or* $x=3$

$S.S=\left\{2,\dfrac{1}{2},3,\dfrac{1}{3}\right\}$

24. $x^4-6x^2+10-\dfrac{6}{x^2}+\dfrac{1}{x^4}=0$

$$\left(x^4+\frac{1}{x^4}\right)-6\left(x^2+\frac{1}{x^2}\right)+10=0 \_\_\_\_\_\_ I$$

Sol. $Put\ x^2+\dfrac{1}{x^2}=y \Rightarrow x^4+\dfrac{1}{x^4}+2=y^2$

$x^4+\dfrac{1}{x^4}=y^2-2$

$then\ \ y^2-2-6y+10=0\ \ (I\ become)$

$y^2-2y-4y+8=0$

$y(y-2)-4(y-2)=0$

$(y-2)(y-4)=0$

$y-2=0 \quad or \quad y-4=0$

$y=2 \quad or \quad y=4$

$when \quad y=2 \ \ then\ x^2+\dfrac{1}{x^2}=2$

$or\ \ x^4+1=2x^2 \quad or \quad x^4-2x^2+1=0$

$(x^2-1)^2=0 \quad or \quad x^2-1=0$

$(x-1)(x+1)=0$

$x-1=0 \quad or \quad x+1=0$

$x=1 \quad or \quad x=-1$

$when \quad y=4 \ \ then\ x^2+\dfrac{1}{x^2}=4$

$or\ \ x^4+1=4x^2 \quad or \quad x^4-4x^2+1=0$

$Put\ x^2=t\ then\ t^2-4t+1=0$

$$t=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)}=\frac{4\pm\sqrt{(16)-4}}{2}$$

$$t=\frac{4\pm\sqrt{(12)}}{2}=\frac{4\pm2\sqrt{(3)}}{2}=\frac{2(2\pm\sqrt{3})}{2}=2\pm\sqrt{3}$$

$when \quad t=2\pm\sqrt{3}\ then\ x^2=2\pm\sqrt{3}$

$x=\pm\sqrt{2\pm\sqrt{3}}$

$S.S=\left\{-1,1,\pm\sqrt{2\pm\sqrt{3}}\right\}$

xercise 4.3

Solve the following equations:

1. $\mathbf{3x^2+2x-\sqrt{3x^2+2x-1}=3}$

Sol. $3x^2+2x-\sqrt{3x^2+2x-1}=3$ -------- I

put $\sqrt{3x^2+2x-1}=y$ --- II $\Rightarrow 3x^2+2x-1=y^2$ *or* $3x^2+2x=y^2+1$

(I *become*) $y^2+1-y=3 \quad \Rightarrow \quad y^2+1-y-3=0$

or $\quad y^2-y-2=0$ *or* $\quad y^2-2y+y-2=0$

$y(y-2)+1(y-2)=0$

$(y-2)(y+1)=0 \quad \Rightarrow \quad y-2=0$ *or* $y+1=0$

$y=2 \qquad$ *or* $\qquad y=-1$

when $y=-1$ *then* $\sqrt{3x^2+2x-1}=-1 \qquad$ (*Use II*)

$\Rightarrow \quad 3x^2+2x-1=1$ *or* $\quad 3x^2+2x-1-1=0$

$3x^2+2x-2=0$

$$x=\frac{-2\pm\sqrt{(2)^2-4(3)(-2)}}{2(3)}=\frac{-2\pm\sqrt{4+24}}{6}=\frac{-2\pm\sqrt{28}}{6}$$

$$x=\frac{-2\pm\sqrt{2\times2\times7}}{6}=\frac{-2\pm2\sqrt{7}}{6}$$

$$x=\frac{\cancel{2}(-1\pm\sqrt{7})}{\cancel{2}\times3}=\frac{-1\pm\sqrt{7}}{3}$$

when $y=2$ *then* $\sqrt{3x^2+2x-1}=2$

$\Rightarrow 3x^2+2x-1=4 \quad \Rightarrow \quad 3x^2+2x-1-4=0$

$3x^2+2x-5=0$

$3x^2+5x-3x-5=0$

$x(3x+5)-1(3x+5)=0$

$(x-1)(3x+5)=0$ *or* $x-1=0$ *or* $3x+5=0$

$x=1 \qquad$ *or* $\qquad x=-5/3$

CHECKING $\qquad$ *For* $x=1$, I *become*

$$3(1)^2+2(1)-\sqrt{3(1)^2+2(1)-1}=3$$

$$3+2-\sqrt{3+2-1}=3$$

$$5-\sqrt{4} \Rightarrow \quad 5-2=3$$

$$3=3 \; TRUE$$

For $x=-5/3$ *, I become*

$$3(-5/3)^2+2(-5/3)-\sqrt{3(-5/3)^2+2(-5/3)-1}=3$$

$$3\left(\frac{25}{9}\right)-\frac{10}{3}-\sqrt{3\left(\frac{25}{9}\right)-\frac{10}{3}-1}=3$$

$$\frac{75}{9}-\frac{10}{3}-\sqrt{\frac{75}{9}-\frac{10}{3}-1}=3$$

$$\frac{75-30}{9}-\sqrt{\frac{75-30-9}{9}}=3$$

$$\frac{45}{9}-\sqrt{\frac{36}{9}}=3 \Rightarrow \quad \frac{45}{9}-\frac{6}{3}=3$$

$$5-2=3 \quad \Rightarrow \quad 3=3 \; TRUE$$

for $x=\dfrac{-1+\sqrt{7}}{3}$ *, I become*

$$3\left(\frac{-1+\sqrt{7}}{3}\right)^2+2\left(\frac{-1+\sqrt{7}}{3}\right)-\sqrt{3\left(\frac{-1+\sqrt{7}}{3}\right)^2+2\left(\frac{-1+\sqrt{7}}{3}\right)-1}=3$$

$$\cancel{3}\left(\frac{1+7-2\sqrt{7}}{3\,\cancel{9}}\right)+\frac{-2+2\sqrt{7}}{3}-\sqrt{\cancel{3}\left(\frac{1+7-2\sqrt{7}}{3\,\cancel{9}}\right)+\frac{-2+2\sqrt{7}}{3}-1}=3$$

$$\frac{1+7-2\sqrt{7}}{3}+\frac{(-2)+2\sqrt{7}}{3}-\sqrt{+\frac{8-2\sqrt{7}}{3}+\frac{-2+2\sqrt{7}-1}{3}}=3$$

$$\frac{8-\cancel{2\sqrt{7}}-2+\cancel{2\sqrt{7}}}{3}-\sqrt{\frac{8-\cancel{2\sqrt{7}}-2+\cancel{2\sqrt{7}}-1}{3}}=3$$

$$2-\sqrt{1}=3 \Rightarrow 1\neq 3 \; FALSE$$

Similarly $x=\dfrac{-1-\sqrt{7}}{3}$ *is FALSE* $\Rightarrow S.S=\left\{1,\dfrac{-5}{3}\right\}$ *and Extraneous roots are* $\dfrac{-1\pm\sqrt{7}}{3}$

2. $x^2 - \frac{x}{2} - 7 = x - 3\sqrt{2x^2 - 3x + 2}$

Sol. $x^2 - \frac{x}{2} - 7 = x - 3\sqrt{2x^2 - 3x + 2}$ ——— I

Multiply by '2'

$2x^2 - x - 14 = 2x - 6\sqrt{2x^2 - 3x + 2}$

$2x^2 - x - 14 = 2x - 6\sqrt{2x^2 - 3x + 2} = 0$

$2x^2 - 3x - 14 + 6\sqrt{2x^2 - 3x + 2} = 0$ ——— II

Put $\sqrt{2x^2 - 3x + 2} = y$ ——— III

$\Rightarrow \quad 2x^2 - 3x + 2 = y^2 \quad or \quad 2x^2 - 3x = y^2 - 2$

$(II\ become)\ y^2 - 2 - 14 + 6y = 0$

$y^2 - 2 - 14 + 6y = 0 \quad \Rightarrow \quad y^2 + 6y - 16 = 0$

$y^2 + 8y - 2y - 16 = 0 \Rightarrow y(y+8) - 2(y+8) = 0$

$(y+8)(y-2) = 0$

$y + 8 = 0 \quad or \quad y - 2 = 0$

$y = -8 \quad or \quad y = 2$

$when\ y = -8\ then\ \sqrt{2x^2 - 3x + 2} = -8 \quad Use\ III \Rightarrow 2x^2 - 3x + 2 = 64$

$2x^2 - 3x - 62 = 0 \Rightarrow x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-62)}}{2(2)}$

$\Rightarrow x = \frac{3 \pm \sqrt{9 + 496}}{4} \Rightarrow x = \frac{3 \pm \sqrt{505}}{4}$

$When\ y = 2; \sqrt{2x^2 - 3x + 2} = 2 \Rightarrow 2x^2 - 3x + 2 = 4 \Rightarrow 2x^2 - 3x + 2 - 4 = 0$

$2x^2 - 3x - 2 = 0 \Rightarrow 2x^2 - 4x + x - 2 = 0$

$2x(x-2) = 1(x-2) = 0$

$(x-2)(2x+1) = 0$

$x - 2 = 0 \quad or \quad 2x + 1 = 0$

$x = 2 \quad or \quad x = \frac{-1}{2}$

Checking *For* $x = 2$, *I become*

$(2)^2 - \frac{2}{2} - 7 = 2 - 3\sqrt{2(2)^2 - 3(2) + 2}$

$$4-1-7=2-3\sqrt{8-6+2}$$

$$-4=2-3\sqrt{4}\ \Rightarrow -4=2-3(2)$$

$$-4=-4\ \ TRUE$$

For $\quad x=\dfrac{-1}{2}, I\ become$

$$\left(\frac{-1}{2}\right)^2-\frac{\left(\frac{-1}{2}\right)}{2}-7=-\frac{1}{2}-3\sqrt{2\left(\frac{-1}{2}\right)^2-3\left(\frac{-1}{2}\right)+2}$$

$$\frac{1}{4}+\frac{1}{4}-7=\frac{1}{2}-3\sqrt{2\left(\frac{1}{4}\right)+\frac{3}{2}+2}$$

$$\frac{2}{4}-7=-\frac{1}{2}-3\sqrt{\frac{1}{2}+\frac{3}{2}+2}$$

$$\frac{1}{2}-7=-\frac{1}{2}-3\sqrt{\frac{1+2+4}{2}}$$

$$\frac{-13}{2}=-\frac{1}{2}-3\sqrt{\frac{8}{2}}$$

$$\frac{-13}{2}=-\frac{1}{2}-3\sqrt{4}$$

$$\frac{-13}{2}=-\frac{1}{2}-6\ \Rightarrow\ \frac{-13}{2}=\frac{-13}{2}\quad TRUE$$

For $\quad x=\dfrac{3+\sqrt{505}}{4},\ I\ become$

$$\left(\frac{3+\sqrt{505}}{4}\right)^2-\frac{1}{2}\left(\frac{3+\sqrt{505}}{4}\right)-7=-\frac{1}{2}-3\sqrt{2\left(\frac{3+\sqrt{505}}{4}\right)^2-3\left(\frac{3+\sqrt{505}}{4}\right)+2}$$

$$\frac{9+505+6\sqrt{505}}{16}-\left(\frac{3+\sqrt{505}}{8}\right)-7=-\frac{1}{2}-3\sqrt{2\left(\frac{9+505+6\sqrt{505}}{16}\right)-\left(\frac{9+3\sqrt{505}}{4}\right)+2}$$

$$\frac{514+6\sqrt{505}-6-2\sqrt{505}-112}{16}=-\frac{1}{2}-3\sqrt{\left(\frac{9+505+6\sqrt{505}}{8}\right)-\left(\frac{9+3\sqrt{505}}{4}\right)+2}$$

$$\frac{396+4\sqrt{505}}{16}=-\frac{1}{2}-3\sqrt{\frac{514+6\sqrt{\cancel{505}}-18-6\sqrt{\cancel{505}}+16}{8}}$$

$$\frac{396+4\sqrt{505}}{16}=-\frac{1}{2}-3\sqrt{\frac{512}{8}}$$

$$\frac{396+4\sqrt{505}}{16}=-\frac{1}{2}-3\sqrt{64}$$

$$\frac{396+4\sqrt{505}}{16}=-\frac{1}{2}-3(8) \qquad \textit{FALSE}$$

$$\textit{Similarly } x=\frac{3-\sqrt{505}}{4} \textit{ is FALSE}$$

$$S.S=\left\{2,-\frac{1}{2}\right\} \textit{ and Extraneous roots}=\frac{3\pm\sqrt{505}}{4}$$

03. $\sqrt{2x+8}+\sqrt{x+5}=7$

Sol. $\sqrt{2x+8}+\sqrt{x+5}=7$ ________ I

Squaring both sides

$$2x+8+x+5+2\sqrt{2x+8}\sqrt{x+5}=49$$
$$3x+13+2\sqrt{(2x+8)(x+5)}=49$$
$$2\sqrt{2x^2+10x+8x+40)}=49-13-3x$$
$$2\sqrt{2x^2+18x+40}=36-3x$$

Again Squaring.

$$4(2x^2+18x+40)=1296+9x^2-216x$$
$$8x^2+72x+160=9x^2-216x+1296$$
$$9x^2-216x+1296-8x^2-72x-160=0$$
$$x^2-288x+1136=0$$
$$x^2-4x-284x+1136=0$$
$$x(x-4)-284(x-4)=0$$
$$(x-4)(x-284)=0$$
$$x-4=0 \quad or \quad x-284=0$$
$$x=4 \quad or \quad x=284$$

CHECKING for x = 4, I become

$$\sqrt{2(4)+8}+\sqrt{4+5}=7$$
$$\sqrt{16}+\sqrt{9}=7 \quad \Rightarrow \quad 4+3=7 \quad \Rightarrow \quad 7=7 \textit{ TRUE}$$

For x=284 I become

$\sqrt{2(284)+8}+\sqrt{284+5}=7$

$\sqrt{256+8}+\sqrt{289}=7$

$\sqrt{276}+\sqrt{289}=7$

$\sqrt{276}+\sqrt{289}=7$

24+17 = 7 **FALSE**

$S.S=\{4\}$ *and* Extraneous Root = 284

04. $\boldsymbol{\sqrt{3x+4}=2+\sqrt{2x-4}}$

Sol. $\sqrt{3x+4}=2+\sqrt{2x-4}$

Squaring both sides

$3x+4=4+2x-4+2(2)\sqrt{2x-4}$

$3x+4-2x=4\sqrt{2x-4}$

$x+4=4\sqrt{2x-4}$

Again Squaring

$x^2+8x+16=16(2x-4)$

$x^2+8x+16=32x-64$

$x^2+8x+16-32x+64=0$

$x^2-24x+80=0$

$x^2-4x-20x+80=0$

$x(x-4)-20(x-4)=0$

$(x-4)(x-20)=0$

$x-4=0 \quad or \quad x-20=0$

$x=4 \quad or \quad x=20$

Checking for x=4 I become

$\sqrt{3(4)+4}=2+\sqrt{2(20)-4}$

$\sqrt{16}=2+\sqrt{4}$

$4=2+2=4 \quad \Rightarrow \quad 4=4$ *True*

For x=20 I become

$\sqrt{3(20)+4}=2+\sqrt{2(4)-4}$

$\sqrt{60+4}=2+\sqrt{40-4}$

$\sqrt{64}=2+\sqrt{36} \quad \Rightarrow \quad 8=2+6$

8 = 8 **TRUE**

$S.S=\{4,20\}$

05. $\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$ **Faisalabad 2007, Sargodha 2008**

Sol. $\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$ ______ I

Squaring both sides

$$x+7+x+2+2\sqrt{x+7}\sqrt{x+2}=6x+13$$

$$2x+9+2\sqrt{(x+7)(x+2)}=6x+13$$

$$2\sqrt{x^2+7x+2x+14}=6x+13-2x-9$$

$$\cancel{2}\sqrt{x^2+9x+14}=4x+4=\cancel{2}(2x+2)$$

$$\sqrt{x^2+9x+14}=2x+2$$

Again Squaring

$$x^2+9x+14=4x^2+4+8x$$

$$4x^2+8x+4-x^2-9x-14=0$$

$$3x^2-x-10=0$$

$$3x^2-6x+5x-10=0$$

$$3x(x-2)+5(x-2)=0$$

$$(x-2)(3x+5)=0$$

$x-2=0$ *or* $3x+5=0$

$x=2$ *or* $x=-5/3$

CHECKING for x = 2 I become

$$\sqrt{2+7}+\sqrt{2+2}=\sqrt{6(2)+13}$$

$$\sqrt{9}+\sqrt{4}=\sqrt{25} \Rightarrow 3+2=5$$

5 = 5 **TRUE**

For x = -5/3 I become

$$\sqrt{\frac{-5}{3}+7}+\sqrt{\frac{-5}{3}+2}=\sqrt{6\left(\frac{-5}{3}\right)+13}$$

$$\sqrt{\frac{-5+21}{3}}+\sqrt{\frac{-5+6}{3}}=\sqrt{\frac{-30}{3}+13}$$

$$\sqrt{\frac{16}{3}}+\sqrt{\frac{1}{3}}=\sqrt{\frac{-30+39}{3}}$$

$$\frac{4}{\sqrt{3}}+\frac{1}{\sqrt{3}}=\sqrt{\frac{9}{3}} \Rightarrow \frac{5}{\sqrt{3}}=\sqrt{3} \quad \textbf{FALSE}$$

$$S.S=\{2\} \quad \textit{and} \text{ Extraneous Root } =\frac{-5}{3}$$

06. $\sqrt{x^2+x+1}-\sqrt{x^2+x-1}=1$

Sol. $\sqrt{x^2+x+1}-\sqrt{x^2+x-1}=1$ ________ I

Put $x^2+x=y$ *then* ________ II

$\sqrt{y+1}-\sqrt{y-1}=1$

Squaring

$$y+1+y-1-2\sqrt{y+1}\sqrt{y-1}=1$$

$$2y-2\sqrt{y^2-1}=1$$

$$-2\sqrt{y^2-1}=1-2y$$

Squaring

$$4(y^2-1)=1+4y^2-4y$$

$$4y^2-4=1+4y^2-4y$$

$$\cancel{4y^2}-4y+1-\cancel{4y^2}+4=0$$

$$-4y+5=0 \quad \Rightarrow \quad y=\frac{5}{4}$$

When $y=\frac{5}{4}$ *then* $x^2+x=\frac{5}{4}$ *Use* II

$$x^2+x-\frac{5}{4}=0$$

'×' *by* 4, $4x^2+4x-5=0$

$$x=\frac{-4\pm\sqrt{(4)^2-4(4)(-5)}}{2(4)}$$

$$x=\frac{-4\pm\sqrt{16+80}}{8}=\frac{-4\pm\sqrt{96}}{8}$$

$$x=\frac{-4\pm\sqrt{16\times 6}}{8}=\frac{-4\pm 4\sqrt{6}}{8}$$

$$x\frac{4(-1\pm\sqrt{6})}{8}=\frac{-1\pm\sqrt{6}}{2}$$

CHECKING for $x=\dfrac{-1+\sqrt{6}}{2}$ I become

$$\sqrt{\left(\frac{-1+\sqrt{6}}{2}\right)^2+\left(\frac{-1+\sqrt{6}}{2}\right)+1}-\sqrt{\left(\frac{-1+\sqrt{6}}{2}\right)^2+\left(\frac{-1+\sqrt{6}}{2}\right)-1}=1$$

$$\sqrt{\frac{1+6-2\sqrt{6}}{4}+\frac{-1+\sqrt{6}}{2}+1}-\sqrt{\frac{1+6-2\sqrt{6}}{4}+\frac{-1+\sqrt{6}}{2}-1}=1$$

$$\sqrt{\frac{7-\cancel{2\sqrt{6}}-2+\cancel{2\sqrt{6}}+4}{4}}-\sqrt{\frac{7-2\sqrt{6}-2+2\sqrt{6}-4}{4}}=1$$

$$\sqrt{\frac{9}{4}}-\sqrt{\frac{1}{4}}=1 \Rightarrow \frac{3}{2}-\frac{1}{2}=1 \Rightarrow 1=1$$

Similarly For $x=\dfrac{-1-\sqrt{6}}{2}$

$$S.S=\left\{\frac{-1\pm\sqrt{6}}{2}\right\}$$

07. $\sqrt{x^2+2x-3}+\sqrt{x^2+7x-8}=\sqrt{5(x^2+3x-4)}$

Sol. $\sqrt{x^2+2x-3}+\sqrt{x^2+7x-8}=\sqrt{5(x^2+3x-4)}$

$$\sqrt{x^2+3x-x-3}+\sqrt{x^2+8x-x-8}=\sqrt{5(x^2+4x-x-4)}$$

$$\sqrt{x(x+3)-1(x+3)}+\sqrt{x(x+8)-1(x+8)}=\sqrt{5(x(x+4)-1(x+4))}$$

$$\sqrt{(x+3)(x-1)}+\sqrt{(x+8)(x-1)}-\sqrt{5(x+4)(x-1)}=0$$

$$\sqrt{x-1}\left[\sqrt{x+3}+\sqrt{x+8}-\sqrt{5(x+4)}\right]=0$$

$$\sqrt{x-1}=0 \quad or \quad \sqrt{x+3}+\sqrt{x+8}-\sqrt{5(x+4)}=0$$

$$x-1=0 \Rightarrow x=1 \quad or \quad \sqrt{x+3}+\sqrt{x+8}=\sqrt{5(x+4)}$$

Squaring both sides

$$x+3+x+8+2\sqrt{x+3}\sqrt{x+8}=5(x+4)$$

$$2\sqrt{(x+3)(x+8)}=5x+20-2x-11$$

$$2\sqrt{x^2+3x+8x+24}=3x+9$$

$$2\sqrt{x^2+11x+24}=3x+9$$

Again Squaring both sides

$$4(x^2+11x+24)=9x^2+81+54x$$

$$4x^2+44x+96=9x^2+54x+81$$

$$9x^2+54x+81-4x^2-44x-96=0$$

$$5x^2+10x-15=0$$

$$\div \text{ by } 5 \Rightarrow x^2+2x-3=0$$

$$x^2+3x-x-3=0$$

$x(x+3)-1(x+3)=0$

$(x+3)(x-1)=0 \quad \Rightarrow \quad x+3=0 \quad or \quad x-1=0$

$x=-3 \quad or \quad x=1$

CHECKING for x=1, I become

$\sqrt{(1)^2+2(1)-3}+\sqrt{(1)^2+7(1)-8}=\sqrt{5((1)^2+3(1)-4)}$

$\sqrt{1+2-3}+\sqrt{1+7-8}=\sqrt{5(1+3-4)}$

$\sqrt{0}+\sqrt{0}=\sqrt{0} \quad \Rightarrow \quad 0=0$ **True**

$For \quad x=-3$

$\sqrt{(-3)^2+2(-3)-3}+\sqrt{(-3)^2+7(-3)-8}=\sqrt{5((-3)^2+3(-3)-4)}$

$\sqrt{9-6-3}+\sqrt{9-21-8}=\sqrt{5(9-9-4)}$

$0+\sqrt{-20}=\sqrt{-20}$ **True**

$S.S=\{1,-3\}$

08. $\sqrt{2x^2-5x-3}+3\sqrt{2x+1}=\sqrt{2x^2+25x+12}$ ______ I

Sol. $\sqrt{2x^2-5x-3}+3\sqrt{2x+1}=\sqrt{2x^2+25x+12}$

$\sqrt{2x^2-6x+x-3}+3\sqrt{2x+1}-\sqrt{2x^2+x+24x+12}=0$

$\sqrt{2x(x-3)+1(x-3)}+3\sqrt{2x+1}-\sqrt{x(2x+1)+12(2x+1)}=0$

$\sqrt{(x-3)(2x+1)}+3\sqrt{2x+1}-\sqrt{(2x+1)(x+12)}=0$

$\sqrt{2x+1}\left[\sqrt{x-3}+3-\sqrt{x+12}\right]=0$

$\sqrt{2x+1} \quad or \quad \sqrt{x-3}+3-\sqrt{x+12}=0$

$2x+1=0 \quad \Rightarrow x=-\frac{1}{2} or \quad \sqrt{x-3}+3=\sqrt{x+12}$

Squaring both sides

$x-3+9+6\sqrt{x-3}=x+12$

$x+6+6\sqrt{x-3}=x+12$

$6\sqrt{x-3}=x+12-x-6=6$

$\sqrt{x-3}=\frac{6}{6}=1 \Rightarrow x-3=1 \Rightarrow x=4$

CHECKING for x = 4 I become

$\sqrt{2(4)^2-5(4)-3}+3\sqrt{2(4)+1}=\sqrt{2(4)^2+25(4)+12}$

$\sqrt{32-20-3}+3\sqrt{8+1}=\sqrt{32+100+12}$

$\sqrt{9}+3\sqrt{9}=\sqrt{144} \quad \Rightarrow \quad 3+3(3)=12$

$$12 = 12 \quad TRUE$$

For $x = \frac{-1}{2}$ I *become*

$$\sqrt{2\left(\frac{-1}{2}\right)^2 - 5\left(-\frac{1}{2}\right) - 3} + 3\sqrt{2\left(-\frac{1}{2}\right) + 1} = \sqrt{2\left(\frac{-1}{2}\right)^2 + 25\left(-\frac{1}{2}\right) + 12}$$

$$\sqrt{2\left(\frac{1}{4}\right) + \frac{5}{2} - 3} + 3\sqrt{-1+1} = \sqrt{2\left(\frac{1}{4}\right) - \frac{25}{2} + 12}$$

$$\sqrt{\frac{1}{2} + \frac{5}{2} - 3} + 3(0) = \sqrt{\frac{1}{2} - \frac{25}{2} + 12}$$

$$\sqrt{\frac{1+5-6}{2}} + 0 = \sqrt{\frac{1-25+24}{2}}$$

$0 = 0$ **TRUE** $\quad \Rightarrow S.S = \left\{\frac{-1}{2}, 4\right\}$

09. $\sqrt{3x^2 - 5x + 2} + \sqrt{6x^2 - 11x + 5} = \sqrt{5x^2 - 9x + 4}$ -------I

Sol. $\sqrt{3x^2 - 5x + 2} + \sqrt{6x^2 - 11x + 5} = \sqrt{5x^2 - 9x + 4}$

$$\sqrt{3x^2 - 3x - 2x + 2} + \sqrt{6x^2 - 6x - 5x + 5} = \sqrt{5x^2 - 5x - 4x + 4}$$

or $\sqrt{3x(x-1) - 2(x-1)} + \sqrt{6x(x-1) - 5(x-1)} = \sqrt{5x(x-1) - 4(x-1)}$

or $\sqrt{(x-1)(3x-2)} + \sqrt{(x-1)(6x-5)} - \sqrt{(x-1)(5x-4)} = 0$

or $\sqrt{x-1}\left[\sqrt{3x-2} + \sqrt{6x-5} - \sqrt{5x-4}\right] = 0$

$\sqrt{x-1} = 0$ *or* $\sqrt{3x-2} + \sqrt{6x-5} - \sqrt{5x-4} = 0$

$x - 1 = 0$ *or* $\sqrt{3x-2} + \sqrt{6x-5} = \sqrt{5x-4}$

Squaring both sides

$x = 1$ *or* $3x - 2 + 6x - 5 + 2\sqrt{3x-2}\sqrt{6x-5} = 5x - 4$

$$\Rightarrow \quad 2\sqrt{(3x-2)(6x-5)} = 5x - 4 - 3x + 2 - 6x + 5$$

$$2\sqrt{18x^2 - 15x - 12x + 10} = -4x + 3 \Rightarrow 4(18x^2 - 27x + 10) = 16x^2 - 24x + 9$$

$$72x^2 - 108x + 140 - 16x^2 + 24x - 9 = 0 \Rightarrow 56x^2 - 84x + 31 = 0$$

$$(a = 56,\ b = -84,\ c = 31)\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-84) \pm \sqrt{(-84)^2 - 4(56)(31)}}{2(56)}$$

$$x = \frac{84 \pm \sqrt{7056 - 6944}}{112} \Rightarrow x = \frac{84 \pm \sqrt{112}}{112} = \frac{84 \pm \sqrt{16 \times 7}}{112}$$

$$x=\frac{84\pm 4\sqrt{7}}{112}=\cancel{4}\left(\frac{21\pm\sqrt{7}}{\cancel{112}}\right)\Rightarrow x=\frac{21\pm\sqrt{7}}{28}$$

CHECKING for x=1 (A) become

$$\sqrt{3(1)^2-5(1)+2}+\sqrt{6(1)^2-11(1)+5}=\sqrt{5(1)^2-9(1)+4}\Rightarrow\sqrt{3-5+2}+\sqrt{6-11+5}=\sqrt{5-9+4}$$

$0+0=0 \;\Rightarrow 0=0$ **True**

For $x=\dfrac{21+\sqrt{7}}{28}$ (A) *become*

$$\sqrt{3\left(\frac{21+\sqrt{7}}{28}\right)^2-5\left(\frac{21+\sqrt{7}}{28}\right)+2}+\sqrt{6\left(\frac{21+\sqrt{7}}{28}\right)^2-11\left(\frac{21+\sqrt{7}}{28}\right)+5}$$

$$\sqrt{5\left(\frac{21+\sqrt{7}}{28}\right)^2-9\left(\frac{21+\sqrt{7}}{28}\right)+4}$$

$$\sqrt{3\left(\frac{441+7+42\sqrt{7}}{784}\right)-5\left(\frac{21+\sqrt{7}}{28}\right)+2}+\sqrt{6\left(\frac{441+7+42\sqrt{7}}{784}\right)-11\left(\frac{21+\sqrt{7}}{28}\right)+5}$$

$$=\sqrt{5\left(\frac{441+7+42\sqrt{2}}{784}\right)-9\left(\frac{21+\sqrt{7}}{28}\right)+4}$$

$$\sqrt{\frac{1323+21+126\sqrt{7}-2940-140\sqrt{7}+1568}{784}}+$$

$$\sqrt{\frac{2646+42+252\sqrt{7}-6468-308\sqrt{7}+3920}{784}}$$

$$=\sqrt{\frac{2205+35+210\sqrt{7}-5292-252\sqrt{7}+3136}{784}}$$

$$\sqrt{\frac{-28-14\sqrt{7}}{28}}+\sqrt{\frac{141-56\sqrt{7}}{28}}=\sqrt{\frac{84-42\sqrt{7}}{28}}$$ Not true

Similarly

$x=\dfrac{21-\sqrt{7}}{28}$ not satisfied

So Extraneous roots are $x=\dfrac{21\pm\sqrt{7}}{28}$

& $S.S=\{1\}$

10. $(x+4)(x+1)=\sqrt{x^2+2x-15}+3x+31$ ______ (A)

Sol. $(x+4)(x+1)=\sqrt{x^2+2x-15}+3x+31$

$x^2+5x+4=\sqrt{x^2+2x-15}+3x+31$

$or\;\; x^2+5x+4-3x-31=\sqrt{x^2+2x-15}$

$x^2+2x-27=\sqrt{x^2+2x-15}$ (1)

$Put\; \sqrt{x^2+2x-15}=y$ (2)

$\Rightarrow\; x^2+2x-15=y^2 \;\Rightarrow\; x^2+2x=y^2+15$

(1) $become\; y^2+15-27=y \;\Rightarrow\; y^2-y-12=0 \;\Rightarrow\; y^2-4y+3y-12=0$

$(y-4)(y+3)=0 \;\Rightarrow y-4=0 \;or\; y+3=0$

$y=4 \;\; or \;\; y=-3$

$When\;\; y=4\; then\; \sqrt{x^2+2x-15}=4$ $By(2)$

$\Rightarrow\; x^2+2x-15=16 \;\Rightarrow\; x^2+2x-31=0$

$$x=\frac{-2\pm\sqrt{(2)^2-4(1)(-31)}}{2(1)}=\frac{-2\pm\sqrt{4+124}}{2}=\frac{-2\pm\sqrt{128}}{2}=\frac{-2\pm2\sqrt{32}}{2}$$

$$=\frac{\cancel{2}(-1\pm\sqrt{32})}{\cancel{2}}\Rightarrow x=(-1\pm\sqrt{32}) \quad\Rightarrow\quad x=-1\pm4\sqrt{2}$$

$When\;\; y=-3\; then\; \sqrt{x^2+2x-15}=-3$ $By(2)$

$\Rightarrow\; x^2+2x-15=9 \;\Rightarrow\; x^2+2x-24=0$

$x^2+6x-4x-24=0 \;\Rightarrow\; x(x+6)-4(x+6)=0$

$(x+6)(x-4)=0 \;\Rightarrow\; x+6=0 \;\; or \;\; x-4=0$

$x=-6 \;\; or \;\; x=4$

CHECKING For $x=-6$ (A) become

$(-6+4)(-6+1)=\sqrt{(-6)^2+2(-6)-15}+3(-6)+31$

$or\;\; (-2)(-5)=\sqrt{36-12-15}-18+31$

$10=\sqrt{9}+13 \;\Rightarrow 10=16$ **False**

For $x=4$ (A) *become*

$(4+4)(4+1)=\sqrt{(4)^2+2(4)-15}+3(4)+31$

$40=\sqrt{16+8-15}+12+31 \;\Rightarrow\; 40=\sqrt{9}+43 \;\Rightarrow\; 40=46$ **False**

For $x=-1+4\sqrt{2}$ (A) *become*

$(-1+4\sqrt{2}+4)(-1+4\sqrt{2}+1)=\sqrt{(-1+4\sqrt{2})^2+2(-1+4\sqrt{2})-15}+3(-1+4\sqrt{2})+31$

$(3+4\sqrt{2})(4\sqrt{2}) = \sqrt{1+32-\cancel{8\sqrt{2}}-2+\cancel{8\sqrt{2}}-15-3+12\sqrt{2}+31}$

$12\sqrt{2}+32 = \sqrt{16}+28+12\sqrt{2}$

$12\sqrt{2}+32 = 4+28+12\sqrt{2} \quad \Rightarrow \quad 32+12\sqrt{2} = 32+12\sqrt{2}$ **True**

Similarly $x = -1-4\sqrt{2}$ *is* **True**

$S.S = \left\{-1 \pm 4\sqrt{2}\right\}$ *and Extraneous roots* $4, -6$

11. $\sqrt{3x^2-2x+9}+\sqrt{3x^2-2x-4} = 13$

Sol. $\sqrt{3x^2-2x+9}+\sqrt{3x^2-2x-4} = 13 \qquad (1)$

Put $\sqrt{3x^2-2x+9} = a$ *and* $\sqrt{3x^2-2x-4} = b$

(1) Become $a + b = 13 \qquad (2)$

Now $a^2-b^2 = (3x^2-2x+9)-(3x^2-2x-4)$

$a^2-b^2 = 3x^2-2x+9-3x^2+2x+4 = 13 \quad (3)$

$(3) \div by\ (2)$

$\frac{a^2-b^2}{a+b} = \frac{13}{13} \quad \Rightarrow \quad \frac{(a-b)\cancel{(a+b)}}{\cancel{(a+b)}} = 1 \quad \Rightarrow \quad a-b = 1 \quad (4)$

Add (2) *and* (4) $a+b = 13$ *put value of a in* (2)

Add $\frac{a-\cancel{b} = 1}{2a = 14} \Rightarrow \boxed{a=7} \quad 7+b = 13 \Rightarrow \boxed{b=6}$

Put value of $a \Rightarrow \sqrt{3x^2-2x+9} = 7$

$3x^2-2x+9 = 49 \quad \Rightarrow 3x^2-2x+9-49 = 0$

$3x^2-2x-40 = 0 \quad \Rightarrow 3x^2-12x+10x-40 = 0$

$3x(x-4)+10(x-4) = 0 \quad \Rightarrow (x-4)(3x+10) = 0$

$x-4 = 0 \quad or \quad 3x+10 = 0 \quad \Rightarrow \quad x = 4 \ or \ x = \frac{-10}{3}$

CHECKING for x =4 (1) become

$\sqrt{3(4)^2-2(4)+9}+\sqrt{3(4)^2-2(4)-4} = 13$

$\sqrt{48-8+9}+\sqrt{48-8-4} = 13 \quad \Rightarrow \quad \sqrt{49}+\sqrt{36} = 13$

$7+6 = 13 \quad \Rightarrow \quad 13 = 13$ **True**

For $x = \frac{-10}{3}$

$\sqrt{3\left(\frac{-10}{3}\right)^2-2\left(\frac{-10}{3}\right)+9}+\sqrt{3\left(\frac{-10}{3}\right)^2-\left(\frac{-10}{3}\right)-4} = 13$

$$\sqrt{3\left(\frac{100}{9}\right)+\frac{20}{3}+9}+\sqrt{3\left(\frac{100}{9}\right)+\frac{20}{3}-4}=13$$

$$\sqrt{\frac{300+60+81}{9}}+\sqrt{\frac{300+60-36}{9}}=13$$

$$\sqrt{\frac{441}{9}}+\sqrt{\frac{324}{9}}=13 \quad \Rightarrow \quad \frac{21}{3}+\frac{18}{3}=13$$

$$7+6=13 \quad \Rightarrow \quad 13=13 \quad \textbf{True}$$

$$S.S=\left\{4,\frac{-10}{3}\right\}$$

12. $\sqrt{5x^2+7x+2}-\sqrt{4x^2+7x+18}=x-4$

Sol. $\sqrt{5x^2+7x+2}-\sqrt{4x^2+7x+18}=x-4$ (1)

Take $\sqrt{5x^2+7x+2}=a$ *and* $\sqrt{5x^2+7x+18}=b$

(1) Become $a-b = x-4$ (2)

Now $a^2-b^2= 5x^2+7x+2-4x^2-7x-18$

$a^2-b^2=x^2-16$ (3)

(3) $\div$ *by* (2) *we get* $\dfrac{a^2-b^2}{a-b}=\dfrac{x^2-16}{x-4}$

$$\Rightarrow \frac{\cancel{(a-b)}(a+b)}{\cancel{(a-b)}}=\frac{\cancel{(x-4)}(x+4)}{\cancel{(x-4)}} \Rightarrow a+b=x+4 \qquad (4)$$

$(2)+(4)$ *we get* $a+b=x+4$ *put a in* (4)

$$\frac{a-b=x-4}{2a=2x} \qquad x+b=x+4\ \boxed{b=4}$$

$$\Rightarrow \boxed{a=x}$$

Put value of $a \Rightarrow \sqrt{5x^2+7x+2}=x \quad \Rightarrow 5x^2+7x+2=x^2$

or $5x^2+7x+2-x^2=0 \quad \Rightarrow 4x^2+7x+2=0 \quad (a=4,\ b=7,\ c=2)$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-7\pm\sqrt{(7)^2-4(4)(2)}}{2(4)}=\frac{-7\pm\sqrt{49-32}}{8}$$

$$\boxed{x=\frac{-7\pm\sqrt{17}}{8}}$$

CHECKING $x=\dfrac{-7+\sqrt{17}}{8}$ then (1) become

$$\sqrt{5\left(\frac{-7+\sqrt{17}}{8}\right)^2+7\left(\frac{-7+\sqrt{17}}{8}\right)+2}+\sqrt{4\left(\frac{-7+\sqrt{17}}{8}\right)^2+7\left(\frac{-7+\sqrt{17}}{8}\right)+18}$$

$$=\frac{-7+\sqrt{17}}{8}-4$$

$$\sqrt{5\left(\frac{49+17-14\sqrt{17}}{64}\right)+7\left(\frac{-49+7\sqrt{17}}{64}\right)+2}-\sqrt{4\left(\frac{49+17-14\sqrt{17}}{64}\right)+\left(\frac{-49+7\sqrt{17}}{64}\right)+18}$$

$$=\frac{-7+\sqrt{17}-32}{8}$$

$$\sqrt{\frac{330-70\sqrt{17}-392+56\sqrt{17}+128}{64}}-\sqrt{\frac{264-\cancel{56\sqrt{17}}-392+\cancel{56\sqrt{17}}+1152}{64}}$$

$$=\frac{-39+\sqrt{17}}{8}$$

$$\sqrt{\frac{66-14\sqrt{17}}{8}}-\sqrt{\frac{1024}{8}}=\frac{39+\sqrt{17}}{8}\Rightarrow\sqrt{\frac{66-14\sqrt{17}}{8}}-\frac{32}{8}=\frac{39+\sqrt{17}}{8}$$

$$\frac{8.27}{8}-4=\frac{-34.87}{8}\quad\Rightarrow\quad -2.96=-4.35\ (\textit{Approximately})$$

$$S.S=\left\{\frac{-7\pm\sqrt{17}}{8}\right\}$$

Exercise 4.4

CUBE Root of unity

Proof : $x = (1)^{1/3} \Rightarrow x^3 = 1 \Rightarrow x^3 - (1)^3 = 0$

$$(x-1)(x^2+x+1)=0$$

$$x-1=0 \text{ or } x^2+x+1=0$$

$$\boxed{x=1} \text{ or } x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{(1)^2-4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2}$$

$$x = \frac{-1 \pm i\sqrt{3}}{2}$$

Hence Cube Root of unity are

$$1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$$

$$\text{Where } \omega = \frac{-1 \pm i\sqrt{3}}{2} \quad \text{and } \omega^2 = \frac{-1-i\sqrt{3}}{2}$$

Sum $= 1+\omega+\omega^2$ Faisalabad 2008, Multan 2009, Sargodha 2009

$$= 1 + \frac{-1+i\sqrt{3}}{2} + \frac{-1-i\sqrt{3}}{2}$$

$$= \frac{2-1+i\sqrt{3}-1-i\sqrt{3}}{2} = \frac{0}{2} = 0$$

Product $= 1.\omega.\omega^2$

$$= 1.\left(\frac{-1+i\sqrt{3}}{2}\right)\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$= 1.\left(\frac{(-1)^2-(i\sqrt{3})^2}{4}\right)$$

$$= 1.\frac{(1-(-3))}{4} = \frac{1+3}{4} = \frac{4}{4} = 1$$

1. **Find the three Cube roots of: 8,– 8, 27, –27, 64**

(i) **Find Cube root of 8.** Multan 2008

Sol. $x = (8)^{1/3} \Rightarrow x^3 = 8 \Rightarrow x^3 - (2)^3 = 0$

$$(x-2)(x^2+2x+4)=0$$

$$x-2=0 \text{ or } x^2+2x+4=0$$

$\boxed{x=2}$ or $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ $\quad (a=1,\ b=2,\ c=4\ )$

$$x=\frac{-2\pm\sqrt{(2)^2-4(1)(4)}}{2(1)}=\frac{-2\pm\sqrt{4-16}}{2}$$

$$x=\frac{-2\pm\sqrt{-12}}{2}=2\left(\frac{-1\pm\sqrt{-3}}{2}\right)=2\left(\frac{-1\pm i\sqrt{3}}{2}\right)$$

$$x=2\left(\frac{-1+i\sqrt{3}}{2}\right) and \ x=2\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$x=2\omega \quad \& \quad x=\omega^2$

Hence Root are $\{2,2\omega,2\omega^2\}$

(ii) $\mathbf{x=(-8)^{1/3} \Rightarrow x^3=-8 \Rightarrow x^3+8=0}$

Sol. $x^3+(2)^3=0 \Rightarrow (x+2)(x^2-2x+4)=0$

$x+2=0 \quad or \quad x^2-2x+4=0$

$\boxed{x=-2}$ or $x=\dfrac{-(-2)\pm\sqrt{(-2)^2-4(1)(4)}}{2(1)}$

$$x=\frac{2\pm\sqrt{4-16}}{2}=\frac{2\pm\sqrt{-12}}{2}$$

$$x=\frac{2\pm\sqrt{(-3)(4)}}{2}=\frac{2\pm2\sqrt{-3}}{2}$$

$$x=\frac{2(1+\sqrt{-3})}{2}\ \&\ x=\frac{2(1-\sqrt{-3})}{2}$$

$$x=\frac{-2(-1-\sqrt{-3})}{2}\ \&\ x=\frac{-2(-1+\sqrt{-3})}{2}$$

$x=-2\omega \quad \& \ x=-2\omega^2$

So roots are $\{-2,-2\omega,-2\omega^2\}$

(iii) *Take* $x=(27)^{1/3} \Rightarrow x^3=27 \Rightarrow x^3-(3)^3=0$

Sol $(x-3)(x^2+3x+9)=0$

$x-3=0 \quad or \quad x^2+3x+9=0$

$\boxed{x=3}$ $\quad x=\dfrac{-3\pm\sqrt{(3)^2-4(1)(9)}}{2(1)}$ $(a=1, b=3,\ c=9)$

$$x=\frac{-3\pm\sqrt{9-36}}{2}=\frac{-3\pm\sqrt{-27}}{2}=\frac{-3\pm\sqrt{9(-3)}}{2}=\frac{-3\pm3\sqrt{-3}}{2}$$

$$x=3\frac{(-1+\sqrt{(-3)})}{2} \ \& \ x=3\frac{(-1-\sqrt{-3})}{2}$$

$$x=3\omega \quad \& \quad x=3\omega^2$$

Hence root are $\{3,3\omega,3\omega^2\}$

iv. *Take* $x=(-27)^{1/3} \Rightarrow x^3=-27 \Rightarrow x^3+(3)^3=0$ **Gujranwala 2009**

Sol. $(x+1)(x^2-3x+9)=0$

$$x+3=0 \ \ or \ \ x^2-3x+9=0$$

$$\boxed{x=3} \quad x=\frac{-3\pm\sqrt{(-3)^2-4(1)(9)}}{2(1)} \ (a=1,\ b=-3,\ c=9)$$

$$x=\frac{3\pm\sqrt{9-36}}{2} \Rightarrow x=\frac{3\pm\sqrt{-27}}{2}$$

$$x=\frac{3\pm3\sqrt{-3}}{2} \Rightarrow x=3\frac{(1+\sqrt{-3})}{2} \ \& \ x=\frac{3(1-\sqrt{-3})}{2}$$

$$x=-3\frac{(-1-\sqrt{-3})}{2} \ \& \ x=-3\frac{(-1+\sqrt{-3})}{2}$$

$$x=-3\omega^2 \quad \& \quad x=-3\omega$$

Hence root are $\{-3,-3\omega,-3\omega^2\}$

v. *Take* $x=(64)^{1/3} \Rightarrow x^3=64 \Rightarrow x^3-64=0$

Sol. $x^3-(4)^3=0 \Rightarrow (x-4)(x^2+4x+16)=0$

$$x-4=0 \ \ or \ \ x^2+4x+16=0$$

$$\boxed{x=4} \ \ or \ \ x=\frac{-4\pm\sqrt{(4)^2-4(1)(16)}}{2(1)} \ (a=1,\ b=4,\ c=16)$$

$$x=\frac{-4\pm\sqrt{16-64}}{2}=\frac{-4\pm\sqrt{-48}}{2}=\frac{-4\pm4\sqrt{-3}}{2}$$

$$x=4\frac{(-1\pm\sqrt{-3})}{2}$$

$$x=4\frac{(-1+\sqrt{-3})}{2} \ \& \ x=4\frac{(-1-\sqrt{-3})}{2}$$

$$x=4\omega \quad \& \quad x=4\omega^2$$

Hence Root are $\{4,4\omega,4\omega^2\}$

2. Evaluate

(i) $(1+\omega+\omega^2)^8$ **Sargodha 2006**

Sol. $(1+\omega+\omega^2)^8$ *Note* $1+\omega+\omega^2=0$

$=(-\omega^2-\omega^2)^8 \rightarrow Use\ \text{I}$ $\Rightarrow 1+\omega=-\omega^2$ ______ I

$=(-2\omega^2)^8=(-2)^8(\omega^2)^8=256\omega^{16}$

$=256.\omega.\omega^{15}=256.\omega.(\omega^3)^5$

$=256.\omega.(1)^5=256\omega$

(ii) $\omega^{28}+\omega^{29}+1$ **Sargodha 2010**

Sol.

$\omega^{28}+\omega^{29}+1=\omega.\omega^{27}+\omega^2.\omega^{27}+1$

$=\omega.(\omega^3)^9+\omega^2.(\omega^3)^9+1$

$=\omega.(1)^9+\omega^2.(1)^9+1$

$=\omega+\omega^2+1=0\ (Use\ 1+\omega+\omega^2=0)$

(iii) $(1+\omega-\omega^2)(1-\omega+\omega^2)$ **Sargodha 2008**

Sol. $=(1+\omega-\omega^2)(1+\omega^2-\omega)$

$=(-\omega^2-\omega^2)(-\omega-\omega)$

$=(-2\omega^2)(-2\omega)=4\omega^3=4(1)=4$

(iv) $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^7+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^7$

Sol. $=(\omega)^7+(\omega^2)^7=\omega^7+\omega^{14}$

$=\omega.\omega^6+\omega^2.\omega^{12}$

$=\omega.(\omega^3)^2+\omega^2.(\omega^3)^4$

$=\omega.(1)^2+\omega^2.(1)^4=\omega+\omega^2=-1$ $(Use\ 1)$

Note II

$$\omega=\frac{-1+\sqrt{-3}}{2}$$

$$\omega^2=\frac{-1-\sqrt{-3}}{2}$$

(v) $=(-1+\sqrt{-3})^5+(-1-\sqrt{-3})^5$

Sol. $=(2\omega)^5+(2\omega^2)^5$

$=32\omega^5+32\omega^{10}$

$=32(\omega^5+\omega^{10})$

$$= 32(\omega^2.\omega^3 + \omega.\omega^9)$$
$$= 32(\omega^2.1 + \omega.(\omega^9))$$
$$= 32(\omega^2 + \omega(1)^3) = 32(\omega + \omega^2)$$
$$= 32(-1) = -32$$

Note

$$\omega = \frac{-1+\sqrt{-3}}{2}$$
$$2\omega = -1+\sqrt{-3}$$
$$\& \quad 2\omega^2 = -1-\sqrt{-3}$$

3. Show that

(i) $x^3 - y^3 = (x-y)(x-\omega y)(x-\omega^2 y)$ **Faisalabad 2008**

Sol: R.H.S. $= (x-y)(x-\omega y)(x-\omega^2 y)$

$$= (x-y)(x-\omega y)(x-\omega^2 y)$$
$$= (x-y)(x^2 - \omega xy - \omega^2 xy + \omega^3 y^2)$$
$$= (x-y)(x^2 - xy(\omega+\omega^2) + \omega^3 y^2)$$
$$= (x-y)(x^2 - xy(-1) + 1.y^2) = (x-y)(x^2 + xy + y^2)$$
$$= x^3 + \cancel{x^2y} + \cancel{xy^2} - \cancel{x^2y} - \cancel{xy^2} - y^3$$
$$= x^3 - y^3 = L.H.S$$

(ii) $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$ **Sargodha 2011**

Sol: R.H.S. $= (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

$$= (x+y+z)(x^2 + \omega^2 xy + \omega xz + \omega xy + w^3 y^2 + \omega^2 zx + \omega^4 yz + \omega^3 z^2)$$
$$= (x+y+z)(x^2 + \omega^2 xy + \omega xz + \omega xy + 1.y^2 + \omega^2 yz + \omega yz + 1.z^2)$$
$$= (x+y+z)(x^2 + xy(\omega+\omega^2) + xz(\omega+\omega^2) + yz(\omega+\omega^2) + y^2 + z^2)$$
$$= (x+y+z)(x^2 + xy(-1) + xz(-1) + yz(-1) + y^2 + z^2)$$
$$= (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)$$
$$= x^3 + y^3 + z^3 - 3xyz$$

(iii) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)..........2n$ *factor* $= 1$ **Lahore 2009**

Sol. $L.H.S = (1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8) ----2n$ *factor*

$$= (1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2) ----2n \text{ factor}$$
$$= (1+\omega+\omega^2+\omega^3)(1+\omega+\omega^2+\omega^3) --n \text{ factor}$$
$$= (0+1)(0+1) ---------------n \text{ factor}$$
$$= 1.1.1. -------n \text{ factor} = 1 = R.H.S$$

Note
$\omega^4 = \omega\omega^3 = \omega 1 = \omega$
$\omega^8 = \omega^2.\omega^6 = \omega^2.1 = \omega^2$

4. If ω is a root of $x^2+x+1=0$, show that its other root is ω^2 and prove that $\omega^3=1$

Sol. $x^2+x+1=0$ ______ I

Given ω is root so put $x=\omega$

$\omega^2+\omega+1=0$ ______ II

To check ω^2 $put\ x=\omega^2\ in$ I

$(\omega^2)^2+\omega^2+1=0 \Rightarrow \omega^4+\omega^2+1=0$ ______ III

$or\ \omega^4+2\omega^2+1-\omega^2=0$ ('+' & '−' ω^2)

$or\ (\omega^2+1)^2-\omega^2=0$

$\Rightarrow (\omega^2+1+\omega)(\omega^2+1-\omega)=0 \Rightarrow 0=0$

$(0)(\omega^2+1-\omega)=0 \Rightarrow 0=0$

Hence ω^2 is other root.

Alternate

$\omega^4+\omega^2+1=0$

$\omega+\omega^2+1=0 \Rightarrow 0=0$

$Now\ III-I$

$$\begin{array}{r} \omega^4+\omega^2+1=0 \\ \underline{\omega^2\pm\omega\pm1=0} \\ \omega^4-\omega=0 \end{array}$$

$\Rightarrow \omega(\omega^3-1)=0$ $but\ \omega\neq0 \Rightarrow \omega^3-1=0 \Rightarrow \boxed{\omega^3=1}$

5. Prove that complex Cube roots of -1 *are* $\frac{1+\sqrt{3i}}{2}$ *and* $\frac{1-\sqrt{3i}}{2}$ and hence:

Prove that $\left(\frac{1+\sqrt{-3}}{2}\right)^9+\left(\frac{1-\sqrt{-3}}{2}\right)^9=-2.$

Sol. $x=(-1)^{\frac{1}{3}} \Rightarrow x^3=-1 \Rightarrow x^3+1=0$ ______ I

$x^3+(1)^3=0 \Rightarrow (x+1)(x^2-x+1)=0$

$x+1\ or\ x^2-x+1=0$

$\boxed{x=-1}\ \ or\ \ x=\dfrac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(2)}$

$x=\dfrac{1\pm\sqrt{1-4}}{2} \Rightarrow x=\dfrac{1\pm\sqrt{-3}}{2}$

Hence Cube Roots are

$-1,\dfrac{1+\sqrt{-3}}{2},\dfrac{1-\sqrt{-3}}{2}$

Now we have to prove $\left(\dfrac{1+\sqrt{-3}}{2}\right)^9+\left(\dfrac{1-\sqrt{-3}}{2}\right)^9=-2$

$$L.S.H=\left(\frac{1+\sqrt{-3}}{2}\right)^9+\left(\frac{1-\sqrt{-3}}{2}\right)^9$$

$$=(-\omega)^9+(-\omega^2)^9$$

$$=-\omega^{18}-\omega^9$$

$$=-(\omega^3)^6-(\omega^3)^3$$

$$=(1)^6-(1)^3$$

$$=-1-1=-2=R.H.S$$

Note

$$\omega=\frac{-1+\sqrt{-3}}{2}$$

$$\Rightarrow\quad -\omega=\frac{1-\sqrt{-3}}{2}$$

$$\omega^2=\frac{-1-\sqrt{-3}}{2}$$

$$\Rightarrow\quad -\omega^2=\frac{1+\sqrt{-3}}{2}$$

6. **If ω is a cube root of unity, from an equation whose roots are 2ω and $2\omega^2$**

Sol. $\alpha=2\omega, \beta=2\omega^2$ **Faisalabad 2007**

$S=\alpha+\beta=2\omega+2\omega^2=2(\omega+\omega^2)$

$=2(-1)\quad\Rightarrow\quad S=-2$

$P=(2\omega)(2\omega^2)=4\omega^3=4(1)=4$

Required Equation is $x^2-Sx+P=0$

$x^2-(-2)x+4=0\ \Rightarrow\ x^2+2x+4=0$

7. **Find four roots of 16, 81, 625**

(i) **Take $x=(16)^{1/4}$**

Sol. $x^4=16\quad\Rightarrow\quad x^4-16=0\quad\Rightarrow\quad (x^2)^2-(4)^2=0$

$(x^2-4)(x^2+4)=0\quad\Rightarrow\quad (x-2)(x+2)(x^2+4)=0$

$x-2=0,\ x+2=0,\ x^2+4=0$

$x=2,\ x=-2,\ x\quad -4\quad\Rightarrow x=\pm\sqrt{-4}=\pm\sqrt{4}i=\pm 2i$

Hence roots are $2,-2,2i,-2i$

(ii) **Take x = $(81)^{1/4}$ $\Rightarrow$ $x^4=81$ $\Rightarrow$ $x^4-81=0$ $\Rightarrow (x^2)^2-(9)^2=0$**

Sol. $(x^2-9)(x^2+9)=0\quad\Rightarrow\quad (x-3)(x+3)(x^2+9)=0$

$x-3=0,\ x+3=0,\ x^2+9=0$

$x=3,\ x=-3,\ x=\pm\sqrt{-9}=\pm 3i$

Hence roots are $3,-3,3i,-3i$

(iii) $x=(25)^{1/4}\quad\Rightarrow\quad x^4=625\quad\Rightarrow\quad x^4-625=0$

Sol. $(x^2)^2-(25)^2=0\quad\Rightarrow\quad (x^2-25)(x^2+25)=0$

$(x-5)(x+5)(x^2+25)=0$

$x-5=0,\ x+5=0,\ x^2+25=0$

$x=5,\ x=-5,\ x^2=-25 \Rightarrow x=\pm\sqrt{-25}$

Roots are $x=\pm 5i$

$S.S=\{5,\ -5,\ 5i,\ 5i\}$

8. Solve the following equations:

(i) $2x^4-32=0$

Sol. $'\div'$ *by* $2 \Rightarrow x^4-16=0$

$x^4-(4)^2=0 \Rightarrow (x^2)^2-(4)^2=0$

$(x^2-4)(x^2+4)=0 \Rightarrow (x-2)(x+2)(x^2+4)=0$

$x-2=0,\ x+2=0,\ x^2+4=0$

$x=2,\ x=-2,\ x^2=-4 \Rightarrow x=\pm\sqrt{-4} \Rightarrow x=\pm 2i$

$S.S=\{\pm 2, \pm 2i\}$

(ii) $3y^5-234y=0$

Sol. $'\div'$ *by* $3 \Rightarrow y^5-81y=0 \Rightarrow y(y^4-81)=0 \Rightarrow y=0$ *or* $y^4-81=0$

$\boxed{y=0}$ *or* $(y^2)^2-(9)^2=0 \Rightarrow (y^2-9)(y^2+9)=0$

$(y^2+9)(y-3)(y+3)=0,\quad y+3=0$

$y=\pm\sqrt{-9}, y=3, y=-3$

$y=\pm 3i, y \Rightarrow \pm 3 \Rightarrow S.S=\{0, \pm 3, \pm 3i\}$

(iii) $x^3+x^2+x+1=0$

Sol. $x^2(x+1)+1(x+1)=0$

$(x+1)(x^2+1)=0 \Rightarrow x+1=0$ *or* $x^2+1=0$

$\boxed{x=-1}$ *or* $x^2=-1$

$x=\pm\sqrt{-1} \Rightarrow x=\pm i \Rightarrow S.S=\{-1, \pm i\}$

(iv) $5x^5-5x=0 \Rightarrow 5x(x^4-1)=0$ **Sargodha 2009, Multan 2010**

Sol. $5x=0$ *or* $x^4-1=0$

$\boxed{x=0}$ *or* $(x^2-1)(x^2+1)=0$

$(x-1)(x+1)(x^2+1)=0$

$x-1=0,\ x+1=0,\ x^2=-1$

$x=1,\ x=-1,\quad x=\pm\sqrt{-1} \Rightarrow x=\pm i \Rightarrow S.S=\{0, \pm 1, \pm i\}$

Remainder Theorem: Sargodha 2009, Faisalabad 2008, Multan 2009, 10

If a Polynomial $f(x)$ of degree $n \geq 1$(n is non negative) is divided by $(x-a)$ till no x term exits in the remainder then $f(a)$ is remainder.

Factor Theorem: Faisalabad 2007, Multan 2010

The polynomial (x-a) is a factor of the polynomial f(x) if and only if f(a)=0

Exercise 4.5

Use the Remainder Theorem to find the remainder when the first polynomial is divided by the second polynomial:

01. $x^2+3x+7,\ x+1$ **Multan 2008,**

Sol. $Let\ f(x)=x^2-3x-7$

$Take\ x+1=0 \quad \Rightarrow x=-1$

$f(-1)=(-1)^2+3(-1)+7=5$

Remainder = 5

02. $x^3-x^2+5x+4,\ x-2$ **Faisalabad 2007**

Sol. $Let\ f(x)=x^3-x^2+5x+4$

$Take\ x-2=0 \quad \Rightarrow x=2$

$f(2)=(2)^3-(2)^2+5(2)+4=18$

Remainder = 18

03. $3x^4+4x^3+x-5,\ x+1$

Sol. $Let\ f(x)=3x^4+4x^3+x-5$

$Take\ x+1=0 \quad \Rightarrow \quad x=-1$

$f(-1)=3(-1)^4+4(-1)^3+(-1)-5$

$f(-1)=3+4(-1)-1-5=-7$

Remainder = –7

04. $x^3-2x^2+3x+3 \quad , \quad x-3$

Sol. $Let\ f(x)=x^3-2x^2+3x+3$

$Take\ x-3=0 \quad \Rightarrow \quad x=3$

$f(3)=(3)^3-2(3)^2+3(3)+3$

$=27-18+9+3=21$

Remainder = 21

Use the factor theorem to determine if the first polynomial is a factor of the second polynomial.

05. $x-1,\ x^2+4x-5$

Sol. Let $f(x)=x^2+4x-5$

Take $x-1=0 \Rightarrow x=1$

$f(1)=(1)^2+4(1)-5=0$

Yes $x-1$ is factor of x^2+4x-5

06. $x-2,\ x^3+x^2-7x+1$ **Sargodha 2008**

Sol. Let $f(x)=x^3+x^2-7x+1$

Take $x-2=0 \Rightarrow x=2$

$f(2)=(2)^3+(2)^2-7(2)+1$

$=8+4-14+1=-1\neq 0$

Hence $x-2$ is not factor of x^3+x^2-7x+1

07. $\omega+2,\ 2\omega^3+\omega^2-4\omega+7$

Sol. Let $f(\omega)=2\omega^3+\omega^2-4\omega+7$

Take $\omega+2=0 \Rightarrow \omega=-2$

$f(-2)=2(-2)^3+(-2)^2-4(-2)+7$

$=2(-8)+4+8+7=3\neq 0$

Hence $\omega+2$ is not of factor

08. $x-a,\ x^n-a^n$ **when n is a positive integer** **Lahore 2009**

Sol. Let $f(x)=x^n-a^n$

Take $x-a=0$ then $x-a=0 \Rightarrow x=a$

$f(a)=a^n-a^n=0$

Yes $x-a$ is factor of x^n-a^n

09. $x+a,\ x^n+a^n$ ***where n is a odd integer.***

Sol. Let $f(x)=x^n+a^n$ **Sargodha 2009, Faisalabad 2008,09, Lahore 2009**

Take $x+a=0 \Rightarrow x=-a$

$f(-a)=(-a)^n+a^n$

Because n is odd.

$=-a^n+a^n=0$

Yes $x+a$ is factor of x^n+a^n

10. ***When $x^4+2x^3+kx^2+3$ is divided by $x-2$, the remainder is 1. Find the value of k.***

Sol. Let $f(x)=x^4+2x^3+kx^2+3$ **Multan 2009**

$x-2=0 \Rightarrow x=2$

Put $x=2$

$f(2) = (2)^4 + 2(2)^3 + k(2)^2 + 3 = 4k + 35$

Given remainder is 1 *so*

$4k + 35 = 1 \quad \Rightarrow \quad 4k = 1 - 35 = -34$

$\Rightarrow \quad k = -34/4 \quad \Rightarrow \quad \boxed{-17/2}$

11. ***When the polynomial $x^3 + 2x^2 + kx + 4$ is divided by $x - 2$, the remainder is 14. Find the value of k.***

Sol. *Let* $f(x) = x^3 + 2x^2 + kx + 4$ **Faisalabad 2008,09**

Take $x - 2 = 0 \quad \Rightarrow \quad x = 2$

$f(2) = (2)^3 + 2(2)^2 + k(2) + k$

$= 8 + 8 + 2k + 4 = 2k + 20$

Given reminder is 14 *then*

$2k + 20 = 14 \quad \Rightarrow \quad 2k = 14 - 20$

$2k = -6 \quad \Rightarrow \quad \boxed{k = -3}$

Use synthetic division to show that x is the solution of the polynomial and use the result to factorize the polynomial completely.

12. $\mathbf{x^3 - 7x + 6 = 0,\ x = 2}$

Sol. *or* $x^3 + 0x^2 - 7x + 6 = 0$

Now

	1	0	-7	6
2		2	4	-6
	1	2	-3	$\boxed{0}$

Remainder is 0 *so* $x = 2$ *is solution*

Also $x^3 - 7x + 6 = (x^2 + 2x - 3x)(x - 2)$

$= (x^2 + 3x - x - 3)(x - 2)$

$= (x(x + 3) - 1(x + 3))(x - 2)$

$= (x - 1)(x + 3)(x - 2)$

13. $\mathbf{x^3 - 28x - 48 = 0,\ x = -4}$ **Sargodha 2008**

Sol. *or* $x^3 + 0x^2 - 28x - 48 = 0$

	1	0	-28	-48
-4		-4	+16	48
	1	-4	-12	$\boxed{0}$

Remainder is o so $x = -4$ *is solution*

$$Also\ \ x^3-28x-48=(x+4)(x^2-4x-12)$$
$$=(x+4)(x^2-6x+2x-12)$$
$$=(x+4)(x(x-6)+2(x-6))$$
$$=(x+4)(x-6)(x+2)$$

14. $\mathbf{2x^4+7x^3-4x^2-27x-18,\ x=2,\ x=-3}$ **Sargodha 2006**

Sol. $x=2,\ x=-3$

	2	7	-4	-27	-18
2		4	22	36	18
	2	11	18	+9	0
-3		-6	-15	-9	
	2	5	3	0	

Hence $x=2,\ x=-3$ *are solutions.*

Now $2x^4+7x^3-4x^2-27x-18$
$$=(x-2)(x+3)(2x^2+5x+3)$$
$$=(x-2)(x+3)(2x^2+2x+3x+3)$$
$$=(x-2)(x+3)(2x(x+1)+3(x+1))$$
$$=(x-2)(x+3)(x+1)(2x+3)$$

15. ***Use synthetic division to find the values of p and q if $x+1$ and $x-2$ are the factors of the polynomial x^3+px^2+qx+6.***

Sol. $x^3+px^2+qx+6=0$ **Multan 2008, 09**

$x+1=0\ \Rightarrow x=-1$

$x-2=0\ \Rightarrow x=2$

-1	1	p	q	6
		-1	-p+1	-q+p-1
	1	p-1	q-p+1	p-q+5

Since $x+1$ *is factor so* $p-q+5=0$ ------- I

2	1	p-1	q-p+1
		2	2p+2
	1	P+1	P+q+3

$p+q+3=0(x-2\ is\ factor)$ ------ II

$I + II$

$p - \cancel{q} + 5 = 0$

$p + \cancel{q} + 3 = 0$

$2p + 8 = 0 \quad \Rightarrow \quad \boxed{p = -4}$

$I - II$

$\cancel{p} - q + 5 = 0$

$-\cancel{p} \pm q \pm 3 = 0$

$-2q + 2 = 0$

$-2q = -2 \Rightarrow \boxed{q = 1}$

16. ***Find the values of a and b if −2 and 2 are roots of the polynomial*** $x^3 - 4x^2 + ax + b$.

Sol. *Let* $f(x) = x^3 - 4x^2 + ax + b$

Put $x = -2$

$f(-2) = (-2)^3 - 4(-2)^2 + a(-2) + b$

$f(-2) = -8 - 16 - 2a + b = -2a + b - 24$

-2 *is root so* $-2a + b - 24 = 0$ ________ I

$f(2) = (2)^3 - 4(2)^2 + a(2) + b$

$= 8 - 16 + 2a + b$

$= 2a + b - 8$ ________ II

2 *is root so* $2a + b - 8 = 0$ 11

$I + II$

$-\cancel{2a} + b - 24 = 0$

$\cancel{2a} + b - 8 = 0$

$2b - 32 = 0$

$\boxed{b = 16}$

$I - II$

$-2a + \cancel{b} - 24 = 0$

$-2a \pm \cancel{b} \mp 8 = 0$

$4a - 16 = 0$

$\boxed{a = -4}$

Exercise 4.6

$$Sum = \alpha + \beta = -\frac{b}{a} \qquad I$$

$$Product = \alpha\beta = \frac{c}{a} \qquad II$$

Equation from roots is

$$x^2 - Sx + p = 0 \qquad III$$

Proof 1 *we know that*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \;\&\; \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$Sum = \alpha + \beta$$

$$= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b + \cancel{\sqrt{b^2 - 4ac}} - b - \cancel{\sqrt{b^2 - 4ac}}}{2a}$$

$$= \frac{-\cancel{2}b}{\cancel{2}a} \Rightarrow \boxed{S = \alpha + \beta = -\frac{b}{a}}$$

Proof II **Product** $= \alpha\beta = {}^{c}\!/_{a}$

$$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{(-b)^2 - \sqrt{(b^2 - 4ac)^2}}{4a^2}$$

$$= \frac{b^2 - (b^2 - 4ac)^2}{4a^2} = \frac{b^2 - b^2 + 4ac}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a} \Rightarrow \boxed{p = \alpha\beta = \frac{c}{a}}$$

Proof III *we know that*

$$ax^2 + bx + c = 0$$

$\div$ *by a we get*

$$x^2 + \frac{b}{a}x + \frac{C}{a} = 0$$

$$or \quad x^2-\left(-\frac{b}{a}\right)x+\frac{c}{a}=0$$

Use result I and II

$$x^2+-Sx+p=0$$

1. ***If α,β are the roots of $3x^2-2x+4=0$, find the values of***

(i) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

Sol. *Given* $3x^2-2x+4=0$

$$\alpha+\beta=-\frac{b}{a}=\frac{-(-2)}{3}=\frac{2}{3}$$

$$\alpha\beta=\frac{c}{a}=\frac{4}{3}$$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}$$

$$=\frac{\alpha^2+\beta^2+2\alpha\beta-2\alpha\beta}{(\alpha\beta)^2}$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}=\frac{(2/3)^2-2(4/3)}{(4/3)^2}$$

$$=\frac{4/9-8/3}{16/9}=(\frac{4-24}{9})\times\frac{9}{16}$$

$$=\frac{-20}{16}=-5/4$$

(ii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ **Sargodha 2011**

Sol. $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}$

$$=\frac{\alpha^2+\beta^2+2\alpha\beta-2\alpha\beta}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

$$=\frac{(2/3)^2-2(4/3)}{4/3}=(\frac{4}{9}-\frac{8}{3})\times\frac{3}{4}$$

$$=(\frac{4-24}{9})\times\frac{3}{4}=(\frac{-20}{9})(\frac{3}{4})=-5/3$$

(iii) $\alpha^4+\beta^4$

Sol. $\alpha^4+\beta^4=(\alpha^2)^2+(\beta^2)^2$

$$= (\alpha^2)^2 + (\beta^2)^2 + 2\alpha^2\beta^2 - 2\alpha^2\beta^2$$
$$= (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$$
$$= (\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta)^2 - 2\alpha^2\beta^2$$
$$= \left[(\alpha+\beta)^2 - 2\alpha\beta\right]^2 - 2(\alpha\beta)^2$$
$$= \left[(\frac{2}{3})^2 - 2(\frac{4}{3})\right]^2 - 2(\frac{4}{3})^2$$
$$= (\frac{4}{9} - \frac{8}{3})^2 - 2(\frac{16}{9}) = (\frac{4-24}{9})^2 - \frac{32}{9}$$
$$= (\frac{-20}{9})^2 + \frac{32}{9} = \frac{400}{81} - \frac{32}{9}$$
$$= \frac{400-288}{81} = \frac{112}{81}$$

(iv) $\alpha^3 + \beta^3$ **Multan 2009**

Sol. $\alpha^3 + \beta^3 = (\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2)$

$$\boxed{+ \& -2\alpha\beta} = (\alpha+\beta)(\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta - \alpha\beta)$$
$$= (\alpha+\beta)((\alpha+\beta)^2 - 3\alpha\beta)$$
$$= (2/3)\left[(2/3)^2 - 3(4/3)\right]$$
$$= (\frac{2}{3})(\frac{4}{9} - 4)$$
$$= (\frac{2}{3})(\frac{-32}{9}) = \frac{-64}{27}$$

(v) $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$

Sol. $\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\alpha^3 + \beta^3}{\alpha^3\beta^3}$

$$= \frac{(\alpha+\beta)(\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta - \alpha\beta}{(\alpha\beta)^3}$$
$$= \left[\frac{(\alpha+\beta)\left[(\alpha+\beta)^2 - 3\alpha\beta\right]}{(\alpha\beta)^3}\right] = \frac{(\frac{2}{3})\left[(\frac{2}{3})^2 - 3(\frac{4}{3})\right]}{(4/3)^3}$$

$$=\frac{\left(\frac{2}{3}\right)\left[\frac{4}{9}-4\right]}{\frac{64}{27}}=(\frac{2}{3})(\frac{4-36}{9})\times\frac{27}{64}$$

$$=(\frac{2}{3})(\frac{-32}{9})\times\frac{27}{64}=-1$$

(*vi*) $\alpha^2-\beta^2$

Sol. $\alpha^2-\beta^2=(\alpha+\beta)(\alpha-\beta)$

We know that

$(\alpha-\beta)^2=\alpha^2+\beta^2-2\alpha\beta$

$=\alpha^2+\beta^2+2\alpha\beta-2\alpha\beta-2\alpha\beta$

$=(\alpha+\beta)^2-4\alpha\beta$

$\Rightarrow(\alpha-\beta)=\sqrt{(\alpha+\beta)^2-4\alpha\beta}$

I become $\alpha^2-\beta^2=(\alpha+\beta)(\alpha-\beta)$

$$=(\alpha+\beta)\sqrt{(\alpha+\beta)^2-4\alpha\beta}$$

$$=(2/3)\sqrt{(2/3)^2-4(4/3)}$$

$$=(\frac{2}{3})\sqrt{\frac{4}{9}-\frac{16}{3}}=(\frac{2}{3})\sqrt{\frac{4-48}{9}}$$

$$=(\frac{2}{3})(\sqrt{\frac{-44}{9}})=(\frac{2}{3})(i\sqrt{\frac{44}{9}})=\frac{2}{3}i\frac{\sqrt{4\times11}}{3}$$

$$=\frac{2}{9}\sqrt{11}i$$

2. ***If α,β are the roots of $x^2-px-c=0$, prove that***

$(1+\alpha)(1+\beta)=1-c$ **Sargodha 2008,2009 Lahore 2009, Rawalpindi 2009**

Sol. *Then* $\alpha+\beta=\frac{-(-p)}{1}=p$

and $\alpha\beta=\frac{(-p-c)}{1}=-p-c$

$L.H.S=(1+\alpha)(1+\beta)$

$=1+\alpha+\beta+\alpha\beta$

$=1+p-p-c$

$=1-c=R.H.S$

3. ***Find condition that one root of $x^2 + px + q = 0$ is*** **Federal**

(i) Double the other

Sol. $x^2 + px + q = 0 \ (a = 1, b = p, c = q)$

According to the given condition x

$\alpha = \alpha$ *and* $\beta = 2\alpha$ *so*

$$\alpha + \beta = \frac{-b}{a} \quad \Rightarrow \quad \alpha + 2\alpha = \frac{-(p)}{1}$$

$$\Rightarrow 3\alpha = -p \quad \Rightarrow \quad \alpha = \frac{-p}{3} \text{\_\_\_\_\_\_\_\_\_\_ } 1$$

$$\text{And } \alpha\beta = \frac{c}{a} \quad \Rightarrow \quad (\alpha)(2\alpha) = \frac{q}{1} \Rightarrow 2\alpha^2 = q - - - - - II$$

$$\Rightarrow 2(-p/3)^2 = q \quad \Rightarrow \quad 2(p^2/9) = q \quad \boxed{2p^2 = 9q}$$

(ii) Square of the other

Sol. *According to the given condition*

$\alpha = \alpha \ \& \ \beta = \alpha^2$ *then*

$$\alpha + \beta = \frac{-b}{a} \quad \Rightarrow \quad \alpha + \alpha^2 = \frac{-p}{1} = -p$$

$$\text{Also } \alpha\beta = \frac{c}{a} \quad \Rightarrow \quad (\alpha)(\alpha^2) = \frac{q}{1}$$

$$\alpha^3 = q \quad \Rightarrow \quad \alpha = q^{1/3}$$

$$I \text{ become } \ \alpha + \alpha^2 = -p \quad \Rightarrow \quad q^{1/3} + (q^{1/3})^2 = -p$$

$$q^{1/3} + q^{2/3} = -p \ - - - - II \text{ or } (q^{1/3} + q^{2/3})^3 = (-p)^3$$

$$(q^{1/3})^3 + (q^{2/3})^3 + 3(q^{1/3})(q^{2/3})(q^{1/3} + q^{2/3}) = -p^3$$

$$q + q^2 + 3q^{1/3+2/3}(-p) = -p^3$$

$$q + q^2 - 3q^1(p) + p^3 = 0$$

$$\Rightarrow \quad \boxed{p^3 + q + q^2 = 3pq}$$

(iii) Additive inverse of the other

Sol. *According to the given condition*

$\alpha = \alpha \ \& \ \beta = -\alpha$

$$\alpha + \beta = \frac{-b}{\alpha} \quad \Rightarrow \quad \alpha + (-\alpha) = \frac{-p}{1}$$

$$\alpha - \alpha = -p \quad \Rightarrow \quad 0 = -p \quad \Rightarrow \boxed{p = 0}$$

(iv) ***Multiplicative inverse of the other*** **Multan 2009**

Sol. *According to the given condition*

$$\alpha = \alpha \ \& \ \beta = \frac{1}{\alpha} \text{ so}$$

$$\alpha\beta = \frac{c}{a} \quad \Rightarrow \quad \alpha\left(\frac{1}{\alpha}\right) = \frac{q}{1} \quad \Rightarrow \boxed{1 = q}$$

4. ***If the roots of the equation $x^2 - px + q = 0$ differ by unity, prove that $p^2 = 4q + 1$.*** **Sargodha 2007**

Sol. $x^2 - px + q = 0 \qquad (a = 1, b = -p, c = q)$

According to the given condition

$\alpha = \alpha \ \& \ \beta = \alpha + 1$ *then*

$$\alpha + \beta = -\frac{b}{a} \quad \Rightarrow \quad \alpha + \alpha + 1 = -\frac{(-p)}{1}$$

$$2\alpha + 1 = p \quad \Rightarrow \quad 2\alpha = p - 1 \quad \Rightarrow \quad \alpha = \frac{p-1}{2}$$

$$\text{And } \alpha\beta = \frac{c}{a} \quad \Rightarrow \quad \alpha(\alpha + 1) = \frac{q}{1}$$

$$\alpha^2 + \alpha = q (\text{Put } 1)$$

$$\left(\frac{p-1}{2}\right)^2 + \frac{p-1}{2} = q \quad \Rightarrow \quad \frac{p^2 - 2p + 1}{4} + \frac{p-1}{2} = q$$

$$\text{'}\times\text{' by } 4 \quad \Rightarrow \quad p^2 - 2p + 1 + 2p - 2 = 4q$$

$$p^2 - 1 = 4q \quad \Rightarrow \quad \boxed{p^2 = 1 + 4q}$$

5. ***Find the condition that $\frac{a}{x-a} + \frac{b}{x-b} = 5$ may have roots equal in magnitude but opposite in signs.***

Sol. $\frac{a}{x-a} + \frac{b}{x-b} = 5$

'$\times$' *both sides by* $(x-a)(x-b)$

$$a(x-b) + b(x-a) = 5(x-a)(x-b)$$

$$ax - ab + bx - ab = 5(x^2 - ax - bx + ab)$$

$$ax + bx - 2ab = 5x^2 - 5ax - 5bx + 5ab$$

$$5x^2 - 5bx - 5ax + 5ab - ax - bx + 2ab = 0$$

$$5x^2 - 6ax - 6bx + 7ab = 0$$

$$5x^2 - 6(a+b)x + 7ab = 0$$

According to the given condition

$A=5, B=-6(a+b), C=7ab$

$\alpha=\alpha$ & $\beta=-\alpha$ *so*

$$\alpha+\beta=\frac{-b}{a} \quad \Rightarrow \alpha+(-\alpha)=\frac{-[-6(\alpha+b)]}{5}$$

$$\alpha-\alpha=\frac{6(\alpha+b)}{5} \quad \Rightarrow \quad 0=\frac{6(\alpha+b)}{5}$$

$$\Rightarrow \quad \boxed{\alpha+b=0}$$

6. ***If the roots of $px^2+qx+q=0$ are α and β then prove that***

$$\sqrt{\frac{\alpha}{\beta}}+\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{q}{p}}=0$$ **Multan 2007, 2010**

Sol. $\alpha+\beta=\frac{-q}{p}$ ________ I & $\alpha\beta=\frac{q}{p}$ ________ II

$\sqrt{\alpha\beta}=\sqrt{\frac{q}{p}}$ ________ III *Take square root of II*

$\frac{\alpha+\beta}{\sqrt{\alpha\beta}}=\frac{-q/p}{\sqrt{q/p}}$ *Divide I by III*

$$\Rightarrow \frac{\alpha}{\sqrt{\alpha\beta}}+\frac{\beta}{\sqrt{\alpha\beta}}=\frac{-q/p}{\sqrt{q/p}}$$

NOTE

$x=\sqrt{x}\sqrt{x}=(\sqrt{x})^2=x$

$$\frac{\sqrt{\alpha}\sqrt{\alpha}}{\sqrt{\alpha}\sqrt{\beta}}+\frac{\sqrt{\beta}\sqrt{\beta}}{\sqrt{\alpha}\sqrt{\beta}}=\frac{-\sqrt{q/p}\sqrt{q/p}}{\sqrt{q/p}}$$

$$\sqrt{\frac{\alpha}{\beta}}+\sqrt{\frac{\beta}{\alpha}}=-\sqrt{\frac{q}{p}}$$

$$\Rightarrow \sqrt{\frac{\alpha}{\beta}}+\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{q}{p}}=0$$

7. ***If α, β are the roots of the equation $ax^2+bx+c=0$, form the equation whose roots are***

(i) α^2, β^2

Sol. $\alpha+\beta=\frac{-b}{\alpha}$ *and* $\alpha\beta=\frac{c}{\alpha}$

are Given (For all parts)

$S=\alpha^2+\beta^2$

$=\alpha^2+\beta^2+2\alpha\beta-2\alpha\beta$

$$=(\alpha+\beta)^2-2\alpha\beta=\left(\frac{-b}{a}\right)^2-2\left(\frac{c}{a}\right)=\frac{b^2}{a^2}-\frac{2c}{a}=\frac{b^2-2ac}{a^2}$$

$$p=\alpha^2\beta^2=(\alpha\beta)^2=\left(\frac{c}{a}\right)^2=\frac{c^2}{a^2}$$

$$y^2-Sy+p=0$$

$$y^2-\left(\frac{b^2-2ac}{a^2}\right)y+\frac{c^2}{a^2}=0$$

'×' by a^2

$$a^2y^2-(b^2-2ac)y+c^2=0$$

(ii) $\frac{1}{\alpha},\frac{1}{\beta}$

Sol. $S=\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{-b/a}{c/a}$

$$=\frac{-b}{a}\times\frac{a}{c}=-\frac{b}{c}$$

$$p=\frac{1}{\alpha}.\frac{1}{\beta}=\frac{1}{\alpha\beta}=\frac{1}{c/a}=\frac{a}{c}$$

$$y^2-Sy+p=0$$

$$y^2-\left(\frac{-b}{c}\right)y+\frac{a}{c}=0$$

'×' by c ....

$$cy^2+by+a=0$$

(iii) $\frac{1}{\alpha^2},\frac{1}{\beta^2}$

Sol. $S=\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\beta^2+\alpha^2}{\alpha^2\beta^2}$

$$=\frac{\alpha^2+\beta^2+2\alpha\beta-2\alpha\beta}{\alpha^2\beta^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}=\frac{\left(\frac{-b}{a}\right)^2-\frac{2c}{a}}{\left(\frac{c}{a}\right)^2}$$

$$=\left(\frac{b^2}{a^2}-\frac{2c}{a}\right)\times\frac{a^2}{c^2}$$

$$=\left(\frac{b^2-2ac}{a^2}\right)\times\frac{a^2}{c^2}=\frac{b^2-2ac}{c^2}$$

$$P=\frac{1}{\alpha^2}.\frac{1}{\beta^2}=\frac{1}{\alpha^2\beta^2}=\frac{1}{(\alpha\beta)^2}=\frac{1}{(c/a)^2}$$

$$P=\frac{1}{c^2/a^2}=\frac{a^2}{c^2}$$

$$y^2-Sy+P=0$$

$$y^2-\left(\frac{b^2-2ac}{c^2}\right)y+\frac{a^2}{c^2}=0$$

'×' by c^2

$$c^2y^2-(b^2-2ac)y+a^2=0$$

(iv) α^3,β^3

Sol. $S=\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$

$$=(\alpha+\beta)(\alpha^2+\beta^2+2\alpha\beta-2\alpha\beta-\alpha\beta)$$

$$=(\alpha+\beta)\left[(\alpha+\beta)^2-3\alpha\beta\right]$$

$$=\left(\frac{-b}{a}\right)\left[\left(\frac{-b}{a}\right)^2-3\left(\frac{c}{a}\right)\right]=(-\frac{b}{a})\left(\frac{b^2-3ac}{a^2}\right)$$

$$=\frac{-b^3+3abc}{a^3}$$

$$P=\alpha^3\beta^3=(\alpha\beta)^3=\left(\frac{c}{a}\right)^3=\frac{c^3}{a^3}$$

$$y^2-Sy+p=0$$

$$y^2-\left(\frac{-b^3+3abc}{a^3}\right)y+\frac{c^3}{a^3}=0$$

'×' by a^3

$$a^3y^2+(b^3-3abc)y+c^3=0$$

(v) $\frac{1}{\alpha^3},\frac{1}{\beta^3}$

Sol. $S=\frac{1}{\alpha^3}+\frac{1}{\beta^3}=\frac{\alpha^3+\beta^3}{\alpha^3\beta^3}$

$$=\frac{(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)}{(\alpha\beta)^3}$$

$$=\frac{(\alpha+\beta)(\alpha^2+\beta^2+2\alpha\beta-2\alpha\beta-\alpha\beta)}{(\alpha\beta)^3}$$

$$=\frac{(\alpha+\beta)((\alpha+\beta)^2-3\alpha\beta)}{(\alpha\beta)^3}$$

$$=\frac{\left(\frac{-b}{a}\right)\left(\left(\frac{-b}{a}\right)^2-\frac{3c}{a}\right)}{\left(\frac{c}{a}\right)^3}=\frac{\left(-\frac{b}{a}\right)\left(\frac{b^2}{a^2}-\frac{3c}{a}\right)}{\frac{c^3}{a^3}}$$

$$=\left(\frac{-b}{a}\right)\left(\frac{b^2-3ac}{a^2}\right)\times\frac{a^3}{c^3}$$

$$=\left(\frac{-b^3+3abc}{a^3}\right)\times\frac{a^3}{c^3}$$

$$=\frac{-b^3+3abc}{c^3}$$

$$\mathbf{P}=\frac{1}{\alpha^3}\cdot\frac{1}{\beta^3}=\frac{1}{(\alpha\beta)^3}=\frac{1}{(c/a)^3}=\frac{a^3}{c^3}$$

$$y^2-Sy+p=0$$

$$y^2-\left(\frac{-b^3+3abc}{c^3}\right)y+\frac{a^3}{c^3}=0$$

$$c^3y^2-(-b^3+3abc)y+a^3=0$$

(vi) $\alpha+\dfrac{1}{\alpha},\beta+\dfrac{1}{\beta}$

Sol. $S=\alpha+\dfrac{1}{\alpha}+\beta+\dfrac{1}{\beta}$

$$=(\alpha+\beta)+\frac{1}{\alpha}+\frac{1}{\beta}$$

$$=(\alpha+\beta)+\frac{\alpha+\beta}{\alpha\beta}=(-\frac{b}{a})+\frac{-b/a}{c/a}$$

$$=-\frac{b}{a}-\frac{b}{a}\times\frac{a}{c}=\frac{-b}{a}-\frac{b}{c}$$

$$=\frac{-bc-ab}{ac}$$

$$\mathbf{P}=(\alpha+\frac{1}{\alpha})(\beta+\frac{1}{\beta})$$

$$=\alpha\beta+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+\frac{1}{\alpha\beta}$$

$$=\alpha\beta+\frac{1}{\alpha\beta}+\frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$=\alpha\beta+\frac{1}{\alpha\beta}+\frac{\alpha^2+\beta^2+2\alpha\beta-2\alpha\beta}{\alpha\beta}$$

$$=\alpha\beta+\frac{1}{\alpha\beta}+\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

$$=\frac{c}{a}+\frac{1}{c/a}+\frac{(\frac{-b}{a})^2-\frac{2c}{a}}{c/a}$$

$$=\frac{c}{a}+\frac{a}{c}+\left(\frac{b^2}{a^2}-\frac{2c}{a}\right)\times\frac{a}{c}$$

$$=\frac{a^2+c^2}{ac}+\left(\frac{b^2-2ac}{a^2}\right)\times\frac{a}{c}$$

$$=\frac{a^2+c^2}{ac}+\frac{(b^2-2ac)}{ac}$$

$$=\frac{a^2+c^2+b^2-2ac}{ac}$$

$$y^2-Sy+P=0$$

$$y^2-\frac{(-bc-ab)}{ac}y+\frac{a^2+c^2+b^2-2ac}{ac}=0$$

'×' by ac

$$=y^2ac+b(c+a)y+a^2+b^2-2ac=0$$

(vii) $(\alpha-\beta)^2,(\alpha+\beta)^2$

Sol. $\mathbf{S}=(\alpha-\beta)^2+(\alpha+\beta)^2$

$$=(\alpha^2+\beta^2-2\alpha\beta)+(\alpha+\beta)^2$$

$$=(\alpha^2+\beta^2+2\alpha\beta-2\alpha\beta-2\alpha\beta)+(\alpha+\beta)^2$$

$$=(\alpha+\beta)^2-4\alpha\beta+(\alpha+\beta)^2$$

$$=2(\alpha+\beta)^2-4\alpha\beta=2\left(\frac{-b}{a}\right)^2-4\frac{c}{a}$$

$$=\frac{2b^2}{a^2}-\frac{4c}{a}=\frac{2b^2-4ac}{a^2}$$

$$P=(\alpha+\beta)^2.(\alpha+\beta)^2$$

$$=(\alpha^2+\beta^2-2\alpha\beta+2\alpha\beta-2\alpha\beta)(\alpha+\beta)^2$$

$$=((\alpha+\beta)^2-4\alpha\beta)(\alpha+\beta)^2$$

$$=\left(\left(\frac{-b}{a}\right)^2-\frac{4c}{a}\right)\left(\frac{-b}{a}\right)^2=\left(\frac{b^2}{a^2}-\frac{4c}{a}\right)\left(\frac{b^2}{a^2}\right)$$

$$=\left(\frac{b^2-4ac}{a^2}\right)\left(\frac{b^2}{a^2}\right)=\frac{b^4-4ab^2c}{a^4}$$

$$y^2-Sy+P=0$$

$$y^2-\left(\frac{2b^2-4ac}{a^2}\right)y+\left(\frac{b^4-4ab^2c}{a^4}\right)=0$$

'×' by a^4

$$a^4y^2-2a^2(b^2-2ac)y+b^4-ab^2c=0$$

(viii) $-\frac{1}{\alpha^3}, -\frac{1}{\beta^3}$

Sol. $S=-\frac{1}{\alpha^3}+\left(\frac{-1}{\beta^3}\right)=-\frac{1}{\alpha^3}-\frac{1}{\beta^3}$

$$=\frac{-\alpha^3-\beta^3}{\alpha^3\beta^3}=-\frac{(\alpha^3+\beta^3)}{(\alpha\beta)^3}$$

$$=-\frac{(\alpha+\beta)(\alpha^2+\beta^2+2\alpha\beta-2\alpha\beta-\alpha\beta)}{(\alpha\beta)^3}$$

$$=-\frac{(\alpha+\beta)((\alpha+\beta)^2-3\alpha\beta)}{\left(\frac{c}{a}\right)^3}=\frac{-\left(\frac{-b}{a}\right)\left(\left(-\frac{b}{a}\right)^2-\frac{3c}{a}\right)}{\left(\frac{c}{a}\right)^3}$$

$$=\left(\frac{b}{a}\right)\left(\frac{b^2}{a^2}-\frac{3c}{a}\right)\frac{a^3}{c^3}=\left(\frac{b}{a}\right)\left(\frac{b^2-3ac}{a^2}\right)\left(\frac{a^3}{c^3}\right)$$

$$=\frac{b^3-3abc}{a^3}\times\frac{a^3}{c^3}=\frac{b^3-3abc}{c^3}$$

$$P=\left(\frac{-1}{\alpha^3}\right)\left(\frac{-1}{\beta^3}\right)=\frac{1}{\alpha^3\beta^3}=\frac{1}{(\alpha\beta)^3}$$

$$=\frac{1}{\left(\frac{c}{a}\right)^3}=\frac{1}{\frac{c^3}{a^3}}=\frac{a^3}{c^3}$$

$$y^2-Sy+p=0$$

$$y^2-\left(\frac{b^3-3abc}{c^3}\right)y+\frac{a^3}{c^3}=0$$

'×' by c^3

$$c^3y^2-(b^3-3abc)y+a^3=0$$

8. ***If α, β are the roots of $5x^2-x-2=0$, form the equation whose roots are $\frac{3}{\alpha}$ and $\frac{3}{\beta}$.*** *Federal, Faisalabad* 2008, 09, *Gujranwala* 2009, *Mul* tan 2009

Sol. $\alpha+\beta=-\left(-\frac{1}{5}\right)=\frac{1}{5}$

$$\alpha\beta=\frac{c}{a}=-\frac{2}{5}$$

Given roots are $\frac{3}{\alpha}$ *and* $\frac{3}{\beta}$

$$S=\frac{3}{\alpha}+\frac{3}{\beta}=3\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)=3\frac{(\alpha+\beta)}{\alpha\beta}$$

$$=\frac{3(1/5)}{-2/5}=\frac{3}{5}\times\frac{5}{-2}=-\frac{3}{2}$$

$$P=\left(\frac{3}{\alpha}\right)\left(\frac{3}{\beta}\right)=\frac{9}{\alpha\beta}=\frac{9}{-2/5}=\frac{9\times5}{-2}=\frac{-45}{2}$$

$$y^2-Sy+P=0$$

$$y^2-\left(\frac{-3}{2}\right)y-\frac{45}{2}=0$$

$$\Rightarrow y^2+\frac{3}{2}y-\frac{45}{2}$$

'×' by 2 $\Rightarrow 2y^2+3y-45=0$

9. *If α and β are the roots of $x^2-3x+5=0$, from the equation whose roots are $\dfrac{1-\alpha}{1+\alpha}$ and $\dfrac{1-\beta}{1+\beta}$*

Sol. $\alpha+\beta=\dfrac{-b}{a}=\dfrac{-(-3)}{1}=3$

$\alpha\beta=\dfrac{c}{a}=\dfrac{5}{1}=5$

Roots are $\dfrac{1-\alpha}{1+\alpha},\dfrac{1-\beta}{1+\beta}$

$$\mathbf{S}=\frac{1-\alpha}{1+\alpha}+\frac{1-\beta}{1+\beta}=\frac{(1-\alpha)(1+\beta)+(1-\beta)(1+\alpha)}{(1+\alpha)(1+\beta)}$$

$$=\frac{1-\alpha+\beta-\alpha\beta+1-\beta+\alpha-\alpha\beta}{1+\alpha+\beta+\alpha\beta}$$

$$=\frac{2-2\alpha\beta}{1+(\alpha+\beta)+\alpha\beta}=\frac{2-2(5)}{1+3+5}=\frac{2-10}{9}=-\frac{8}{9}$$

$$\mathbf{P}=\left(\frac{1-\alpha}{1+\alpha}\right)\left(\frac{1-\beta}{1+\beta}\right)=\frac{1-\alpha-\beta+\alpha\beta}{1+\alpha+\beta+\alpha\beta}$$

$$=\frac{1-(\alpha+\beta)+\alpha\beta}{1+(\alpha+\beta)+\alpha\beta}=\frac{1-3+5}{1+3+5}=\frac{3}{9}$$

$$y^2-Sy+p=0 \quad\Rightarrow\quad y^2-\left(\frac{-8}{9}\right)y+\frac{3}{9}=0$$

'×' by 9 $\quad 9y^2+8y+3=0$

Exercise 4.7

$$\left.\begin{array}{l} i.\ b^2-4ac<0 \ \textit{Roots are complex} \\ ii.\ b^2-4ac=0 \ \textit{Roots are equal} \\ \left.\begin{array}{l} iii.\ b^2-4ac>0 \ (\textit{Not Square})\ \textit{Irrational} \\ iv.\ b^2-4ac>0 \ (\textit{Square})\ \textit{Rational} \end{array}\right\}\mathrm{Re}\,al \end{array}\right\}$$ **Faisalabad 2009**

01. Discuss the nature of the roots of the following equations:

i. $\mathbf{4x^2+6x+1=0}$ **Faisalabad 2009**

Sol. $4x^2+6x+1=0 \qquad (a=4,\ b=5,\ c=1)$

$Take\ x+1=0 \quad \Rightarrow x=-1$

$b^2-4ac=(6)^2-4(4)(1)$

$=36-16=20$ **Irrational and unequal**

ii. $\mathbf{x^2-5x+6=0}$ **Multan 2010**

Sol. $b^2-4ac=(-5)^2-4(1)(6)$

$=25-24=1=(1)^2$ **Rational & unequal**

iii. $\mathbf{2x^2-5x+1=0}$ **Multan 2009**

Sol. $b^2-4ac=(-5)^2-4(2)(1)=25-8=17$ **Irrational unequal**

iv. $\mathbf{25x^2-30x+9=0}$ **Multan 2009**

Sol. $a=25,\ b-30,\ c=9$

$b^2-4ac=(-30)^2-4(25)(9)=900-900=$ **Equal**

02. Show that the roots of the following equations will be real:

i. $\mathbf{x^2-2\left(m+\frac{1}{m}\right)x+3=0;\ m\neq 0}$ **Rawalpindi 2009**

Sol. $a=1,\ b=-2\left(m+\frac{1}{m}\right), c=3$

$$b^2-4ac=\left[-2\left(m+\frac{1}{m}\right)\right]^2-4(1)(3)$$

$$=4\left(m^2+\frac{1}{m^2}+2\right)-12=4m^2+\frac{4}{m^2}+8-12$$

$$=4m^2+\frac{4}{m^2}-4=4\left(m^2+\frac{1}{m^2}-1\right)$$

$$= 4\left(m^2 + \frac{1}{m^2} - 1 - 1 + 1\right) = 4\left(m^2 + \frac{1}{m^2} - 2 + 1\right)$$

$$= 4\left[\left(m - \frac{1}{m}\right)^2 + 1\right] > 0 \text{ Hence } \textbf{Real}$$

ii. $(b-c)x^2 + (c-a)x + (a-b) = 0;\ a,b,c, \in Q$

Sol. $A = b - c,\ B = c - a, C = a - b$

$$B^2 - 4AC = (c-a)^2 - 4(b-c)(a-b)$$
$$= c^2 + a^2 - 2ac - 4ab + 4b^2 + 4ac - 4bc$$
$$= a^2 + c^2 + 2ac - 4ab - 4bc + 4b^2$$
$$= (a + c - 2b)^2 > 0 \text{ Hence } \textbf{Real}$$

03. **Show that the roots of the following equations will be rational:**

i. $(p+q)x^2 - px - q = 0$ *Sgd* 2009, *Lahore* 2009, *Multan* 2010, *Fsd* 2007, 08

Sol. $a = p + q, b = -p, c = -q$

$$b^2 - 4ac = (-p)^2 - 4(p+q)(-q)$$
$$= p^2 + 4pq + 4q^2 = (p + 2q)^2 \text{ Hence } \textbf{Rational}$$

ii. $px^2 - (p-q)x - q = 0$ *Sargodha* 2009, *Multan* 2008

Sol. $a = p,\ b = -(p-q),\ c = -q$

$$b^2 - 4ac = [-(p-q)]^2 - 4(p)(-q)$$
$$= p^2 - 2pq + q^2 + 4pq$$
$$= p^2 + 2pq + q^2 = (p+q)^2 \text{ Hence } \textbf{Rational}$$

04. **For what values of m will the roots of the following equations be equal?**

i. $(m+1)x^2 + 2(m+3)x + m + 8 = 0$ *Sargodha* 2006

Sol. $a = m + 1,\ b = 2(m+3),\ C = m + 8$

$$b^2 - 4ac = [2(m+3)]^2 - 4(m+1)(m+8)$$
$$= 4(m^2 + 6m + 9) - 4(m^2 + m + 8m + 8)$$
$$b^2 - 4ac = \cancel{4m^2} + 24m + 36 - \cancel{4m^2} - 4m - 32m - 32 = -12m + 4$$

Given roots are equal *i.e* $b^2 - 4ac = 0 \Rightarrow -12m + 4 = 0$

$$12m = 4$$
$$m = 4/12 \Rightarrow m = \tfrac{1}{3}$$

ii. $x^2 - 2(1+3m)x + 7(3+2m) = 0$ *Lahore* 2009

Sol. $a = 1,\ b = -2(1+3m),\ c = 7(3+2m)$

$$b^2-4ac=\left[-2(1+3m)\right]^2-4(1)(7(3+2m))=4(1+6m+9m^2)-28(3+2m)$$
$$=4+24m+36m^2-84-56m$$
$$=36m^2-32m-80$$
$$=9m^2-8m-20\,(\div\text{ by }4)$$
$$=9m^2-18m+10m-20$$
$$=9m(m-2)+10(m-2)=(m-2)(9m+10)$$

Given roots are equal so

$(m-2)(9m+10)=0$

$m-2=0 \quad or \quad 9m+10=0$

$m=2 \quad or \quad m=-10/9$

iii. $\mathbf{(1+m)x^2-2(1+3m)x+(1+8m)=0}$ ***Mul*** **tan 2008**

Sol. $a=1+m,\ b=-2(1+3m),\ c=1+8m$

Roots are equal if $b^2-4ac=0$

$\Rightarrow\left[-2(1+3m)\right]^2-4(1+m)(1+8m)=0$

$4(1+9m^2+6m)-4(1+m+8m+8m^2)=0$

$4(9m^2+6m+1)-4(8m^2+9m+1)=0$

$\div$ *by* 4 ; $(9m^2+6m+1)-(8m^2+9m+1)=0$

$9m^2+6m+1-8m^2-9m-1=0$

$m^2-3m=0 \quad\Rightarrow\quad m(m-3)=0$

$\boxed{m=0}$ or $\boxed{m-3}$

05. **Show that the roots of** $x^2+(mx+c)^2=a^2$ **will be equal if** $c^2=a^2(1+m^2)$

i. $\mathbf{x^2+(mx+c)^2=a^2}$ ***Federal, Fsd*** **2007, 2009,** ***Mul*** **tan 2008,** ***Sgd*** **2007, 08**

Sol. $x^2+m^2x^2+2mcx+c^2-a^2=0$

$(1+m^2)x^2+2mcx+(c^2-a^2)=0$

$A=1+m^2,\ B=2mc,\ C=c^2-a^2$

Given roots are equal so $B^2-4AC=0$

$(2mc)^2-4(1+m^2)(c^2-a^2)=0$

$\cancel{4m^2c^2}-4c^2+4a^2-\cancel{4m^2c^2}+4m^2a^2=0$

$-4c^2+4a^2+4m^2a^2=0$

$\div$ *by* 4 $\quad\Rightarrow -c^2+a^2+a^2m^2=0$

$or -c^2+a^2(1+m^2)=0$

$\Rightarrow \boxed{c^2=a^2(1+m^2)}$

06. Show that the roots of $(mx+c)^2 = 4ax$ will be equal if $c = a/m$; $m \neq 0$

Sargodha 2010

Sol. $(mx+c)^2 = 4ac$

$m^2x^2 + 2mcx + c^2 - 4ax = 0$

or $m^2x^2 + 2mcx - 4ax + c^2 = 0$

$m^2x^2 + (2mc - 4a)x + c^2 = 0$

$A = m^2,\ B = 2mc - 4a,\ C = c^2$

Roots are equal so $B^2 - 4AC = 0$

$(2mc-4a)^2 - 4m^2c^2 = 0$

$\cancel{4m^2c^2} + 16a^2 - 16amc - \cancel{4m^2c^2} = 0 \Rightarrow 16a^2 - 16amc = 0$

'$\div$' by $16a \Rightarrow a - mc = 0 \Rightarrow mc = a \Rightarrow \boxed{c = a/m}$

07. *Prove that $\dfrac{x^2}{a^2} + \dfrac{(mx+c)^2}{b^2} = 1$ will have equal roots if $c^2 = a^2m^2 + b^2$; $a \neq 0, b \neq 0$*

Faisalabad 2008, Multan 2007

Sol. $\dfrac{x^2}{a^2} + \dfrac{(mx+c)^2}{b^2} = 1$ or $\dfrac{x^2b^2 + a^2(mx+c)^2}{a^2b^2} = 1$

$x^2b^2 + a^2(m^2x^2 + 2mcx + c^2) = a^2b^2$

$x^2b^2 + a^2m^2x^2 + 2a^2mcx + a^2c^2 - a^2b^2 = 0$

$(b^2 + a^2m^2)x^2 + 2a^2mcx + (a^2c^2 - a^2b^2) = 0$

$A = b^2 + a^2m^2, B = 2a^2mc, C = a^2c^2 - a^2b^2$

Roots are equal so $B^2 - 4AC = 0$

$\Rightarrow (2a^2mc)^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) = 0$

$\cancel{4a^4m^2c^2} - 4a^2b^2c^2 + 4a^2b^4 - \cancel{4a^4m^2c^2} + 4a^4m^2b^2 = 0$

'$\div$' by $2a^2b^2 \Rightarrow -4a^2b^2c^2 + 4a^2b^4 + 4a^4m^2b^2 = 0 \Rightarrow -c^2 + b^2 + a^2m^2 = 0$

$\Rightarrow \boxed{c^2 = a^2m^2 + b^2}$

08. Show that the roots of the equation $(a^2 - bc)x^2 + 2(b^2 - ca)x + c^2 - ab = 0$ will be equal, if either $a^3 + b^3 + c^3 = 3abc$ or $b = 0$

Sol. $(a^2 - bc)x^2 + 2(b^2 - ca)x + (c^2 - ab) = 0$

$A = a^2 - bc, B = 2(b^2 - ca), C = c^2 - ab$

Roots are equal so $B^2 - 4AC = 0$

$\left[2(b^2 - ca)\right]^2 - 4(a^2 - bc)(c^2 - ab) = 0$

$4(b^4 + c^2a^2 - 2ab^2c) - 4a^2c^2 + 4a^3b + 4bc^3 - 4ab^2c = 0$

$4b^4+\cancel{4c^2a^2}-8ab^2c-\cancel{4c^2a^2}+4a^3b+4bc^3-4ab^2c=0$

$4b^4-12ab^2c+4a^3b+4bc^3=0$ '÷' by 4

$\Rightarrow b(b^3-3abc+a^3+c^3)=0 \Rightarrow b=0$ or $a^3+b^3+c^3-3abc=0$

$\Rightarrow a^3+b^3+c^3=3abc$ or $b=0$

Exercise 4.8

Solve the following systems of equations:

01. $\mathbf{2x-y=4; 2x^2-4xy-y^2=6}$

Sol. $2x-y=4$ ——— i, $2x^2-4xy-y^2=6$ ——— ii

From i $-y=4-2x \quad \Rightarrow y=2x-4$ ——— iii

(Put iii in ii) $2x^2-4x(2x-4)-(2x-4)^2=6$

$2x^2-8x^2+16x-(4x^2-16x+16)-6=0$

$2x^2-8x^2+16x-4x^2+16x-16-6=0$

$-10x^2+32x-22=0 \quad \Rightarrow 10x^2-32x+22=0$ ('×' by -1)

÷ by 2 both sides

$5x^2-16x+11=0 \quad \Rightarrow 5x^2-5x-11x+11=0$

$5x(x-1)-11(x-1)=0 \Rightarrow (x-1)(5x-11)=0$

$x-1=0$ or $5x-11=0 \Rightarrow x=1$ or $x=\dfrac{11}{5}$

When $x=1$ *then* $y=2(1)-4=2-4=-2$

When $x=\dfrac{11}{5}$ *then* $y=2\left(\dfrac{11}{5}\right)-4=\dfrac{22}{5}-4=\dfrac{22-20}{5}=\dfrac{2}{5}$

$S.S=\left\{(1,-2),(\dfrac{11}{5},\dfrac{2}{5})\right\}$

02. $\mathbf{x+y=5;\ x^2+2y^2=17}$ **Multan 2010, Sargodha 2011**

Sol. $x+y=5$ ———i, $x^2+2y^2=17$ ———ii

From i $x=5-y$ ——— iii

Put value of x in $ii (5-y)^2+2y^2=17$

$25-10y+y^2+2y^2-17=0 \quad \Rightarrow \quad 3y^2-10y+8=0$

$3y^2-6y-4y+8=0 \quad \Rightarrow \quad 3y(y-2)-4(y-2)=0$

$(y-2)(3y-4)=0 \quad \Rightarrow \quad y-2=0$ or $3y-4=0$

$y=2$ or $y=4/3$ *when* $y=2$ *then* $x=5-2=3$

whin $y = 4/3$ *thin* $x = 5 - 4/3 = \frac{15-4}{3} = \frac{11}{3}$

$S.S = \left\{(3,2), (\frac{11}{3}, \frac{4}{3})\right\}$

03. $\mathbf{3x + 2y = 7;\ 3x^2 = 25 + 2y^2}$

Sol. $3x + 2y = 7$ —— I 1, $3x^2 = 25 + 2y^2$ —— II

(*From I*) $2y = 7 - 3x \quad \Rightarrow \quad y = \frac{7-3x}{2}$ —— III

Put value of y *in ii*

$3x^2 = 25 + 2(\frac{7-3x}{2})^2 \quad \Rightarrow \quad 3x^2 = 25 + \cancel{2}(\frac{49 - 42x + 9x^2}{\cancel{4}_2})^2$

$3x^2 = 25 + \frac{9x^2 - 42x + 49}{2}$

(x) *both sides by* 2 $\qquad 6x^2 = 50 + 9x^2 - 42x + 49$

or $9x^2 - 42x + 99 - 6x^2 = 0 \Rightarrow 3x^2 - 42x + 99 = 0$

$\div$ *both sides by* 3 $\qquad x^2 - 14x + 33 = 0$

$x^2 - 3x - 11x + 33 = 0 \Rightarrow x(x-3) - 11(x-3) = 0$

$(x-3)(x-11) = 0 \quad \Rightarrow \quad x - 3 = 0$ *or* $x - 11 = 0$

$x = 3$ *or* $x = 11$

When $x = 3$ *then* $y = \frac{7-3(3)}{2} = \frac{7-9}{2} = \frac{\cancel{2}}{\cancel{2}} = -1$

When $x = 11$ *then* $y = \frac{7-3(11)}{2} = \frac{7-33}{2} = \frac{-26}{2} = -13$

$S.S = \{(3,-1), (11,-13)\}$

04. $\mathbf{x + y = 5;\ \frac{2}{x} + \frac{2}{y} = 2,\ x \neq 0, y \neq 0}$

Sol. $x + y = 5$ —— I $\qquad \frac{2}{x} + \frac{2}{y} = 2$ —— II

From 1 $x = 5 - y$ —— III (×) II *by* xy *both sides*

Put value of x *in* $\qquad 2y + 3x = 2xy$ —— IV

$2y + 3(5-y) = 2(5-y)y \Rightarrow 2y + 15 - 3y = 10y - 2y^2$

$\Rightarrow \quad 2y + 15 - 3y - 10y + 2y^2 = 0 \Rightarrow 2y^2 - 11y + 15 = 0$

$2y^2 - 6y - 5y + 15 = 0 \quad \Rightarrow \quad 2y(y-3) - 5(y-3) = 0$

$(y-3)(2y-5) = 0 \quad \Rightarrow \quad y - 3 = 0$ *or* $2y - 5 = 0$

$(y-3)(2y-5)=0 \quad \Rightarrow \quad y-3=0 \quad or \quad 2y-5=0$

$y=3 \quad or \quad y=\dfrac{5}{2}$

$When \;\; y=3 \;\; then \;\; x=5-3=2$

$When \;\; y=\dfrac{5}{2} \quad then \;\; x=5-\dfrac{5}{2}=\dfrac{10-5}{2}=\dfrac{5}{2}$

$S.S=\left\{(2,3),(\dfrac{5}{2},\dfrac{5}{2})\right\}$

05. $x+y=a+b; \dfrac{a}{x}+\dfrac{b}{y}=2$

Sol. $x+y=a+b$ ———I $\quad \dfrac{a}{x}+\dfrac{b}{y}=2$ ———II

$(From\ 1)\ x=a+b-y$ ———III $\quad (\times)\ II.\ by\ \ xy\ \ we\ get$

$(Put\ value\ of\ x\ \ in\ \ IV) \qquad ay+bx=2xy$ ———IV

$ay+b(a+b-y)=2(a+b-y)y. \quad \Rightarrow ay+ab+b^2-by=2ay+2by-2y^2$

$\Rightarrow \quad ay+ab+b^2-by-2ay-2by+2y^2=0$

$2y^2-ay-3by+ab+b^2=0 \quad \Rightarrow \quad 2y^2-(a+3b)y+ab+b^2=0$

$A=2,\ B=-(a+3b),\ \ C=ab+b^2$

$$y=\frac{-[(-a+3b)]\pm\sqrt{[-(a+3b)]^2-4(2)(ab+b^2)}}{2(2)}$$

$$y=\frac{(a+3b)\pm\sqrt{a^2+6ab+9b^2-8ab-8b}}{4}$$

$$y=\frac{(a+3b)\pm\sqrt{a^2-2ab+b^2}}{4}=\frac{(a+3b)\pm\sqrt{(a-b)^2}}{4}$$

$$y=\frac{(a+3b)\pm(a-b)}{4}$$

$$y=\frac{a+3b+a-b}{4} \quad and \quad y=\frac{\cancel{a}+3b-\cancel{a}+b}{4}$$

$$y=\frac{2a+2b}{4}=\frac{\cancel{2}(a+b)^2}{4}=\frac{a+b}{2} \quad and \quad y=\frac{4b}{4}=b$$

$When \;\; y=\dfrac{a+b}{2} \;\; then\ x=a+b-\dfrac{a+b}{2}=\dfrac{2a+2b-a-b}{2}=\dfrac{a+b}{2}$

$When \;\; y=b \;\; then\ x=a+\cancel{b}-\cancel{b}=a$

$$S.S=\left\{(a,b),(\frac{a+b}{2},\frac{a+b}{2})\right\}$$

06. $3x+4y=25;\ \frac{3}{x}+\frac{4}{y}=2$ ***Rawalpindi* 2009**

Sol. $3x+4y=25$ ———I; $\frac{3}{x}+\frac{4}{y}=2$ ———II

(From 1) $x=\frac{25-4y}{3}$ $(\times)$ II by $xy \Rightarrow 3y+4x=2xy$ ———III

Put value of x in III

$$3y+4\left(\frac{25-4y}{3}\right)=2\left(\frac{25-4y}{3}\right)y$$

$$3y+\frac{100-16y}{3}=\frac{50y-8y^2}{3}$$

$(\times)$ *both sides by 3*

$9y+100-16y=50y-8y^2 \Rightarrow 9y+100-16y-50y+8y^2=0$

$8y^2-57y+100=0 \Rightarrow 8y^2-32y-25y+100=0$

$8y(y-4)-25(y-4)=0 \Rightarrow (y-4)(8y-25)=0$

$y-4=0$ *or* $8y-25=0 \Rightarrow y=4$ *or* $y=\frac{25}{8}$

When $y=4$ *then* $x=\frac{25-(4)4}{3}=\frac{25-16}{3}=\frac{9}{3}=3$

when $y=25/8$ *then* $x=\frac{25-4\left(\frac{25}{8}\right)}{3}=\frac{25-\frac{25}{2}}{3}=\frac{25}{2}\times\frac{1}{3}=\frac{25}{6}$

$$S.S=\left\{(3,4),(\frac{25}{6},\frac{25}{8})\right\}$$

07. $(x-3)^2+y^2=5; 2x=y+6$

Sol. $(x-3)^2+y^2=5$, $2x=y+6$ ———I

or $x^2-6x+9+y^2-5=0$ (From I) $y=2x-6$ ———III

or $x^2+y^2-6x+4=0$ ———II

Put value of III in II

$x^2+(2x-6)^2-6x+4=0$

$x^2+4x^2-24x+36-6x+4=0 \Rightarrow 5x^2-30x+40=0 \Rightarrow x^2-6x+8=0 \div$ *by* 5

$\Rightarrow x^2-2x-4x+8=0$

$x(x-2)-4(x-2)=0 \quad \Rightarrow \quad (x-2)(x-4)=0$

$x-2=0 \quad or \quad x-4=0 \quad \Rightarrow \quad x=2 \quad or \quad x=4$

When $x=2$ *then* $y=2(2)-6=4-6=-2$

When $x=4$ *then* $y=2(4)-6=8-6=2$

$S.S=\{(2,-2),(4,2)\}$

08. $(x+3)^2+(y-1)^2=5 \;;\; x^2+y^2+2x=9$

Sol. $x^2+6x+9+y^2-2y+1-5=0 \;;\; x^2+y^2+2x-9=0$

$x^2+y^2+6x-2y+5=0$ ——— I ; $x^2+y^2+2x-9=0$ ——— II

Subtracting II from I

$$\cancel{x^2+y^2}+6x-2y+5=0$$
$$\cancel{x^2\pm y^2}\pm 2x \quad \mp 9=0$$

$4x-2y+14=0 \quad \Rightarrow 2x-y+7=0 \quad \Rightarrow \quad y=2x+7$ ——— III

Put value of III in II $\quad x^2+(2x+7)^2+2x-9=0$

$x^2+4x^2+28x+49+2x-9=0 \;\Rightarrow\; 5x^2+30x+40=0 \Rightarrow\; x^2+6x+8=0$

$\Rightarrow x^2+2x+4x+8=0$

$x(x+2)+4(x+2)=0 \quad \Rightarrow \quad (x+2)(x+4)=0$

$x+2=0 \quad or \quad x+4=0 \quad \Rightarrow \quad x=-2,\ x=-4$

When $x=-2$ *then* $y=2(-2)+7=-4+7=3$.

When $x=-4$ *then* $y=2(-4)+7=-8+7=-1 \Rightarrow S.S=\{(-2,3),(-4,-1)\}$

09. $x^2+(y+1)^2=18 \;;\; (x+2)^2+y^2=21$

Sol. *or* $x^2+y^2+2y+1-18=0 \;;\; x^2+4x+4+y^2-21=0$

$x^2+y^2+2y-17=0$ ____ I ; $x^2+y^2+4x-17=0$ ____ II

$II-I \Rightarrow$

$$\cancel{x^2+y^2}+4x-\cancel{17}=0$$
$$\cancel{x^2\pm y^2}\pm 2y\mp\cancel{17}$$

$4x-2y=0 \;\Rightarrow\; 4x=2y \;\Rightarrow\; 2x=y$ ____ III

Put value of III in II

$x^2+(2x)^2+4x-17=0 \quad \Rightarrow \quad x^2+4x^2+4x-17=0$

$5x^2+4x-17=0 \quad \Rightarrow \quad x=\dfrac{-4\pm\sqrt{(4)^2-4(5)(-17)}}{2(5)}$

$$=\frac{-4\pm\sqrt{16+340}}{10}=\frac{-4\pm\sqrt{356}}{10}$$

$$=\frac{-4+\sqrt{4\times 89}}{10}=\frac{-4\pm 2\sqrt{89}}{10}=\frac{\cancel{2}(-2\pm\sqrt{89})}{\cancel{10}^{5}}$$

$$x=\frac{-2+\sqrt{89}}{5}\ \&\ x\frac{-2-\sqrt{89}}{5}$$

$$When\ \ x=\frac{-2+\sqrt{89}}{5}\ then\ y=2\left(\frac{-2+\sqrt{89}}{5}\right)=\frac{-4+2\sqrt{89}}{5}$$

$$When\ \ x=\frac{-2-\sqrt{89}}{5}\ then\ y=2\left(\frac{-2-\sqrt{89}}{5}\right)=\frac{-4-2\sqrt{89}}{5}$$

$$S.S=\left\{\left(\frac{-2+\sqrt{89}}{5}\right),\left(\frac{-4+\sqrt{89}}{5}\right)\right\},\left\{\left(\frac{-2-\sqrt{89}}{5}\right);\left(\frac{-4+2\sqrt{89}}{5}\right)\right\}$$

10. $x^2+y^2+6x=1\ ;\ x^2+y^2+2(x+y)=3$

Sol. $x^2+y^2+6x=1$ ——I, $\quad x^2+y^2+2x+2y=3$ ——II

$$II-I\qquad \cancel{x^2+y^2}+2x+2y=3$$
$$\underline{\cancel{x^2\pm y^2}\pm 6x\qquad =1}$$

$-4x+2y=2 \quad\Rightarrow\quad -2x+y=1\ \Rightarrow\ y=2x+1$

$(Put\ value\ of\ y\ in\ \ I)\qquad x^2+(2x+1)^2+6x=1$

$x^2+4x^2+4x+\cancel{1}+6x-\cancel{1}=0\ \Rightarrow\ 5x^2+10x=0$

$5x(x+2)=0\ \Rightarrow\ 5x=0\ \ or\ \ x+2=0$

$x=0\ \ or\ \ x=-2$

$When\ x=0\qquad then\quad y=2\ (0)+1=0+1=1$

$When\ x=-2\qquad then\quad y=2\ (-2)+1=-4+1=-3\Rightarrow S.S=\{(0,1),(-2,-3)\}$

Example-1 (Exercise 4.9): $x^2+y^2=25$ ———I $\qquad$ ***Sargodha* 2010**

Sol. $\qquad 2x^2+3y^2=66$ ______ II

$$II-2\times I\Rightarrow\qquad 2x^2+3y^2=66$$
$$\underline{\underline{2x^2}\pm 2y^2=\underline{50}}$$
$$y^2=16\Rightarrow y=\pm 4$$

$(Put\ in\ I)\ x^2+(\pm 4)^2=25\ \Rightarrow\ x^2+16=25$

$x^2=25-16=9\Rightarrow\ x=\pm 3\Rightarrow S.S=\{(\pm 3,\pm 4)\}$

Exercise 4.9

Show the following systems of Equations:

01. $2x^2 = 6 + 3y^2;\ 3x^2 - 5y^2 = 7$

Sol. $\Rightarrow 2x^2 - 3y^2 = 6$ —— I & $3x^2 - 5y^2 = 7$ —— II

(×) I by3 II by 2, then subtracting

$$\cancel{6x^2} - 9y^2 = 18$$

$$\underline{-\cancel{6x^2} \mp 10y^2 = 14}$$

$y^2 = 4 \quad \Rightarrow \quad y \pm 2$

Put $y = x \pm 2$ in I

$2x^2 - 3(\pm 2)^2 = 6$

$2x^2 - 3(4) = 6$

$2x^2 = 6 + 12 = 18$

$x^2 = 9 \quad \Rightarrow \quad x = \pm 3$

$S.S = \{(3,2),(3,-2),(-3,2),(-3,-2)\}$ or $\{(\pm 3, \pm 2)\}$

02. $8x^2 = y^2 \quad ; \quad x^2 + 2y^2 = 19$

Sol. $8x^2 = y^2 \quad \Rightarrow \quad 8x^2 - y^2 = 0$ —— I

$x^2 + 2y^2 = 19$ —— II

(×) I by 2 and add in II

$$16x^2 - \cancel{2y^2} = 0$$

$$\underline{x^2 + \cancel{2y^2} = 19}$$

$17x^2 = 19 \quad \Rightarrow \quad x^2 = \frac{19}{17} \Rightarrow \quad x = \pm\sqrt{\frac{19}{17}}$

$8\left(\pm\sqrt{\frac{19}{17}}\right)^2 - y^2 = 0 \qquad$ (I become)

$\Rightarrow \quad 8\left(\frac{19}{17}\right) - y^2 \Rightarrow y^2 = 4 \times \frac{38}{17} \Rightarrow y = \pm 2\sqrt{\frac{38}{17}}$

$S.S = \left\{\pm\sqrt{\frac{19}{17}}. \pm 2\sqrt{\frac{38}{17}}\right\}$

03. $2x^2-8=5y^2 \quad ; \quad x^2-13=-2y^2$

Sol. $2x^2-8=5y^2 \quad , x^2-13=-2y^2$

$\Rightarrow 2x^2-5y^2=8 \text{ —— } I$

$x^2+2y^2=13 \text{ —— } II \quad \Rightarrow \quad x^2+4y^2=26 \text{ —— } III$

$I-III \Rightarrow$

$\cancel{2x^2}-5y^2=8$

$\underline{\cancel{-2x^2}\pm 4y^2=26}$

$-9y^2=-18 \quad \Rightarrow \quad y^2=2 \quad \Rightarrow \quad y=\pm\sqrt{2}$

$Put \; in \; II \quad x^2+2(\pm\sqrt{2})^2=13$

$x^2=13-2(2)=13-4=9 \Rightarrow x=\pm 3$

$S.S=\left\{(\pm 3,\pm\sqrt{2})\right\}$

04. $x^2-5xy+6y^2=0 \quad ; \quad x^2+y^2=45$

Sol. $x^2-5xy+6y^2=0 \text{ ——— } I \; , x^2+y^2=45 \text{ ——— } II$

$(from\ I)\, x^2-5xy+6y^2=0$

$x^2-2xy-3xy+6y^2=0$

$x(x-2y)-3y(x-2y)=0$

$(x-2y)(x-3y)=0$

$x-2y=0 \text{ ——— } III \; or \quad x-3y=0 \text{ ——— } IV$

$\Rightarrow x=2y \; Put\ value\ of\ x\ in \text{ ——— } II$

$(2y)^2+y^2=45$

$4y^2+y^2=45$

$5y^2=45 \quad \Rightarrow \quad y=9 \quad \Rightarrow \quad y=\pm 3$

$When \quad y=3 \; then \; x=2(3)=6$

$When \quad y=-3 \; then \; x=2(-3)=-6$

$from \;\; IV \quad x=3y$

$Put \;\; Value \;\; of \;\; x \; in \quad II$

$(3y)^2+y^2=45$

$9y^2+y^2=45 \quad \Rightarrow \quad 10y^2=45$

$y^2=\frac{\cancel{45}}{\cancel{10}} \quad \Rightarrow \quad y^2=\frac{9}{2} \Rightarrow y=\frac{\pm 3}{\sqrt{2}}$

$When \quad y=\frac{3}{\sqrt{2}},\; x=3\left(\frac{3}{\sqrt{2}}\right)=\frac{9}{\sqrt{2}}$

$$When \quad y=\frac{-3}{\sqrt{2}} \quad then \ x=3\left(-\frac{3}{\sqrt{2}}\right)=\frac{-9}{\sqrt{2}}$$

$$S.S=\left\{(6,3),(-6,-3),\left(\frac{9}{\sqrt{2}},\frac{3}{\sqrt{2}}\right),\left(\frac{-9}{\sqrt{2}},\frac{-3}{\sqrt{2}}\right)\right\}$$

05. $\mathbf{12x^2-25xy+12y^2=0 \quad ; \quad 4x^2+7y^2=148}$ ***Mul* tan 2010**

Sol. $12x^2-25xy+12y^2=0$ —— I, $4x^2+7y^2=148$ —— II

$$(from\ I)\ 12x^2-25xy+12y^2=0$$

$$12x^2-16xy-9xy+12y^2=0$$

$$4x(3x-4y)-3y(3x-4y)=0$$

$$(3x-4y)(4x-3y)=0$$

$$3x-4y=0 \text{——} III \quad or \quad 4x-3y=0 \text{——} IV$$

$$from\ III\ \ 3x=4y \quad \Rightarrow \quad x=\frac{4}{3}y$$

Put value of x in——II

$$4\left(\frac{4}{3}y\right)^2+7y^2=148$$

$$4\left(\frac{16}{9}y^2\right)+7y^2=148$$

$$\frac{64y^2}{9}+7y^2=148$$

(×) by 9 both sides

$$64y^2+63y^2=1332 \quad \Rightarrow \quad 127y^2=1332$$

$$y^2=\frac{1332}{127}=\frac{4\times 333}{127}$$

$$y=\pm 2\sqrt{\frac{333}{127}}=\pm 2\sqrt{\frac{9\times 37}{127}}=\pm 6\sqrt{\frac{37}{127}}$$

$$When \quad y=\pm 2\sqrt{\frac{333}{127}} \quad then \ x=\frac{4}{3}\left(\pm 2\sqrt{\frac{333}{127}}\right)$$

$$x=\pm\frac{8}{3}\sqrt{\frac{333}{127}}=\pm\frac{8}{3}\sqrt{\frac{9\times 37}{127}}=\pm\frac{8\times 3}{3}\sqrt{\frac{37}{127}}\Rightarrow x=\pm 8\sqrt{\frac{37}{127}}$$

$$from\ V \Rightarrow \quad 4x=3y \quad \Rightarrow \quad x=\frac{3}{4}y$$

Put Value of x in II

$$4\left(\frac{3}{4}y\right)^2 + 7y^2 = 148$$

$$4\left(\frac{9}{16}y^2\right) + 7y^2 = 148$$

$$\frac{9}{4}y^2 + 7y^2 = 148$$

(x) *by* 4

$$9y^2 + 28y^2 = 592$$

$$37y^2 = 592$$

$$y^2 = \frac{\cancel{592}}{\cancel{37}} = 16 \Rightarrow y = \pm 4$$

When $y = 4$ *then* $x = \frac{3}{4}(4) = 3$

When $y = -4$ *then* $x = \frac{3}{4}(-4) = -3$

$$S.S = \left\{(3,4),(-3,-4),\left(\pm 8\sqrt{\frac{37}{127}}, \pm 6\sqrt{\frac{37}{127}}\right)\right\}$$

06. $\mathbf{12x^2 - 11xy + 2y^2 = 0 \;\; ; \;\; 2x^2 + 7xy = 60}$

Sol. $12x^2 - 11xy + 2y^2 = 0$ —— I

$2x^2 + 7xy = 60$ —— II

$(from\ I)$ $12x^2 - 11xy + 2y^2 = 0$

$12x^2 - 8xy - 3xy + 2y^2 = 0$

$4x(3x - 2y) - y(3x - 2y) = 0$

$(3x - 2y)(4x - y) = 0$

$3x - 2y = 0$ ____ III *or* $4x - y = 0$ ______ IV

from III $3x = 2y \Rightarrow x = \frac{2}{3}y$

Put $x = \frac{2}{3}y$ *in* ____ II

$$2\left(\frac{2}{3}y\right)^2 + 7\left(\frac{2}{3}y\right)y = 60$$

$$2\left(\frac{4}{9}y^2\right) + \frac{14}{3}y^2 = 60$$

$$\frac{8y^2}{9}+\frac{14}{3}y^2=60$$

(x) *by* 9 *we get*

$$8y^2+42y^2=540$$

$$50y^2=540$$

$$\Rightarrow y=\frac{\cancel{540}}{\cancel{50}}=\frac{54}{5}$$

$$y=\pm\sqrt{\frac{54}{5}}=\pm3\sqrt{\frac{6}{5}}$$

When $y=\pm3\sqrt{\frac{6}{5}}$ *then* $x=\frac{2}{\cancel{3}}\left(\cancel{\pm3}\sqrt{\frac{6}{5}}\right)=\pm2\sqrt{\frac{6}{5}}$

from IV $\Rightarrow$ $y=4x$

Put Value of y *in* II

$$2x^2+7x(4x)=60$$

$$2x^2+28x^2=60$$

$30x^2=60 \quad\Rightarrow\quad x^2=2 \Rightarrow \quad x=\pm\sqrt{2}$

When $x=\pm\sqrt{2}$ *then* $y=\pm4\sqrt{2}$

$$S.S=\left\{(\pm\sqrt{2},\pm4\sqrt{2}),\left(\pm2\sqrt{\frac{6}{5}},\pm3\sqrt{\frac{6}{5}}\right)\right\}$$

07. $x^2-y^2=16$; $xy=15$

Sol. $x^2-y^2=16$ ——— I $\quad xy=15$ ——— II

$(from\ II)\ y=\frac{15}{x}$

Put value of y *in* I

$$x^2-\left(\frac{15}{x}\right)^2=16$$

$$x^2-\frac{225}{x^2}=16$$

or $x^4-225=16x^2$

$$x^4-16x^2-225=0$$

$$x^4+9x^2-25x^2-225=0$$

$$x^2(x^2+9)-25(x^2+9)=0$$

$$(x^2+9)(x^2-25)=0$$

$$x^2+9=0 \quad or \quad x^2-25=0$$

$$x^2=-9 \quad or \quad x^2=25$$

$$x=\pm\sqrt{-9} \quad or \quad x=\pm 5$$

$$x=\pm 3i \quad or \quad x=\pm 5$$

$$When\ \ x=\pm 5\ then\ y=\pm\frac{\cancel{15}}{\cancel{5}}=\pm 3$$

$$When\ \ x=\pm 3i\ then\ y=\pm\frac{15}{3i}$$

$$y=\pm\frac{5}{i}=\pm\frac{5}{i}\times\frac{i}{i}=\pm\frac{5i}{i^2}=\pm\frac{5i}{-1}=\pm 5i$$

$$S.S=\{(\pm 5,\pm 3),(\pm 3i,\pm 5i)\}\ or\ S.S=\{(5,3),(-5,-3),(3i,5i),(-3i,-5i)\}$$

08. $x^2+xy=9 \;\; ; \;\; x^2-y^2=2$

Sol. $x^2+xy=9$ ______ I, $x^2-y^2=2$ ______ II

(x) I *by* 2 *and* II *by* 9 *we get*

$2x^2+2xy=18$ ____ III ; $9x^2-9y^2=18$ ____ IV

Solving III & IV

$$9x^2-9y^2=\cancel{18}$$

$$\underline{\underline{2}x^2\pm 2xy=\pm\cancel{18}}$$

$$7x^2-2xy-9y^2=0$$

$$7x^2-9xy+7xy-9y^2=0$$

$$x(7x-9y)+y(7x-9y)=0$$

$$(7x-9y)(x+y)=0$$

$7x-9y=0$ ——— III *or* $x+y=0$ ——— IV

(*from* IV) $x=-y$ *put in* II.

$(-y)^2-y^2=2 \;\Rightarrow\; y^2-y^2=2 \;\Rightarrow\; 0=2$ (*Not possible*)

from III $7x-9y=0 \;\Rightarrow\; 7x=9y \;\Rightarrow\; x=\frac{9y}{7}$ ______ V

Put in II

$$x^2-y^2=2 \quad\Rightarrow\quad \left(\frac{9y}{7}\right)^2-y^2=2$$

$$\frac{81y^2}{49}-y^2=2\Rightarrow 81y^2-49y^2=98\Rightarrow 32y^2=98\Rightarrow y^2=\frac{98}{32}\Rightarrow y^2=\frac{49}{16}\Rightarrow y=\pm\frac{7}{4}$$

$When\ \ y=\frac{7}{4}\ then\quad x=\frac{9}{7}\left(\frac{7}{4}\right)=\frac{9}{4}\quad use\ V$

$When\ \ y=\frac{7}{4}\ then\quad x=\frac{9}{4}\left(\frac{-7}{4}\right)=\frac{-9}{4}$

$S.S=\left\{\left(\frac{9}{4},\frac{7}{4}\right),\left(\frac{-9}{4},\frac{-7}{4}\right)\right\}$

09. $y^2-7=2xy\quad;\quad 2x^2+3=xy$ **Sargodha 2010**

Sol. $y^2-7=2xy$ —— I ; $2x^2+3=xy$ —— II

$or\ y^2-2xy=7$ —— III or $2x^2-xy=-3$ —— IV

(X) III by 3 & IV by 7

$3y^2-6xy=21$ —— V & $14x^2-7xy=-21$ —— VI

$VI+V$

$$\begin{array}{r} 14y^2-7xy\quad=-\cancel{21}\\ -6xy+3y^2=\cancel{21}\\ \hline \end{array}$$

$14x^2-13xy+3y^2=0$

$14x^2-7xy-6xy+3y^2=0$

$7x(2x-y)-3y(2x-y)=0$

$(2x-y)(7x-3y)=0$

$2x-y=0$ —— VII or $7x-3y=0$ —— $VIII$

$(from\ \ VII)\quad y=2x\ \ put\ in$ —— II

$2x^2+3=x(2x)\quad\Rightarrow 2x^2-2x^2+3=0\quad\Rightarrow 3=0$

$from\ \ VIII\quad 7x-3y=0\quad\Rightarrow\quad 3y=7x\quad\Rightarrow\quad y=7x/3$

$Put\ value\ of\ \ y\ in\ II\quad 2x^2+3=x(7x/3)$

$2x^2+3=\frac{7x^2}{3}$ 'x' by 3 we get $6x^2+9=7x^2$

$-6x^2+7x^2=9\quad\Rightarrow\quad x^2=9\quad\Rightarrow\quad x=\pm3$

$When\ \ x=3\ then\ \ y=\frac{7}{3}(3)\quad\Rightarrow y=7$

$When\ \ x=-3\ then\ \ y=\frac{7}{3}(-3)\quad\Rightarrow y=-7\Rightarrow S.S=\{(3,7),(-3,-7)\}$

10. $x^2 + y^2 = 5$; $xy = 2$ **Multan 2010**

Sol. $x^2 + y^2 = 5$ __ I ; $xy = 2$ _____ II $\Rightarrow y = 2/x$ _____ III

Put III in I

$$x^2 + \left(\frac{2}{x}\right)^2 = 5$$

$$x^2 + \frac{4}{x^2} = 5 \quad \text{'×' by } x^2 \text{ we get}$$

$$x^4 + 4 = 5x^2 \Rightarrow \quad x^4 - 5x^2 + 4 = 0$$

$$x^4 - x^2 - 4x^2 + 4 = 0 \quad \Rightarrow \quad x^2(x^2-1) - 4(x^2-1) = 0$$

$$(x^2-4)(x^2-1) = 0 \quad \Rightarrow \quad x^2 - 4 = 0 \quad or \quad x^2 - 1 = 0$$

$$(x^2-4)(x^2-1) = 0 \quad \Rightarrow \quad x = \pm 2 \quad or \quad x = \pm 1$$

$$When\ x = 2 \quad then \quad y = \frac{\cancel{2}}{\cancel{2}} = 1$$

$$When\ x = -2 \quad then \quad y = \frac{\cancel{2}}{\cancel{2}} = -1$$

$$When\ x = 1 \quad then \quad y = \frac{2}{1} = 2$$

$$When\ x = -1 \quad then \quad y = \frac{1}{-1} = -2$$

$$S.S = \{(2,1), (-2,-1), (1,2), (-1,-2)\}$$

Exercise 4.10

01. The product of one less than a certain positive number and two less than three time the number is 14 find the number? Multan 2007

Sol *Let x be the positive number then one less then positive number* $= x - 1$ *two less then three time* $= 3x - 2$ *then*

According to the given condition equation is

$$(x-1)(3x-2) = 14 \quad \Rightarrow 3x^2 - 2x - 3x + 2 = 14$$

$$\Rightarrow 3x^2 - 5x - 12 = 0 \Rightarrow 3x^2 - 9x + 4x - 12 = 0$$

$$\Rightarrow 3x(x-3) + 4(x-3) = 0 \Rightarrow (x-3)(3x+4) = 0$$

$$\Rightarrow x - 3 = 0 \quad or \quad 3x + 4 = 0 \quad \Rightarrow \quad x = 3 \quad or \quad x = -4/3 \ (ignore)$$

$$-4/3 \text{ is not possible so } \boxed{x = 3}$$

02. The sum of a positive number and its square is 380.find the number.

Sol. Let a positive number $= x$, Square $= x^2$

According to the given condition equation is

$x + x^2 = 380 \Rightarrow x^2 + x - 380 = 0$

$\Rightarrow x^2 + 20x - 19x - 380 = 0 \Rightarrow x(x+20) - 19(x+20) = 0$

$\Rightarrow (x+20)(x-19) = 0 \Rightarrow x + 20 = 0 \ \ or \ \ x - 19 = 0$

$\Rightarrow x = -20 \ \ or \ \ x = 19 \qquad -20 \ is \ not \ positive \ \ so \ \boxed{x = 19}$

03. Divide 40 into two parts such that the sum of their square is greater than 2 times their product by 100.

Sol. Let one part $= x$, Another part $= 40 - x$

According to the given condition equation is

$(x)^2 + (40 - x)^2 = 2x(40 - x) + 100$

$x^2 + 1600 - 80x + x^2 = 80x - 2x^2 + 100$

$\Rightarrow x^2 + 1600 - 80x + x^2 - 80x + 2x^2 - 100 = 0$

$\Rightarrow 4x^2 - 160x + 1500 = 0$ (Dividing by 4 both sides)

$\Rightarrow x^2 - 40x + 375 = 0 \Rightarrow x^2 - 25x - 15x + 375 = 0$

$\Rightarrow x(x - 25) - 15(x - 25) \Rightarrow (x - 25)(x - 15) = 0$

$\Rightarrow x - 25 = 0 \ \ or \ \ x - 15 = 0 \Rightarrow x = 25 \ \ or \ \ x = 15$ are required numbers.

04. The sum of positive number and its reciprocal is $\frac{26}{5}$ Find the number.

Sol. Let the number $= x$, Its reciprocal $= 1/x$ **Faisalabad 2008**

According to the given condition equation is

$$x + \frac{1}{x} = \frac{26}{5} \Rightarrow \frac{x^2 + 1}{x} = \frac{26}{5}$$

$\Rightarrow 5(x^2 + 1) = 26x \Rightarrow 5x^2 + 5 - 26x = 0$

$\Rightarrow 5x^2 - 26x + 5 = 0 \Rightarrow 5x^2 - x - 25x + 5 = 0$

$\Rightarrow x(5x - 1) - 5(5x - 1) = 0 \Rightarrow (5x - 1)(x - 5) = 0$

$\Rightarrow 5x - 1 = 0 \ \ or \ \ x - 5 = 0 \Rightarrow \boxed{x = 1/5} \ \ or \ \ \boxed{x = 5}$ are required number.

05. A number exceeds its square roots by 56. Find the number. **Sargodha 2008**

Sol. Let the number $= x$, Its square root $= \sqrt{x}$

According to the given condition equation is

$x = \sqrt{x} + 56 \ I \Rightarrow x - 56 = \sqrt{x}$

Squaring both sides

$x^2 - 112x + 3136 = x \Rightarrow x^2 - 112x - x + 3136 = 0$

$\Rightarrow x^2 - 113x + 3136 = 0 \Rightarrow x^2 - 64x - 49x + 3136 = 0$

$x(x-64)-49(x-64)=0 \Rightarrow (x-64)(x-49)=0$

$\Rightarrow x-64=0 \ or \ x-49=0 \Rightarrow x=64 \ or \ x=49$

$(Put\ x=64\ in\ I)\ 64=\sqrt{64}+56 \Rightarrow 64=64$

$(Put\ x=49\ in\ I)\ 64=\sqrt{49}+56 \Rightarrow 64=63$ (not possible) Hence $\boxed{x=64}$

06. Find two consecutive number, whose product is 132. **Faisalabad 2007**

Sol. Let two consecutive number are $x \ and \ x+1$ then

According to the given condition equation is

$x(x+1)=132 \Rightarrow x^2+x-132=0$

$\Rightarrow x^2+12-11x-132=0 \Rightarrow x(x+12)-11(x+12)=0$

$\Rightarrow (x+12)(x-11)=0 \Rightarrow x+12=0 \ or \ x-11=0$

$\Rightarrow x=-12 \ or \ x=11$ $\boxed{ignor \quad x=-12}$

$IF \ x=11 \quad then \quad x+1=12$

Hence Required number 11, 12.

07. The difference between the cubes of two consecutive even number is 296. Find them.

Sol. Let two consecutive numbers are $x \ and \ x+2$ then

According to the given condition equation is

$(x+2)^3-x^3=296$

$\Rightarrow \cancel{x^3}+6x^2+12x+8-\cancel{x^3}-296=0$

$\Rightarrow 6x^2+12x-288=0 \quad (\div) by\ 6 \Rightarrow x^2+2x-48=0$

$\Rightarrow x^2+8x-6x-48=0 \Rightarrow x(x+8)-6(x+8)=0$

$\Rightarrow (x+8)(x-6)=0 \Rightarrow x+8=0 \quad or \quad x-6=0$

$\Rightarrow x=-8 \quad or \quad x=6 \quad (Ignor \ x=-8)$

$IF \ x=6 \quad then \quad x+2=6+2=8$

$Required \ number \ 6,8$

08. A former bought some sheep for Rs.9000. If he had paid Rs.100 less for each, he would have got 3 sheep more for the same money. How many sheep did he buy, when the rate in each case is uniform?

Sol. Let number of sheep $= x$

Rate of $x \ sheep = 9000$

Rate of one $sheep = \dfrac{9000}{x}$

If three sheep are more, then rate of one sheep $= \dfrac{9000}{x+3}$

According to the given condition equation is

$$\frac{9000}{x}-100=\frac{9000}{x+3} \Rightarrow \frac{9000-100x}{x}=\frac{9000}{x+3}$$

$$\Rightarrow (x+3)(9000-100x)=9000x \Rightarrow 9000x-100x^2+27000-300x=9000x$$

$$\Rightarrow \cancel{9000x}-\cancel{9000x}+100x^2-27000+300x=0$$

$$\Rightarrow 100x^2+300x-27000=0 \quad (\div)\ by\ 100$$

$$x^2+3x-270=0 \Rightarrow x^2+18x-15x-270=0$$

$$\Rightarrow x(x+18)-15(x+18)=0 \Rightarrow (x+18)(x-15)=0 \Rightarrow x=-18\ or\ x=15$$

$$-18\ Not\ possible\ so\ x=15=number\ of\ sheep$$

09. A man sold his stock of eggs for Rs.240. If he had 2 dozen more, he would have got the same money be selling the whole for Rs.0.50 per dozen cheaper. How many dozen eggs did he sell.

Sol. Let number of eggs $= x\ dozen$

Rate of x dozen eggs $= 240$

Rate of 1 dozen eggs $=\dfrac{240}{x}$

If 2 dozen are more then rate of one dozen $=\dfrac{240}{x+2}$

According to the given condition equation is

$$\frac{240}{x}-0.5=\frac{240}{x+2} \Rightarrow \frac{240-0.5x}{x}=\frac{240}{x+2}$$

$$\Rightarrow (x+2)(240-0.5x)=240x \Rightarrow 240x-0.5x^2+480-x=240x$$

$$or\quad \cancel{240x}-\cancel{240x}+0.5x^2-480+x=0 \Rightarrow 0.5x^2+x-480=0$$

$$(x)\ by\ 2\quad x^2+2x-960=0$$

$$\Rightarrow x^2+32x-30x-960=0 \Rightarrow x(x+32)-30(x+32)=0$$

$$\Rightarrow (x+32)(x-30)=0 \Rightarrow x+32=0\ or\ x-30=0$$

$$\Rightarrow x=-32\ Not\ possible \Rightarrow x=30\ dozen=number\ of\ eggs$$

10. A cyclist travelled 48km at a uniform speed. Hed he travelled 2 km/hour slower, he would have taken 2 hours more to perform the journey. How long did he take to cover 48km?

Sol Let speed=V ; Time=t

According to the given condition equation is :

$$Distance=S=vt=48\text{—}I\ \&\ (v-2)(t+2)=48\text{—}II\ (two\ km\ slow=v-2\ two\ hour\ more=t+2)$$

$$II \Rightarrow vt+2v-2t-4=48$$

$$(put\ vt=48)\ \cancel{48}+2v-2t-4-\cancel{48}=0$$

$$2v-2t-4=0\quad ((\div)\ by\ 2)\ v-t-2=0 \Rightarrow v=t+2\text{——}III$$

$$Put\ III\ in\ I \Rightarrow vt=48 \Rightarrow (t+2)t=48$$

$$t^2+2t-48=0$$
$$t^2+8t-6t-48=0$$
$$t(t+8)-6(t+8)=0$$
$$(t+8)(t-8)=0$$
$$(ignore, t=-8)\Rightarrow \boxed{t=6}$$

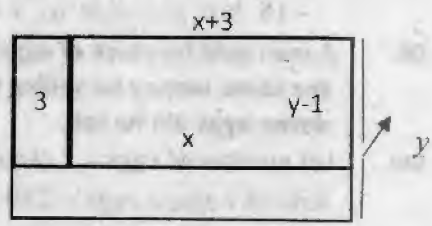
11. The area of a rectangular field is 297 square meters. Had it been 3 meters longer and one meter shorter, the area would have been 3 square meters more. Find its length and breadth. **Federal**

Sol Length = x breadth = y

IF 3 meters longer then length $= x+3$

IF 1 meter shorter then breadth $= y-1$

According to the given condition equatio is.



$$xy=297-I \;\&\; (x+3)(y-1)=297+3$$
$$\Rightarrow xy-x+3y-3=300$$
$$(put\ xy=297)\ 297-x+3y-3=300\Rightarrow -x+3y+297-3-300=0$$
$$\Rightarrow -x+3y-6=0 \Rightarrow x=3y-6$$
$$Put\ in\ I \quad (3y-6)y=297\Rightarrow 3y^2-6y-297=0\Rightarrow y^2-2y-99=0(\div by 3)$$
$$y^2-11y+9y-99=0 \Rightarrow y(y-11)+9(y-11)=0$$
$$\Rightarrow (y-11)(y+9)=0 \Rightarrow y-11=0\ or\ y+9=0$$
$$\Rightarrow y=11\ or\ y=-9\ (Not\ possible)\ When\ y=11\ then\ x=\frac{297}{11}=27$$

So length $=27$; *breadth* $=11$

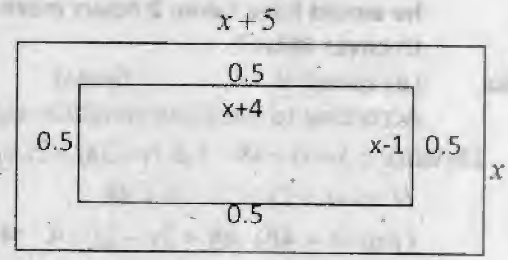
12. The length of a rectangular piece of paper exceeds its breadth by 5cm. If a strip 0.5cm wide be cut all around the piece of paper, the area of the remaining part would be 500 square cms. Find its original dimensions.

Sol *Let breadth* $= x$

then length $= x+5$

New length $= x+5-0.5-0.5 = x-1$

According to the given Condition equation is



$$(x-1)(x+4)=500 \Rightarrow x^2+4x-x-4=500$$

$\Rightarrow x^2+3x-4=500 \Rightarrow x^2+3x-504=0$

$\Rightarrow x^2+24x-21x-504=0 \Rightarrow x(x+24)-21(x+24)=0$

$\Rightarrow (x+24)(x-21)=0 \Rightarrow x+24=0$ or $x-21=0$

$\Rightarrow x=-24$ or $x=21$ then $length = x+5=21+5=24$

Not possible $\quad$ $breadth = x = 21$

13. A number consists of two digits whose product is 18. If the digits are interchanged, the new number 27 less than the original number. Find the number.

Sol *Let digits are x & y then*

$xy=18$ —— I

$Number = 10x+y$

$Reverse = x+10y$

According to the given Condition equation is

$x+10y=10x+y-27 \Rightarrow 10x+y-x-10y-27=0$

$\Rightarrow 9x-9y-27=0$ Divide by 9 $\Rightarrow x-y-3=0 \Rightarrow y=x-3$

Put in $I \Rightarrow x(x-3)=18$

$\Rightarrow x^2-3x-18=0 \Rightarrow x^2-6x+3x-18=0$

$\Rightarrow x(x-6)+3(x-6)=0 \Rightarrow (x-6)(x+3)=0$

$\Rightarrow x-6=0$ or $x+3=0 \Rightarrow x=6$ or ($x=-3$ ignore)

Use I When $x=6$ then $6y=18 \Rightarrow y=3$ So $Number = 10x+y=10(6)+3=63$

14. A number consists of two digits whose product is 14. If the digits are interchanged, the resulting number will exceed the original number by 45. Find the number.

Sol *Let digits are x & y then*

Two digit $Number = 10x+y$

$Reversed = x+10y$

According to the given Condition equation is

$xy=14$ ——— I

Also $(x+10y)=(10x+y)+45$

$\Rightarrow 10x+y+45-x-10y=0 \Rightarrow 9x-9x+45=0$

('÷' by 9) $x-y+5=0 \Rightarrow y=x+5$

Put in $I \Rightarrow (x+5)x=14 \Rightarrow x^2+5x-14=0$

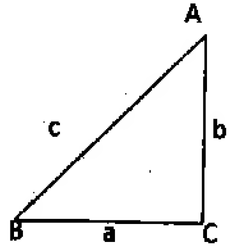
$\Rightarrow x^2+7x-2x-14=0 \Rightarrow x(x+7)-2(x+7)=0$

$\Rightarrow (x+7)(x-2)=0 \Rightarrow x+7=0$ or $x-2=0$

$\Rightarrow x=-7$ or $x=2$ (Ignore $x=-7$) So $x=2$ put in I $2y=14 \Rightarrow y=7$

So required $number = 10x+y=10(2)+7=27$

15. The area of a right triangle is 210 square meters. If its hypotenuse is 37 meters long. Find the length of the base and the altitude.



Sol $Let\ base = a \quad \& \quad Altitude = b$

$then\ Area = \frac{1}{2}(base)(altitude) = \frac{1}{2}ab = 210$

$\Rightarrow \frac{1}{2}ab = 210 \ \Rightarrow \ ab = 420 - I \Rightarrow 2ab = 840 - II$

$(by\ pythagoras\ theorem) \quad a^2 + b^2 = c^2 \quad \Rightarrow \quad a^2 + b^2 = (37)^2$

$\Rightarrow a^2 + b^2 = 1369 - III \qquad III - II \quad a^2 + b^2 = 1369$

$$\begin{array}{r} 2ab = 840 \\ \hline a - 2ab + b^2 = 529 \Rightarrow (a-b)^2 = (23)^2 \end{array}$$

$\Rightarrow a - b = 23 \Rightarrow b = a - 23\ put\ in\ I \Rightarrow (a-23)\ a = 420 \Rightarrow a^2 - 23a - 420 = 0$

$a^2 - 35a + 12a - 420 = 0 \Rightarrow a(a-35) + 12(a-35) = 0$

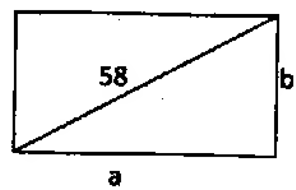
$(a-35)(a+12) = 0 \Rightarrow a - 35 = 0\ or\ a + 12 = 0$

$\Rightarrow a = 35\ or\ a = -12\ (not\ possible)$

$When\ a = 35m\ then\ b = \frac{\cancel{420}}{\cancel{35}} = 12m\ So\ a = base = 35\ \ b = Altitide = 12$

16. The are of a rectangle is 1680 square meters. If its diagonal is 58 meters long, find the length and the breadth of the rectangle.

Sol $Let\ base = a \quad \& \quad Altitude = b$



$then\ Area = (length)(breadth) \Rightarrow 1680 = ab —— I$

$\Rightarrow 2ab = 3360 \quad II \quad By\ pythagoras\ theorem$

$a^2 + b^2 = c^2 \quad \Rightarrow \quad I = a^2 + b^2 = (58)^2$

$\Rightarrow \quad a^2 + b^2 = 3364 —— III$

$$III - II \quad \begin{array}{l} a^2 + b^2 = 3364 \\ 2ab \quad = 3364 \\ \hline \end{array}$$

$a^2 - 2ab + b^2 = 4 \Rightarrow (a-b)^2 - 4 \Rightarrow a - b = 2 \Rightarrow a = b + 2 —— IV$

$I \Rightarrow ab = 1680 \Rightarrow (b+2)b = 1680 \Rightarrow b^2 + 2b - 1680 = 0$

$b^2 + 42b - 40b - 1680 = 0 \Rightarrow b(b+42) - 40(b+42) = 0 \Rightarrow (b+42)(b-40) = 0$

$b + 42 = 0\ or\ b - 40 = 0 \Rightarrow b = 40 \quad or \quad (b = -42\ ignore)$

$II \Rightarrow a = b + 2 \Rightarrow a = 40 + 2 \Rightarrow \boxed{a = 42}$

$So\ \ a = length = 42 \qquad b = breadth = 40$

17. To do a piece of work, A takes 10 days more than B. together they finish the work in 12 days. How long would B take to finish it alone?

Sol *Let B finishes in* $= x$ *days.*

Let A finishes in $= x+10$

$B^{'S}$ *one days work* $= \dfrac{1}{x}$

$A^{'S}$ *one days work* $= \dfrac{1}{x+10}$

$(A+B)^{'S}$ *one days work* $= \dfrac{1}{x+10} + \dfrac{1}{x}$

According to the given Conditions equation is

$$\frac{1}{x+10} + \frac{1}{x} = \frac{1}{12} \Rightarrow \frac{x+x+10}{x(x+10)} = \frac{1}{12} \Rightarrow \frac{2x+10}{x(x+10)} = \frac{1}{12}$$

(By cross multiplication) $12(2x+10) = x(x+10)$

$\Rightarrow 24x + 120x = x^2 + 10x \Rightarrow x^2 + 10x - 24x - 120x = 0$

$\Rightarrow x^2 - 14x + 120 = 0 \Rightarrow x^2 - 20x + 6x - 120 = 0$

$\Rightarrow x(x-20) + 6(x-20) = 0 \Rightarrow (x-20)(x+6) = 0$

$x - 20 = 0$ *or* $x + 6 = 0 \Rightarrow x = 20$ *or* $x = -6$

$\Rightarrow x = 20 = B$ *finish work* -6 *not possible*

So in 20 days B finish his work.

18. To complete a job, A and B take 4 days working together, A alone takes twice as long as B alone to finish the same job. How long would each one alone take to do the job?

Sol *Let B finishes in* $= x$ *days.*

Let A finishes in $= 2x$ *days*

$B^{'S}$ *one days work* $= \dfrac{1}{x}$

$A^{'S}$ *one days work* $= \dfrac{1}{2x}$ **Federal**

$(A+B)^{'S}$ *one days work* $= \dfrac{1}{x} + \dfrac{1}{2x}$

According to the given Conditions

$$\frac{1}{x} + \frac{1}{2x} = \frac{1}{4} \Rightarrow \frac{2+1}{2x} = \frac{1}{4} \Rightarrow \frac{3}{2x} = \frac{1}{4}$$

$\Rightarrow (2x) = 12 \Rightarrow x = 6$ *B finish in* 6 *days A finish in* 12 *days*

19. An open box is to be made from a square piece of tin by cutting a piece 2 dm square from each corner and then folding the sides of the remaining piece. If the capacity of the box is to be 128c dm, find the length of the side of the piece.

Sol *Let the sides are* $(x-4), (x-4), 2$

This volume $= 2(x-4)(x-4)$

$\Rightarrow 128 = 2(x-4)(x-4) \Rightarrow (x-4)^2 = 64$

$\Rightarrow (x-4) = \pm 8 \Rightarrow x = 4 \pm 8$

$\Rightarrow x = 4+8 \;\; and \;\; x = 4-8$

$\Rightarrow x = 12 \;\; and \;\; (x = -4 \;\; (not\ +ve)\ so\ ignore)$

$\Rightarrow x = 12dm\ Hence\ sides\ are\ 2, 8, 8\ (x-4 = 12-4 = 8)$

20. A man invests Rs.1,00,000 in to companies. His total profit is Rs.2080. If he receive Rs.1980 from one company and at the rate 1% more from the other, find the amount of each investment.

Sol. Suppose investment inIcompany $= x$, investment inIIcompany $= 100000 - x$

$Profit\ from\ I\ at\ y\% = 1980,\ profit\ from\ II\ (y+1)\% = 3080$

then according to the given condition equition is $xy\% = 1980$

$\Rightarrow x\left(\dfrac{y}{100}\right) = 1980 \Rightarrow xy = 198000 - I$

$and\ [(y+1)\%][100000 - x] = 3080$

$\Rightarrow \left(\dfrac{y+1}{100}\right)(100000 - x) = 3080$

$\Rightarrow (y+1)(100000 - x) = 308000$

$\Rightarrow 100000y - xy + 100000 - x = 308000$

$\Rightarrow 100000y + 198000 + 100000 - x = 308000\ use\ I$

$\Rightarrow 100000y - 198000 + 100000 - x - 308000 = 0$

$\Rightarrow 100000y - x = 406000 \text{———} II$

$(from - I) \Rightarrow x = \dfrac{198000}{y}\ put\ in\ II$

$100000y - \dfrac{198000}{y} = 406000$

$\Rightarrow 100000y^2 - 198000 = 406000y \Rightarrow 100000y^2 - 406000y = 198000$

$(\div\ by\ 2000)\ 50y^2 - 203y - 99 = 0$

$$y = \frac{-(-203) \pm \sqrt{(-203)^2 - 4(50)(-99)}}{2(50)}$$

$$y=\frac{203\pm\sqrt{41209+19800}}{100}=\frac{203\pm\sqrt{61009}}{100}$$

$$y=\frac{203\pm 247}{100} \Rightarrow y=\frac{203+247}{100} \ \& \ y=\frac{203-247}{100}$$

$$y=\frac{450}{100} \quad and \quad y=\frac{-44}{100}$$

$$y=4.5 \quad and \quad (y=-0.44 \quad not \ possible)$$

$$When \ y=4.5 \quad then \quad x=\frac{198000}{4.5}=44000$$

$$Amount \ invested \ in \ one=44000$$

$$two=100000-44000=56000$$

TEST YOUR SKILLS

Marks: 50

Q # 1. Select the Correct Option **(10)**

i. If polynomial $f(x)$ is divided by its factor $x-a$, then remainder is:

a)	0	b)	$f(a)$
c)	$\frac{1}{f(a)}$	d)	$f(-a)$

ii. If b^2-4ac is positive and not perfect square then roots of $ax^2+bx+c=0$ are:

a)	Irrational	b)	Equal
c)	Rational	d)	Imaginary

iii. If α,β are roots of $x^2-px-p-c=0$, value of $\alpha.\beta$ is:

a)	$-2p$	b)	p/c
c)	$p+c$	d)	$-(p+c)$

iv. If polynomial x^3-x^2+3x+3 is divided by $x-2$ then remainder is:

a)	13	b)	-15
c)	6	d)	-2

v. If ω is cube root of unity then $\omega^2=?$ is equal to:

a)	1	b)	-1
c)	ω^{-1}	d)	ω^{-2}

vi. Factor of x^3-y^3 are

a)	$(x-y)(x^2+xy+y^2)$	b)	$(x-y)(x^2+y^2)$
c)	$(x-y)(x^2-y^2)$	d)	$(x+y)(x^2+y^2)$

vii. Let α,β are roots of $4x^2+5x-6=0$ then value of $4\alpha+4\beta$ are

a)	$-5/4$	b)	-5
c)	$5/2$	d)	$-2/5$

viii. If ω is cube root of unity then form equation whose roots are 2ω & $2\omega^2$

a)	$x^2+2x+4=0$	b)	$x^2-2x+4=0$
c)	$x^2+2x-4=0$	d)	$x^2-2x-4=0$

ix. A quadratic equation $Ax^2+Bx+C=0$ become linear equation if:

a)	$C=0$	b)	$A=0$
c)	$B=0$	d)	$A=B$

x. The product of four forth roots of unity is:

a)	1	b)	-1
c)	0	d)	i

Q # 2. Short Questions: **(10 X 2 = 20)**

i. Solve the equation $2^x + 2^{-x+6} - 20 = 0$

ii. Show that $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

iii. If α, β are roots of $3x^2 - 2x + 4 = 0$ find value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

iv. Solve $x + y = 5$; $x^2 + 2y^2 = 17$

v. Evaluate $(\omega^{28} + \omega^{29} + 1)$

vi. Show that roots of $(mx + c)^2 = 4ax$ will be equal if $c = \frac{a}{m}$

vii. State Remainder Theorem:

viii. If α, β are roots of $x^2 - px - p - c = 0$ prove that $(1+\alpha)(1+\beta) = 1 - c$

ix. Prove that sum cube roots of unity is zero:

x. Use synthetic division to prove that $x = -4$ is a solution of $x^3 - 28x - 48 = 0$

Long Questions: **(2 X 10 = 20)**

Q # 3. (a) Solve the Equation $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$

(b) Solve the equation $x^2 + y^2 = 25,\ 2x^2 + 3y^2 = 66$

Q # 4. (a) Show that the roots of $x^2 + (mx + c)^2 = a^2$ will be equal if $c^2 = a^2(1+m^2)$

(b) Solve $\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

PARTIAL FRACTIONS 5

Exercise 5.1

Partial Fraction Resolution:

Expressing a rational fraction as a sum of partial fractions is called partial fraction resolution.

Conditional Equation: **Sargodha 2008**

It is an equation which is true for particular values of the variable. e.g.

$2x = 3$ is true only if $x = \frac{3}{2}$

Identity:

It is an equation which holds good for all values of variable e.g.

$(a+b)x = ax + bx$

Rational Fraction : **Multan 2007**

The Quotient of two polynomials $\frac{P(x)}{Q(x)}$ Where $Q(x) \neq 0$ with no common factor is called Rational fraction.

Proper Rational Fraction:

A Rational fraction $\frac{P(x)}{Q(x)}$ is called a Proper Rational fraction If the degree of polynomial $P(x)$ is less then degree of polynomial $Q(x)$. e.g. $\frac{3}{x+1}, \frac{2x-5}{x^2+4}$.

Improper Rational Fraction: **Multan 2008, Sargodha 2008**

A Rational fraction $\frac{P(x)}{Q(x)}$ is called an improper rational fraction if the degree of polynomial $P(x)$ is greater then or equal to the degree of polynomial $Q(x)$. e.g. $\frac{3x^2+1}{x-1}$.

Example 1: $\text{Re}\,solve \dfrac{7x+25}{(x+3)(x+4)}\, into\ partial\ Fractions$

Faisalabad 2007, Sargodha 2008, Federal, Multan 2007, 2008

Sol. $Suppose\ \dfrac{7x+25}{(x+3)(x+4)} = \dfrac{A}{x+3} + \dfrac{B}{x+4} \quad \text{——} I$

$"\times"\ by\,(x+3)(x+4)\, both\ sides\ we\ get$

$7x+25 = A(x+4)+B(x+3) \quad \text{——} II$

$put\ x+3=0 \Rightarrow x=-3\ in\ II$

$7(-3)+25 = A(-3+4)+B(-3+3)$

$-21+25 = A(1)+B(0) \Rightarrow \boxed{A=4}$

$put\ x+4=0 \Rightarrow x=-4\ in\ II$

$7(-4)+25 = A(-4+4)+B(-4+3)$

$-28+25 = A(0)+B(-1) \Rightarrow -3=-B \Rightarrow \boxed{B=3}$

$I\ become \dfrac{7x+25}{(x+3)(x+4)} = \dfrac{4}{x+3} + \dfrac{3}{x+4}$

EXERCISE 5.1

Resolve the following into Partial Fractions:

1. $\dfrac{1}{x^2-1}$ **Faisalabad 2008**

Sol $\dfrac{1}{x^2-1} = \dfrac{1}{(x-1)(x+1)}$

Now $\dfrac{1}{(x-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} \quad \longrightarrow I$

Multiply both sides by $(x-1)(x+1)$ we get.

$$1 = A(x+1)+(x-1) \quad \longrightarrow II$$

Put $x-1=0 \Rightarrow x=1\ in\ II$

$$1 = A(1+1)+B(1-1) \Rightarrow 1 = 2A+0 \Rightarrow \boxed{A=1/2}$$

Put $x+1=0 \Rightarrow x=-1\ in\ II$

$$1 = A(-1+1)+B(-1-1) \Rightarrow 1 = 0-2B \Rightarrow \boxed{B=-1/2}$$

Put values of A and B in I.

$$\frac{1}{(x-1)(x+1)}=\frac{1/2}{x-1}+\frac{-1/2}{x+1}$$

Hence $\dfrac{1}{x^2-1}=\dfrac{1}{(x-1)(x+1)}=\dfrac{1}{2(x-1)}-\dfrac{1}{2(x+1)}$

2. $\dfrac{\mathbf{x^2+1}}{\mathbf{(x+1)(x-1)}}$

Sol. $\dfrac{x^2+1}{(x+1)(x-1)}=\dfrac{x^2+1}{x^2-1}$ Improper so.

$$x^2-1\,\overline{\big)\,x^2+1}$$
$$\underline{\underline{x^2}\mp 1}$$
$$2$$

$$\frac{x^2+1}{(x^2-1)}=1+\frac{2}{x^2-1}\qquad\longrightarrow I$$

Now Take $\dfrac{2}{x^2-1}=\dfrac{2}{(x-1)(x+1)}$

$$\frac{2}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}\qquad\longrightarrow II$$

'$\times$' *by* $(x-1)(x+1)$ we get.

$$2=A(x+1)+B(x-1)\qquad\longrightarrow III$$

Put $x-1=0\Rightarrow x=1$ *in III.*

$$2=A(x+1)+B(1-1)\Rightarrow 2=2A+0\Rightarrow\boxed{A=1}$$

Put $x+1=0\Rightarrow x=-1$ *in III.*

$$2=A(-1+1)+B(-1-1)$$

$$2=0-2B\Rightarrow\boxed{B=-1}$$

Put values in II.

$$\frac{2}{(x-1)(x+1)}=\frac{1}{x-1}+\frac{-1}{x+1}$$

$$\frac{x^2+1}{(x+1)(x-1)}=1+\frac{2}{(x-1)(x+1)}=1+\frac{1}{x-1}+\frac{-1}{x+1}\ (Put\ in\ I)$$

3. $\dfrac{\mathbf{2x+1}}{\mathbf{(x-1)(x+2)(x+3)}}$

Sol. Suppose

$$\frac{2x+1}{(x-1)(x+2)(x+3)}=\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{x+3}\qquad\longrightarrow I$$

'X' both sides by $(x-1)(x+2)(x+3)$

$$2x+1 = A(x+2)(x+3)+B(x-1)(x+3)+C(x-1)(x+2) \longrightarrow II$$

Put $x-1=0 \Rightarrow x = 1$ *in II*

$$2(1)+1 = A(1+2)(1+3)+B(1-1)(1+3)+C(1-1)(1+2)$$

$$3 = A(3)(4)+0+0 \Rightarrow 3 = 12A$$

$$\Rightarrow A = \frac{3}{12} = \frac{1}{4} \Rightarrow \boxed{A = \frac{1}{4}}$$

Put $x+2=0 \Rightarrow x=-2$ *in II*

$$2(-2)+1 = A(-2+2)(-2+3)+B(-2-1)(-2+3)+C(-2-1)(-2+2)$$

$$-4+1 = 0+B(-3)(1)+0$$

$$-3 = -3B \Rightarrow B = \frac{\cancel{-3}}{\cancel{-3}} = 1 \Rightarrow \boxed{B=1}$$

Put $x+3=0 \Rightarrow x=-3$ *in II*

$$2(-3)+1 = A(-3+2)(-3+3)+B(-3-1)(-3+3)+C(-3-1)(-3+2)$$

$$-5 = 0+0+C(-4)(-1)$$

$$-5 = 4C \Rightarrow \boxed{C = -5/4}$$

I become

$$\frac{2x+1}{(x-1)(x+2)(x+3)} = \frac{1/4}{x-1}+\frac{1}{x+2}+\frac{-5/4}{x+3} = \frac{1}{4(x-1)}+\frac{1}{x+2}-\frac{5}{4(x+3)}$$

4. $$\frac{3x^2-4x-5}{(x-2)(x^2+7x+10)}$$

Sol $$= \frac{3x^2-4x-5}{(x-2)(x^2+2x+5x+10)} = \frac{3x^2-4x-5}{(x-5)(x(x+2)+5(x+2))}$$

$$= \frac{3x^2-4x-5}{(x-2)(x+2)(x+5)}$$

Suppose

$$\frac{3x^2-4x-5}{(x-2)(x+2)(x+5)} = \frac{A}{x-2}+\frac{B}{x+2}+\frac{C}{x+5} \quad \text{———} I$$

'X' by $(x-2)(x+2)(x+5)$

$$3x^2-4x-5 = A(x+2)(x+5)+B(x-2)(x+5)+C(x+2)(x-2) \text{ ———} II$$

Put $x-2=0 \Rightarrow x=2$ *in II*

$$3(2)^2-4(2)-5 = A(2+2)(2+5)+B(2-2)(2+5)+C(2+2)(2-2)$$

$$12-8-5 = A(4)(7)+0+0$$

$$12-8-5 = A(4)(7)+0+0$$

$-1 = 28A \Rightarrow \boxed{A = -1/28}$

Put $x+2=0 \Rightarrow x=-2$ in II.

$3(-2)^2 - 4(-2) - 5 = A(-2+2)(-2+5) + B(-2-2)(-2+5) + C(-2-2)(-2+2)$

$12+8-5 = 0 + B(-4)(3) + 0$

$15 = -12B \Rightarrow \boxed{B = -5/4}$

Put $x+5=0 \Rightarrow x=-5$ in II.

$3(-5)^2 - 4(-5) - 5 = A(-5+2)(-5+5) + B(-5-2)(-5+5) + C(-5-2)(-5+2)$

$75+20-5 = A(0) + B(0) + C(-7)(-3)$

$90 = 21C \Rightarrow C \Rightarrow \dfrac{90}{21} \Rightarrow \boxed{C = \dfrac{30}{7}}$

I become.

$$\frac{3x^2-4x-5}{(x-2)(x^2+7x+10)} = \frac{3x^2-4x-5}{(x-2)(x+2)(x+5)} = \frac{-1}{28(x-2)} + \frac{30}{7(x+5)} - \frac{5}{4(x+2)}$$

5. $\dfrac{1}{(x-1)(2x-1)(3x-1)}$ **Sargodha 2009, Faisalabad 2008**

Sol. Suppose

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{A}{x-1} + \frac{B}{2x-1} + \frac{C}{3x-1} \longrightarrow I$$

'X' by $(x-1)(2x-1)(3x-1)$ $\rightarrow II$

$1 = A(2x-1)(3x-1) + B(x-1)(3x-1) + C(x-1)(2x-1) \longrightarrow II$

Put $x-1=0 \Rightarrow x=1$ in II.

$1 = A(2-1)(3-1) + B(1-1)(3-1) + C(1-1)(2-1)$

$1 = A(1)(2) + 0 + 0 \Rightarrow 1 = 2A \Rightarrow \boxed{A = \dfrac{1}{2}}$

Put $2x-1=0 \Rightarrow x = 1/2$ in II.

$1 = A(2(\frac{1}{2})-1)(3(\frac{1}{2})-1) + B(\frac{1}{2}-1)(3(\frac{1}{2})-1) + C(\frac{1}{2}-1)(2(\frac{1}{2})-1)$

$1 = A(1-1)(\frac{3}{2}-1) + B(\frac{1}{2}-1)(\frac{3}{2}-1) + C(\frac{1}{2}-1)(1-1)$

$1 = 0 + B(-\frac{1}{2})(\frac{1}{2}) + 0 \Rightarrow 1 = -\frac{1}{4}B \Rightarrow \boxed{B = -4}$

Put $3x-1=0 \Rightarrow x = 1/3$ in II.

$$1 = A(2(\frac{1}{3})-1)(3(\frac{1}{3})-1) + B(\frac{1}{3}-1)(3(\frac{1}{3})-1) + C(\frac{1}{3}-3)(2(\frac{1}{3})-1)$$

$$1 = A(\frac{2}{3}-1)(1-1) + B(\frac{1}{3}-1)(1-1) + C(\frac{1}{3}-3)(\frac{2}{3}-1)$$

$$1 = 0+0+C(-\frac{-8}{3})(-\frac{1}{3}) \Rightarrow 1 = \frac{8}{9}C \Rightarrow \boxed{C = 9/8}$$

Put values in I.

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{1}{2(x-1)} - \frac{4}{2x-1} + \frac{9}{8(3x-1)}$$

6. $\dfrac{x}{(x-a)(x-b)(x-c)}$ **Multan 2009**

Sol. Suppose

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \qquad \longrightarrow I$$

'X' by $(x-a)(x-b)(x-c)$

$$x = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b) \qquad \longrightarrow II$$

Put $x-a=0 \Rightarrow x=a$ in II.

$$a = A(a-b)(a-c) + B(a-a)(a-c) + C(a-a)(a-b)$$

$$a = A(a-b)(a-c) + 0 + 0$$

$$\boxed{A = \frac{a}{(a-b)(a-c)}}$$

Put $x-b=0 \Rightarrow x=b$ in II.

$$b = A(b-b)(b-c) + B(b-a)(b-c) + C(b-a)(b-b)$$

$$b = 0 + B(b-a)(b-c) + 0$$

$$\boxed{B = \frac{b}{(b-a)(b-c)}}$$

Put $x-c=0 \Rightarrow x=c$ in II

$$c = A(c-b)(c-c) + B(c-a)(c-c) + C(c-a)(c-b)$$

$$c = 0 + 0 + C(c-a)(c-b)$$

$$\boxed{C = \frac{c}{(c-a)(c-b)}}$$

Put values in I.

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{a}{(a-b)(a-c)(x-a)}$$

$$+\frac{b}{(b-a)(b-c)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}$$

7. $\dfrac{6x^3+5x^2-7}{2x^2-x-1}$ (Improper) **Federal**

Sol $\dfrac{6x^3+5x^2-7}{2x^2-x-1}=3x+4+\dfrac{7x-3}{2x^2-x-1}$ ——— I

$$\begin{array}{r|l} & 3x+4 \\ 2x^2-x-1 & \cancel{6x^3}+5x^2-7 \\ & \underline{-\cancel{6x^3}\mp 3x^2\mp 3x} \\ & \cancel{8x^2}+3x-7 \\ & \underline{-8x^2\mp 4x\mp 4} \\ & 7x-3 \end{array}$$

Now $\dfrac{7x-3}{2x^2-x-1}=\dfrac{7x-3}{2x^2-2x+x-1}$

$=\dfrac{7x-3}{2x(x-1)+(x-1)}=\dfrac{7x-3}{(x-1)(2x+1)}$

$\dfrac{7x-3}{(x-1)(2x+1)}=\dfrac{A}{x-1}+\dfrac{B}{2x+1}$ ——— II

'X' by $(x-1)(2x+1)$

$7x-3=A(2x+1)+B(x-1)$ ——— III

Put $x-1=0 \Rightarrow x=1$ in III.

$7(1)-3=A(2(1)+1)+B(1-1)$

$4=A(3)+0 \Rightarrow \boxed{A=4/3}$

Put $2x+1=0 \Rightarrow x=-\dfrac{1}{2}$ in III.

$7(-\dfrac{1}{2})-3=A(2(-\dfrac{1}{2})+1)+B(-\dfrac{1}{2}-1)$

$-\dfrac{7}{2}-3=0+B(-\dfrac{3}{2})$

$-\dfrac{13}{2}=0+B(-\dfrac{3}{2})$

$-\dfrac{13}{2}=-\dfrac{3}{2}B \Rightarrow B=(-\dfrac{13}{2})(-\dfrac{2}{3})$

$\boxed{B=\dfrac{13}{3}}$

$\dfrac{7x-3}{(x-1)(2x+1)}=\dfrac{4}{3(x-1)}+\dfrac{13}{3(2x+1)}$ (*II become*)

HENCE

$\dfrac{6x^3+5x^2-7}{2x^2-x-1}=3x+4+\dfrac{4}{3(x-1)}+\dfrac{13}{3(2x-1)}$ (*I become*)

8. $$\frac{2x^3+x^2-5x+3}{2x^3+x^2-3x}$$ Improper

$$\begin{array}{r|l} & \quad 1 \\ 2x^3+x^2-3x & \cancel{2x^3}+\cancel{x^2}-5x+3 \\ & \underline{\cancel{2x^3}\pm\cancel{x^2}\mp 3x} \\ & -2x+3 \end{array}$$

Sol. $$\frac{2x^3+x^2-5x+3}{2x^3+x^2-3x}=1+\frac{-2x+3}{2x^3+x^2-3x} \longrightarrow I$$

Now $$\frac{-2x+3}{2x^3+x^2-3x}=\frac{3-2x}{x(2x^2+x-3)}$$

$$=\frac{3-2x}{x(2x^2+3x-2x-3)}$$

$$=\frac{3-2x}{x((2x+3)x-1(2x+3))}$$

$$=\frac{3-2x}{x(2x+3)(x-1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{2x+3} \longrightarrow II$$

'×' by $x(x-1)(2x+3)$

$$3-2x=A(x-1)(2x+3)+Bx(2x+3)+Cx(x-1) \longrightarrow III$$

Put $x=0$ in III.

$$3-2(0)=A(0-1)(0+3)+0+0$$

$$3=A(-3)\Rightarrow \boxed{A=-1}$$

Put $x-1=0\Rightarrow x=1$ in III.

$$3-2(1)=A(1-1)(2(1)+3)+B(1)(2(1)+3)+C(1)(1-1)$$

$$3-2=0+B(2+3)+0\Rightarrow 1=5B\Rightarrow \boxed{B=1/5}$$

Put $2x+3=0\Rightarrow x=-\frac{3}{2}$ in III.

$$3-2(-\frac{3}{2})=A(-\frac{3}{2}-1)(2(-\frac{3}{2})+3)+B(-\frac{3}{2})(2(-\frac{3}{2}+3))+C(-\frac{3}{2})(-\frac{3}{2}-1)$$

$$3+3=A(-\frac{5}{2})(0)+B(-\frac{3}{2})(0)+C(-\frac{3}{2})(-\frac{5}{2})$$

$$6=0+0+C(\frac{15}{4})\Rightarrow C=6\times\frac{4}{15}\Rightarrow \boxed{C=\frac{8}{5}}$$

Now I become.

$$\frac{2x^3+x^2-5x+3}{2x^3+x^2-3x}=1+\frac{3-2x}{2x^3+x^2-3x}$$

$$=1-\frac{1}{x}+\frac{1}{5(x-1)}+\frac{8}{5(2x+3)}$$

9. $\dfrac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}$

Sol $\dfrac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}=\dfrac{(x-1)(x^2-8x+15)}{(x-2)(x^2-10x+24)}$ Improper

$$=\frac{x^3-8x^2+15x-x^2+8x-15}{x^3-10x^2+24x-2x^2+20x-48}=\frac{x^3-9x^2+23x-15}{x^3-12x^2+44x-48}$$

$$=1+\frac{3x^2-21x+33}{x^3-12x^2+44x-48}$$

$$=1+\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)}$$

$$\begin{array}{r|l} & \quad 1 \\ \hline x^3-12x^2+44x-48 & \cancel{x^3}-9x^2+23x-15 \\ & \cancel{x^3}\mp 12x^2\pm 44x\mp 48 \\ \hline & \quad 3x^2-21x+33 \end{array}$$

Now

$$\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)}=\frac{A}{x-2}+\frac{B}{x-4}+\frac{C}{x-6} \longrightarrow II$$

'×' by $(x-2)(x-4)(x-6)$ we get.

$$3x^2-21x+33=A(x-4)(x-6)+B(x-2)(x-6)+C(x-2)(x-4) \longrightarrow III$$

Put $x-2=0 \Rightarrow x=2$ in III.

$$3(2)^2-21(2)+33=A(2-4)(2-6)+B(2-2)(2-6)+C(2-2)(2-4)$$

$$12-42+33=A(-2)(-4)+0+0 \Rightarrow 3=8A \Rightarrow \boxed{A=\frac{3}{8}}$$

Put $x-4=0 \Rightarrow x=4$ in III.

$$3(4)^2-21(4)+33=A(4-4)(4-6)+B(4-2)(4-6)+C(4-2)(4-4)$$

$$48-84+33=0+(2)(-2)+0 \Rightarrow -3=-4B \Rightarrow \boxed{B=\frac{3}{4}}$$

Put $x-6=0 \Rightarrow x=6$ in III.

$$3(6)^2-21(6)+33=A(6-4)(6-6)+B(6-2)(6-6)+C(6-2)(6-4)$$

$$108-126+33=0+0+C(4)(2) \Rightarrow 15=8C \Rightarrow \boxed{C=\frac{15}{8}}$$

I become.

$$\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}=1+\frac{3}{8(x-2)}+\frac{3}{4(x-4)}+\frac{15}{8(x-6)}$$

10. $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$

Sol $\dfrac{1}{(1-ax)(1-bx)(1-cx)}=\dfrac{A}{1-ax}+\dfrac{B}{1-bx}+\dfrac{C}{1-cx} \longrightarrow I$

'×' $(1-ax)(1-bx)(1-cx)$ we get.

$1=A(1-bx)(1-cx)+B(1-ax)(1-cx)+C(1-ax)(1-bx) \longrightarrow II$

Put $1-ax=0 \Rightarrow ax=1 \Rightarrow x=\dfrac{1}{a}$ in II.

$1=A(1-b(\frac{1}{a}))(1-c(\frac{1}{a}))+B(0)+C(0) \Rightarrow 1=A(\frac{a-b}{a})(\frac{a-c}{a})$

$\boxed{A=\dfrac{a^2}{(a-b)(a-c)}}$ *Put* $1-bx=0 \Rightarrow x=\dfrac{1}{b}$ in II.

$1=A(0)+B(1-a(\frac{1}{b}))(1-c(\frac{1}{b}))+C(0) \Rightarrow 1=B\left(\dfrac{b-a}{b}\right)\left(\dfrac{b-c}{b}\right)$

$\boxed{B=\dfrac{b^2}{(b-a)(b-c)}}$ *Put* $1-cx=0 \Rightarrow x=\dfrac{1}{c}$

$1=A(0)+B(0)+C(1-a(\frac{1}{c}))(1-b(\frac{1}{c})) \Rightarrow 1=C\left(\dfrac{c-a}{c}\right)\left(\dfrac{c-b}{c}\right)$

$\boxed{C=\dfrac{c^2}{(c-a)(c-b)}}$ I become.

$$\frac{1}{(1-ax)(1-bx)(1-cx)}=\frac{a^2}{(1-ax)(a-b)(a-c)}+\frac{b^2}{(1-bx)(b-a)(b-c)}+\frac{c^2}{(1-cx)(c-a)(c-b)}$$

11. $\dfrac{\mathbf{x^2+a^2}}{\mathbf{(x^2+b^2)(x^2+c^2)(x^2+d^2)}}$

Replace x^2 by y.

Sol. $\dfrac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)}=\dfrac{(y+a^2)}{(y+b^2)(y+c^2)(y+d^2)}$

$\dfrac{(y+a^2)}{(y+b^2)(y+c^2)(y+d^2)}=\dfrac{A}{(y+b^2)}+\dfrac{B}{(y+c^2)}+\dfrac{C}{(y+d^2)}$

'×' by $(y+b^2)(y+c^2)(y+d^2)$ we get.

$y+a^2=A(y+c^2)(y+d^2)+B(y+b^2)(y+d^2)+(y+b^2)(y+c^2) \longrightarrow II$

Put $y+b^2=0 \Rightarrow y=-b^2$ in II.

$$-b^2+a^2 = A(-b^2+c^2)(-b^2+d^2)+B(0)+C(0) \Rightarrow \boxed{A=\frac{a^2-b^2}{(c^2-b^2)(d^2-b^2)}}$$

Put $y+c^2=0 \Rightarrow y=-c^2$ in II.

$$-c^2+a^2 = A(0)+B(-c^2+b^2)(-c^2+d^2)+C(0) \Rightarrow \boxed{B=\frac{a^2-c^2}{(b^2-c^2)(d^2-c^2)}}$$

Put $y+d^2=0 \Rightarrow y=-d^2$ in II.

$$-d^2+a^2 = A(0)+B(0)+C(-d^2+b^2)(-d^2+c^2) \Rightarrow \boxed{C=\frac{a^2-d^2}{(b^2-d^2)(c^2-d^2)}}$$

I become.

$$\frac{y+a^2}{(y+b^2)(y+c^2)(y+d^2)}=\frac{(a^2-b^2)}{(y+b^2)(c^2-b^2)(d^2-b^2)}+\frac{(a^2-c^2)}{(y+c^2)(b^2-c^2)(d^2-c^2)}+\frac{(a^2-d^2)}{(y+d^2)(b^2-d^2)(c^2-d^2)}$$

Replace y by x^2

$$\frac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)}=\frac{a^2-b^2}{(x^2+b^2)(c^2-b^2)(d^2-b^2)}+\frac{a^2-c^2}{(x^2+c^2)(b^2-c^2)(d^2-c^2)}+\frac{a^2-d^2}{(x^2+d^2)(b^2-d^2)(c^2-d^2)}$$

EXERCISE 5.2

Resolve the following into Partial Fractions:

1. $\dfrac{2x^2-3x+4}{(x-1)^3}$

Sol Suppose

$$\frac{2x^2-3x+4}{(x-1)^3}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}$$

'X' by $(x-1)^3$ we get.

$$2x^2-3x+4=A(x-1)^2+B(x-1)+C \qquad \longrightarrow II$$

Put $x-1=0 \Rightarrow x=1$ in II.

$$2(1)^2-3(1)+4=A(1-1)^2+B(1-1)+C$$

$$3=0+0+C \Rightarrow \boxed{C=3}$$

Rearrange II.

$$2x^2-3x+4=Ax^2-2Ax+A+Bx-B+C$$

Comparing Co-efficient

$x^2;\boxed{2=A}$

$x;\ -3=-2A+B \Rightarrow -3=-2(2)+B \Rightarrow -3=-4+B \Rightarrow \boxed{B=1}$

I become

$$\frac{2x^2-3x+4}{(x-1)^3}=\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{3}{(x-1)^3}$$

2. $\dfrac{5x^2-2x+3}{(x+2)^3}$ **Faisalabad 2009**

Sol $$\frac{5x^2-2x+3}{(x+2)^3}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{(x+2)^3}$$

Multiply by $(x+2)$ both sides.

$$5x^2-2x+3=A(x+2)^2+B(x+2)+C \qquad \longrightarrow I$$

Put $x+2=0 \Rightarrow x=-2$ in I.

$$5(-2)^2-2(-2)+3=A(-2+2)+B(-2+2)+C$$

$$20+4+3=C \Rightarrow \boxed{C=27}$$

$\text{Re}\,arrange \longrightarrow I$

$$5x^2-2x+3=Ax^2+4Ax+4A+Bx+2B+C$$

Comparing Co-efficients

$x^2;\quad \boxed{5=A}$

$x;\ -2=4A+B$

$Or\quad -2=4(5)+B \Rightarrow \boxed{B=-22}$

Put values of A. B, C. we get.

$$\frac{5x^2-2x+3}{(x+2)^3}=\frac{5}{x+2}-\frac{22}{(x+2)^2}+\frac{27}{(x+2)^3}$$

3. $\dfrac{4x}{(x+1)^2(x-1)}$ **Federal, Sargodha 2006, 2010,2011 Multan 2010, Lahore 2009**

Sol Suppose

$$\frac{4x}{(x+1)^2(x-1)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-1} \longrightarrow I$$

'X' by $(x+1)^2(x-1)$ we get.

$$4x=A(x+1)(x-1)+B(x-1)+C(x+1)^2 \longrightarrow II$$

Put $x-1=0 \Rightarrow x=1$ in II.

$4(1)=A(1+1)(1-1)+B(1-1)+C(1+1)^2$

$4=0+0+4C \Rightarrow \boxed{C=1}$

Put $x+1=0 \Rightarrow x=-1$ in II.

$4(-1)=A(-1+1)(-1-1)+B(-1-1)+C(-1+1)^2$

$-4=0+(-2)B+0 \Rightarrow B=\dfrac{-4}{-2} \Rightarrow \boxed{B=2}$

Rearrange II.

$4x=Ax^2-A+Bx-B+Cx^2+2Cx+C$

Comparing Co-efficient

$x^2;\quad 0=A+C \Rightarrow 0=A+1 \Rightarrow \boxed{A=-1}$

I become

$$\frac{4x}{(x+1)^2(x-1)}=\frac{-1}{x+1}+\frac{2}{(x+1)^2}+\frac{1}{(x-1)}$$

4. $\dfrac{9}{(x+2)^2(x-1)}$ **Sargodha 2011**

Sol Suppose

$$\frac{9}{(x+2)^2(x-1)}=\frac{A}{(x+2)}+\frac{B}{(x+2)^2}+\frac{C}{x-1} \longrightarrow I$$

'×' by $(x+2)^2(x-1)$

$$9 = A(x+2)(x-1)+B(x-1)+C(x+2)^2 \longrightarrow II$$

Put $x-1=0 \Rightarrow x=1$ in II.

$$9 = A(1+2)(1-1)+B(1-1)+C(1+2)^2$$

$$9 = 0+0+9C \Rightarrow \boxed{C=1}$$

Put $x+2=0 \Rightarrow x=-2$ in II.

$$9 = A(-2+2)(-2-1)+B(-2-1)+C(-2+2)^2$$

$$9 = 0+B(-3)+0 \Rightarrow B=\frac{9}{-3} \Rightarrow \boxed{B=-3}$$

Rearrange II.

$$9 = Ax^2-Ax+2Ax-2A+Bx-B+Cx^2+4Cx+4C$$

Comparing Co-efficient

x^2; $\quad 0 = A+C \Rightarrow 0 = A+1 \Rightarrow \boxed{A=-1}$

I become

$$\frac{9}{(x+2)^2(x-1)} = \frac{1}{x-1}-\frac{1}{x+2}-\frac{3}{(x+2)^2}$$

5. $\dfrac{1}{(x-3)^2(x+1)}$ **Sargodha 2009, Rawalpindi 2009, Gujranwala 2009**

Sol Suppose

$$\frac{1}{(x-3)^2(x+1)} = \frac{A}{(x-3)}+\frac{B}{(x-3)^2}+\frac{C}{x+1} \longrightarrow I$$

'X' by $(x-3)^2(x+1)$ We get.

$$1 = A(x-3)(x+1)+B(x+1)+C(x-3)^2 \longrightarrow II$$

Put $x-3=0 \Rightarrow x=3$ in II.

$$1 = A(3-3)(3+1)+B(3+1)+C(3-3)^2$$

$$1 = 0+4B+0 \Rightarrow \boxed{B=1/4}$$

Put $x+1=0 \Rightarrow x=-1$ in II.

$$1 = A(-1-3)+(-1+1)+B(-1+1)+C(-1-3)^2$$

$$1 = A(0)+B(0)+C(16) \Rightarrow \boxed{C=1/16}$$

Rearrange II.

$$1 = Ax^2+Ax-3Ax-3A+Bx+B+Cx^2-6Cx+9C$$

Comparing Co-efficient.

$$0 = A+C \Rightarrow 0 = A+1/16 \boxed{A=-1/16}$$

I become

$$\frac{1}{(x-3)^2(x+1)}=\frac{-1}{16(x-3)}+\frac{1}{4(x-3)^2}+\frac{1}{16(x+1)}$$
$$=\frac{1}{16(x+1)}-\frac{1}{16(x-3)}+\frac{1}{4(x-3)^2}$$

6. $\dfrac{x^2}{(x-2)(x-1)^2}$ **Multan 2007**

Sol Suppose

$$\frac{x^2}{(x-2)(x-1)^2}=\frac{A}{(x-2)}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2} \longrightarrow II$$

'×' $by\,(x-2)(x-1)^2$

$x^2=A(x-1)^2+B(x-2)(x-1)+C(x-2)$

Put $x-1=0\Rightarrow x=1$ in II.

$(1)^2=A(1-1)^2+B(1-2)(1-1)+C(1-2)$

$1=0+0-C\Rightarrow \boxed{C=-1}$

Put $x-2=0\Rightarrow x=2$ in II

$(2)^2=A(2-1)^2+B(2-2)(2-1)+C(2-2)$

$4=A(1)+B(0)+C(0)\Rightarrow \boxed{A=4}$

Rearrange II

$x^2=Ax^2-2Ax+A+Bx^2-Bx-2Bx+2B+Cx-2C$

$x^2;\ A+B=1\Rightarrow 4+B=1\Rightarrow \boxed{B=-3}$

I become

$$\frac{x^2}{(x-2)(x-1)^2}=\frac{4}{x-2}-\frac{3}{(x-1)}-\frac{1}{(x-1)^2}$$

7. $\dfrac{1}{(x-1)^2(x+1)}$

Sol Suppose

$$\frac{1}{(x-1)^2(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)} \longrightarrow I$$

'×' by $(x-1)^2(x+1)$

$$1=A(x-1)(x+1)+B(x+1)+C(x-1)^2 \longrightarrow II$$

Put $x-1=0\Rightarrow x=1$ in II.

$1=A(1-1)(1+1)+B(1+1)+C(1-1)^2$

$1 = 0 + 2B + 0 \Rightarrow \boxed{B = 1/2}$

Put $x+1=0 \Rightarrow x=-1$ in II.

$1 = A(-1-1)(-1+1) + B(-1+1) + C(-1-1)^2$

$1 = 0 + 0 + C(-2)^2 \Rightarrow 1 = 4C \Rightarrow \boxed{C = 1/4}$

Rearrange II

$1 = Ax^2 - A + Bx + B + Cx^2 - 2Cx + C$

Comparing Co-efficient.

$x^2;\ 0 = A + C \Rightarrow 0 = A + \dfrac{1}{4} \Rightarrow \boxed{A = -1/4}$

I become

$$\frac{1}{(x-1)^2(x+1)} = -\frac{1}{4(x-1)} + \frac{1}{2(x-1)^2} + \frac{1}{4(x+1)}$$

$$= \frac{1}{4(x+1)} - \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2}$$

8. $\dfrac{x^2}{(x-1)^3(x+1)}$

Sol Suppose

$$\frac{x^2}{(x-1)^3(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x+1)} \longrightarrow I$$

'X' by $(x-1)^3(x+1)$

$$x^2 = A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3 \longrightarrow II$$

Put $x-1=0 \Rightarrow x=1$ in II.

$(1)^2 = A(1-1)^2(1+1) + B(1-1)(1+1) + C(1+1) + (1-1)^3$

$1 = 0 + 0 + 2C + 0 \Rightarrow \boxed{C = 1/2}$

Put $x+1=0 \Rightarrow x=-1$ in II.

$(-1)^2 = A(-1+1)^2(-1+1) + B(-1-1)(-1+1) + C(-1+1) + (-1-1)^3$

$1 = 0 + 0 + 0 + D(-8) \Rightarrow \boxed{D = -1/8}$

Rearrange II

$x^2 = A(x^2 - 2x + 1)(x+1) + B(x^2 - 1) + C(x+1) + D(x^3 - 3x^2 + 3x - 1)$

$x^2 = Ax^3 - 2Ax^2 + Ax + Ax^2 - 2Ax + A + Bx^2 - B + Cx + C + Dx^3 - 3Dx^2 + 3Dx - D$

$x^3;\ 0 = A + D \Rightarrow 0 = A - 1/8 \Rightarrow \boxed{A = 1/8}$

$x^2;\ 1 = -2A + A + B - 3D \longrightarrow III$

$$1=-A+B-3D$$

$$1=-\frac{1}{8}+B-3(-\frac{1}{8})\Rightarrow 1=-\frac{1}{8}+B+\frac{3}{8}$$

$$1=B+\frac{-1+3}{8}\Rightarrow 1=B+\frac{2}{8}\Rightarrow B=1-\frac{1}{4}\Rightarrow \boxed{B=\frac{3}{4}}$$

I become

$$\frac{x^2}{(x-1)^3(x+1)}=\frac{1}{8(x-1)}+\frac{3}{4(x-1)^2}+\frac{1}{2(x-1)^3}-\frac{1}{8(x+1)}$$

9. $$\frac{x-1}{(x-2)(x+1)^3}$$

Sol Suppose

$$\frac{x-1}{(x-2)(x+1)^3}=\frac{A}{x-2}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{D}{(x+1)^3} \longrightarrow II$$

'X' by $(x-2)(x+1)$

$$x-1=A(x+1)^3+B(x-2)(x+1)^2+C(x-2)(x+1)+D(x-2) \longrightarrow II$$

Put $x-2=0\Rightarrow x=2$ in II.

$$2-1=A(2+1)^3+B(2-2)(2+1)^2+C(2-2)(2+1)+D(2-2)^2$$

$$1=A(27)+0+0++0\Rightarrow \boxed{A=1/27}$$

Put $x+1=0\Rightarrow x=-1$ in II.

$$-1-1=A(-1+1)^3+B(-1-2)(-1+1)^2+C(-1-2)(-1+1)+D(-1-2)$$

$$-2=0+0+D(-3)\Rightarrow \boxed{D=2/3}$$

Rearrange II

$$x-1=Ax^3+3Ax^2+3Ax+A+B(x-2)(x^2+2x+1)+C(x^2+x-2x-2)+D(x-2)$$

$$x-1=Ax^3+3Ax^2+3Ax+A+Bx^3+\cancel{2Bx^2}+Bx-\cancel{2Bx^2}-4Bx-2B+Cx^2-2Cx+Cx-2C+Dx-2D$$

Comparing Co-efficient

$x^3;\ 0=A+B \longrightarrow III$

$x^2;\ 0=3A+C \longrightarrow IV$

$x;\ 0=3A+B-4B-2C+C+D\Rightarrow 0=3A-3B-C+D \longrightarrow V$

$Constant;\ -1=A-2B-2C-2D \longrightarrow VI$

$$III\Rightarrow 0=\frac{1}{27}+B\Rightarrow \boxed{B=-\frac{1}{27}}$$

$$IV \Rightarrow 0 = 3(\frac{1}{27}) + C \Rightarrow 0 = \frac{1}{9} + C \Rightarrow \boxed{C = -\frac{1}{9}}$$

I become

$$\frac{x-1}{(x-2)(x+1)^3} = \frac{1}{27(x-2)} - \frac{1}{27(x+1)} - \frac{1}{9(x+1)^2} + \frac{2}{3(x+1)^3}$$

10. $$\frac{4x^3}{(x^2-1)(x+1)^2}$$

Sol $$\frac{4x^3}{(x^2-1)(x+1)^2} = \frac{4x^3}{(x-1)(x+1)(x+1)^2} = \frac{4x^3}{(x-1)(x+1)^3}$$

Suppose

$$\frac{4x^3}{(x-1)(x+1)^3} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3} \text{———} I$$

'X' by $(x-1)(x+1)^3$

$$4x^3 = A(x+1)^3 + B(x-1)(x+1)^2 + C(x-1)(x+1) + D(x-1) \longrightarrow II$$

Put $x-1=0 \Rightarrow x=1$ in II

$$4(1)^3 = A(1+1)^3 + B(1-1)(1+1)^2 + C(1-1)(1+1) + D(1-1)$$

$$4 = A(8)+0+0+0 \Rightarrow A = \frac{4}{8} \Rightarrow \boxed{A = \frac{1}{2}}$$

Put $x+1=0 \Rightarrow x=-1$ in II

$$4(-1)^3 = A(-1+1)^3 + B(-1-1)(-1+1)^2 + C(-1-1)(-1+1) + D(-1-1)$$

$$-4 = 0+0+0+D(-2) \Rightarrow \boxed{D=2}$$

Rearrange II

$$4x^3 = A(x^3+3x^2+3x+1) + B(x-1)(x^2+2x+1) + C(x^2-1) + D(x-1)$$

$$4x^3 = Ax^3 + 3Ax^2 + 3Ax + A + Bx^3 + 2Bx^2 + Bx - Bx^2 - 2Bx - B + Cx^2 - C - Dx - D$$

Comparing Co-efficient s

$$x^3;\ 4 = A+B \Rightarrow 4 = \frac{1}{2} + B \Rightarrow B = 4 - \frac{1}{2} \Rightarrow \boxed{B = \frac{7}{2}}$$

$$x^2;\ 0 = 3A+2B-B+C \Rightarrow 0 = 3A+B+C \Rightarrow 0 = 3(\frac{1}{2}) + \frac{7}{2} + C$$

$$0 = \frac{3}{2} + \frac{7}{2} + C \Rightarrow 0 = \frac{10}{2} + C \Rightarrow 0 = 5 + C \Rightarrow \boxed{C=-5}$$

I become

$$\frac{4x^3}{(x^2-1)(x+1)^2}=\frac{4x^3}{(x-1)(x+1)^3}=\frac{1}{2(x-1)}+\frac{7}{2(x+1)}-\frac{5}{(x+1)^2}+\frac{2}{(x+1)^3}$$

11. $\dfrac{2x+1}{(x+3)(x-1)(x+2)^2}$

Sol $$\frac{2x+1}{(x+3)(x-1)(x+2)^2}=\frac{A}{x+3}+\frac{B}{x-1}+\frac{C}{x+2}+\frac{D}{(x+2)^2} \longrightarrow I$$

'X' by $(x+3)(x-1)(x+2)^2$ we get.

$$2x+1=A(x-1)(x+2)^2+B(x+3)(x+2)^2+C(x+3)(x-1)(x+2)+D(x+3)(x-1) \longrightarrow II$$

Put $x-1=0 \Rightarrow x=1$ in II.

$$2(1)+1=A(1-1)(1+2)^2+B(1+3)(1+2)^2+C(1+3)(1-1)(1+2)+D(1+3)(1-1)$$

$$3=0+B(4)(9)+0+0 \Rightarrow 3=36B \Rightarrow B=3/36 \Rightarrow \boxed{B=1/12}$$

Put $x+3=0 \Rightarrow x=-3$ in II.

$$2(-3)+1=A(-3-1)(-3+2)^2+B(-3+3)(-3+2)^2+C(-3+3)(-3-1)(-3+2)+D(-3+3)(-3-1)$$

$$-5=A(-4)(1)+0+0+0 \Rightarrow \boxed{A=5/4}$$

Put $x+2=0 \Rightarrow x=-2$ in II.

$$2(-2)+1=A(-2-1)(-2+2)^2+B(-2+3)(-2+2)^2+C(-2+3)(-2-1)(-2+2)+D(-2+3)(-2-1)$$

$$-3=0+0+0+D(1)(-3) \Rightarrow \boxed{D=1}$$

Rearrange II.

$$2x+1=A(x-1)(x^2+4x+4)+B(x+3)(x^2+4x+4)+C(x+3)(x^2+x-2)+D(x^2+2x-3)$$

$$2x+1=Ax^3+4Ax^2+4Ax-Ax^2-4Ax-4A+Bx^3+4Bx^2+4Bx+3Bx^2+12Bx$$
$$+12B+Cx^3+Cx^2-2Cx+3Cx^2+3Cx-2C+Dx^2+2Dx-3D$$

Comparing Co-efficients

$$x^3;\ 0=A+B+C \Rightarrow 0=\frac{5}{4}+\frac{1}{12}+C=0=\frac{15+1}{12}+C$$

$$C=\frac{-16}{12} \Rightarrow \boxed{C=\frac{-4}{3}}$$ I become

$$\frac{2x+1}{(x+3)(x-1)(x+2)^2}=\frac{5}{4(x+3)}+\frac{1}{12(x-1)}-\frac{4}{3(x+2)}+\frac{1}{(x+2)^2}$$

12. $\dfrac{2x^4}{(x-3)(x+2)^2}$

Sol $$\frac{2x^4}{(x-3)(x+2)^2}=\frac{2x^4}{(x-3)(x^2+4x+4)}=\frac{2x^4}{x^3+4x^2+4x-3x^2-12x-12}=\frac{2x^4}{x^3+x^2-8x-12}$$

$$
\begin{array}{r}
2x-2 \\
x^3+x^2-8x-12\overline{\big)\,\cancel{2x^4}\qquad\qquad\qquad\qquad} \\
\underline{\cancel{-2x^4} \pm 2x^3 \mp 16x^2 \mp 24x} \\
\cancel{-2x^3}+16x^2+24x \\
\underline{\cancel{-2x^3} \mp 2x^2 \pm 16x \pm 24} \\
18x^2+8x-24
\end{array}
$$

$$= 2x-2+\frac{18x^2+8x-24}{x^3+x^2-8x-12} \qquad \longrightarrow I$$

$$\frac{18x^2+8x^2-24}{x^3+x^2-8x^2-12}=\frac{18x^2+8x-24}{(x-3)(x+2)^2}=\frac{A}{x-3}+\frac{B}{x+2}+\frac{C}{(x+2)^2} \qquad \longrightarrow II$$

'X' by $(x-3)(x+2)^2$

$$18x^2+8x-24=A(x+2)^2+B(x-3)(x+2)+C(x-3) \qquad \longrightarrow III$$

Put $x-3=0 \Rightarrow x=3$ in III.

$$18(3)^2+8(3)-24=A(3+2)^2+B(3-3)(3+2)+C(3-3)$$

$$162+24-24=25A \Rightarrow \boxed{A=162/25}$$

Put $x+2=0 \Rightarrow x=-2$ in III.

$$18(-2)^2+8(-2)-24=A(-2+2)^2+B(-2-3)(-2+2)+C(-2-3)$$

$$72-16-24=A(0)+B(0)+C(-5)$$

$$32=+0+0-5C \Rightarrow \boxed{C=-32/5}$$

Rearrange III

$$18x^2+8x-24=Ax^2+4Ax+4A+Bx^2-Bx-6B+Cx-3C$$

Comparing Co-efficients

$$x^2;\ 18=A+B \Rightarrow 18=\frac{162}{25}+B \Rightarrow B=18-\frac{162}{25} \Rightarrow \boxed{B=\frac{288}{25}}$$

I become

$$\frac{2x^4}{(x-3)(x+2)^2}=2x-2+\frac{162}{25(x-3)}+\frac{288}{25(x+2)}-\frac{32}{5(x+2)^2}$$

Example 2: $\dfrac{4x^2+8x}{x^4+2x^2+9}$ *Federal*

Here $x^4+2x^2+9=x^4+2x^2+9+4x^2-4x^2=x^4+6x^2+9-4x^2$

$$=(x^2+3)^2-(2x)^2=(x^2+3+2x)(x^2+3-2x)$$

$$=(x^2+2x+3)(x^2-2x+3)$$

Now $\dfrac{4x^2+8x}{x^4+2x^2+9}=\dfrac{4x^2+8x}{(x^2+2x+3)(x^2-2x+3)}=\dfrac{Ax+B}{x^2+2x+3}+\dfrac{Cx+D}{x^2-2x+3}$ ——I

$'\times'$ *by* $(x^2+2x+3)(x^2-2x+3)$ *we get*

$4x^2+8x=(Ax+B)(x^2-2x+3)+(Cx+D)(x^2+2x+3)$

$4x^2+8x=Ax^3-2Ax^2+3Ax+Bx^2-2Bx+3B+Cx^3+2Cx^2+3Cx+Dx^2+2Dx+3D$

Comparing Co – efficents

x^3; $0=A+C$ ——II

x^2; $4=-2A+B+2C+D$——III

x; $8=3A-2B+3C+2D$——IV

cons tan *t*; $0=3B+3D\Rightarrow 0=B+D$——V

$IV\Rightarrow$ $8=3A+3C-2B+2D$

$8=3(A+C)-2B+2D$

$(use\ II)$ $8=3(0)-2B+2D$

$8=-2B+2D$

$\div by\ 2$

$4=-B+D$——VII

$V+VII\Rightarrow 0=\cancel{B}+D$

$\underline{4=-\cancel{B}+D}$

$4=2D\Rightarrow \boxed{D=2}$

$V\Rightarrow 0=B+2\Rightarrow \boxed{B=-2}$

$III\Rightarrow 4=-2A+2C+B+D$

$4=-2A+2C+0\,(use\,V)$

$2=-A+C$ ——VI $(\div by\ 2)$

$II+VI\Rightarrow 0=\cancel{A}+C$

$\underline{2=-\cancel{A}+C}$

$2=2C\Rightarrow \boxed{C=1}$

$II\Rightarrow A+C=0$

$A+1=0\Rightarrow \boxed{A=-1}$

I become $\dfrac{4x^2+8x}{x^4+2x^2+9}=\dfrac{(-1)x-2}{x^2+2x+3}+\dfrac{(1)x+2}{x^2-2x+3}$

$$=\frac{-x-2}{x^2+2x+3}+\frac{x+2}{x^2-2x+3}$$

$$=\frac{x+2}{x^2-2x+3}-\frac{(x+2)}{x^2+2x+3}$$

EXERCISE 5.3

Resolve the following into Partial Fractions:

1. $\dfrac{9x-7}{(x^2+1)(x+3)}$ **Lahore 2009**

Sol Suppose

$$\frac{9x-7}{(x^2+1)(x+3)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+1} \longrightarrow I$$

'X' by $(x^2+1)(x+3)$ we get.

$$9x-7=(Ax+B)(x+3)+C(x^2+1) \longrightarrow II$$

Put $x+3=0 \Rightarrow x=-3$ in II we get.

$$9(-3)-7=(A(-3)+B)((-3)+3)+C((-3)^2+1)$$

$$-27-7=0+10C \Rightarrow C=-34/10 \Rightarrow \boxed{C=-17/5}$$

Rearrange II.

$$9x-7=Ax^2+3Ax+Bx+3B+Cx^2+C$$

Comparing co-efficients

x^2; $\quad 0=A+C \Rightarrow 0=A-17/5 \Rightarrow \boxed{A=17/5}$

x; $\quad 9=3A+B \Rightarrow 9=3(\frac{17}{5})+B \Rightarrow B=9-\frac{51}{5}=\frac{45-51}{5} \Rightarrow \boxed{B=-6/5}$

I become $\dfrac{9x-7}{(x^2+1)(x+3)}=\dfrac{\frac{17x}{5}-\frac{6}{5}}{x^2+1}-\dfrac{\frac{17}{5}}{x+3}=\dfrac{17x-6}{5(x^2+1)}-\dfrac{17}{5(x+3)}$

2. $\dfrac{1}{(x^2+1)(x+1)}$ **Multan 2009**

Sol Suppose

$$\frac{1}{(x^2+1)(x+1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+1} \longrightarrow I$$

'X' by $(x^2+1)(x+1)$ We get.

$$1=(Ax+B)(x+1)+C(x^2+1) \longrightarrow II$$

Put $x+1=0 \Rightarrow x=-1$ in II

$$1=(Ax+B)(-1+1)+C((-1)^2+1)$$

$$1=0+2C \Rightarrow \boxed{C=1/2}$$

Rearrange II

$$1=Ax^2+Ax+Bx+B+Cx^2+C$$

Comparing Co-efficients

$x^2;\ 0 = A + C \Rightarrow 0 = A + 1/2 \Rightarrow \boxed{A = -1/2}$

$x;\ 0 = A + B \Rightarrow 0 = -1/2 + B \Rightarrow \boxed{B = 1/2}$

I become

$$\frac{1}{(x^2+1)(x+1)} = \frac{\frac{-1}{2}x + \frac{1}{2}}{x^2+1} + \frac{\frac{1}{2}}{x+1} = \frac{\frac{(-x+1)}{2}}{x^2+1} + \frac{\frac{1}{2}}{x+1}$$

$$= \frac{(-x+1)}{2(x^2+1)} + \frac{1}{2(x+1)} = \frac{1}{2(1+x)} + \frac{1-x}{2(1+x^2)}$$

3. $\dfrac{3x+7}{(x^2+4)(x+3)}$ **Faisalabad 2009**

Sol Suppose

$$\frac{3x+7}{(x^2+4)(x+3)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+3} \qquad \text{——}I$$

'X' by $(x^2+4)(x+3)$

$$3x+7 = (Ax+B)(x+3) + C(x^2+4) \qquad \text{——}II$$

Put $x+3 = 0 \Rightarrow x = -3$ in II

$$3(-3)+7 = (Ax+B)(-3+3) + C((-3)^2+4)$$

$$-2 = 0 + 13C \Rightarrow \boxed{C = -2/13}$$

Rearrange – II

$$3x+7 = Ax^2 + 3Ax + Bx + 3B + Cx^2 + 4C$$

Comparing Co-efficients.

$x^2;\quad 0 = A + C \Rightarrow 0 = A - 2/13 \Rightarrow \boxed{A = 2/13}$

$x;\quad 3 = 3A + B \Rightarrow 3 = 3(\frac{2}{13}) + B$

$$B = 3 - \frac{6}{13} \Rightarrow \boxed{B = \frac{33}{13}}$$

I become

$$\frac{3x+7}{(x^2+4)(x+3)} = \frac{\frac{2x}{13} + \frac{33}{13}}{x^2+4} + \frac{\frac{-2}{13}}{x+3} = \frac{2x+33}{13(x^2+4)} - \frac{2}{13(x+3)}$$

4. $\dfrac{x^2+15}{(x^2+2x+5)(x-1)}$

Sol Suppose

$$\frac{x^2+15}{(x^2+2x+5)(x-1)}=\frac{Ax+B}{x^2+2x+5}+\frac{C}{x-1} \longrightarrow I$$

'X' by $(x^2+2x+5)(x-1)$ We get.

$$x^2+15=(Ax+B)(x-1)+C(x^2+2x+5) \longrightarrow II$$

Put $x-1=0 \Rightarrow x=1$

$$(1)^2+15=(A(1)+B)(1-1)+C((1)^2+2(1)+5)$$

$$16=0+C(8) \Rightarrow \boxed{C=2}$$

Rearrange II we get.

$$x^2+15=Ax^2-Ax+Bx-B+Cx^2+2Cx+5C$$

Comparing Co-efficients

x^2; $1=A+C \Rightarrow 1=A+2 \Rightarrow \boxed{A=-1}$

x; $0=-A+B+2C$

$0=-(-1)+B+2(2)$

$0=1+B+4 \Rightarrow \boxed{B=-5}$

I become

$$\frac{x^2+15}{(x^2+2x+5)(x-1)}=\frac{-x-5}{x^2+2x+5}+\frac{2}{x-1}=\frac{2}{x-1}-\frac{(x+5)}{x^2+2x+5}$$

5. $\dfrac{x^2}{(x^2+4)(x+2)}$

Sol Suppose

$$\frac{x^2}{(x^2+4)(x+2)}=\frac{Ax+B}{x^2+4}+\frac{C}{x+2} \longrightarrow I$$

'X' by $(x^2+4)(x+2)$ We get.

$$x^2=(Ax+B)(x+2)+C(x^2+4) \longrightarrow II$$

Put $x+2=0 \Rightarrow x=-2$ in II.

$$(-2)^2=(A(-2)+B)(-2+2)+C((-2)^2+4)$$

$$4=0+8C \Rightarrow \boxed{C=1/2}$$

Rearrange II.

$$x^2=Ax^2+2Ax+Bx+2B+Cx^2+4C$$

Comparing Co-efficients

x^2; $1=A+C \Rightarrow 1=A+1/2$

$A=1-1/2 \Rightarrow \boxed{A=1/2}$

$x;\ 0 = 2A + B \Rightarrow 0 = 2(1/2) + B$

$0 = 1 + B \Rightarrow \boxed{B = -1}$

I become

$$\frac{x^2}{(x^2+4)(x+2)} = \frac{\frac{1}{2}x - 1}{x^2+4} + \frac{\frac{1}{2}}{x+2} = \frac{x-2}{2(x^2+4)} + \frac{1}{2(x+2)} = \frac{1}{2(x+2)} + \frac{x-2}{2(x^2+4)}$$

6. $\dfrac{x^2+1}{x^3+1}$

Sol $\dfrac{x^2+1}{x^3+1} = \dfrac{x^2+1}{(x+1)(x^2-x+1)}$

Suppose

$$\frac{x^2+1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \longrightarrow I$$

'X' by $(x+1)(x^2-x+1)$ we get.

$$x^2+1 = A(x^2-x+1) + (Bx+C)(x+1) \longrightarrow II$$

Put $x+1=0 \Rightarrow x=-1$ in II.

$(-1)^2 + 1 = A((-1)^2 - (-1) + 1) + (B(-1) + C)(-1+1)$

$2 = A(1+1+1) + 0 \Rightarrow 2 = 3A \Rightarrow \boxed{A = 2/3}$

Rearrange II.

$x^2 + 1 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$

Comparing Co-efficients

$x^2;\ 1 = A + B \Rightarrow 1 = 2/3 + B \Rightarrow B = 1 - \dfrac{2}{3} \Rightarrow \boxed{B = 1/3}$

$x;\quad 0 = -A + B + C$

$x;\quad 0 = \dfrac{-2}{3} + \dfrac{1}{3} + C \Rightarrow 0 = \dfrac{-1}{3} + C \Rightarrow \boxed{C = 1/3}$

I become

$$\frac{x^2+1}{x^3+1} = \frac{2}{3(x+1)} + \frac{\frac{1}{3}x + \frac{1}{3}}{x^2-x+1} = \frac{2}{3(x+1)} + \frac{(x+1)}{3(x^2-x+1)}$$

7. $$\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}$$

Sol $$\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}=\frac{Ax+B}{x^2+3}+\frac{C}{x+1}+\frac{D}{x-1} \longrightarrow I$$

'×' by $(x^2+3)(x+1)(x-1)$ we get.

$$x^2+2x+2=(Ax+B)(x+1)(x-1)+C(x^2+3)(x-1)+D(x^2+3)(x+1) \longrightarrow II$$

Put $x-1=0 \Rightarrow x=1$ in II

$$(1)^2+2(1)+2=(A(1)+B)(1+1)(1-1)+C(1^2+3)(1-1)+D(1^2+3)(1+1)$$

$$1+2+2=0+C(4)(0)+D(1^2+3)(1+1) \Rightarrow 5=8D \Rightarrow \boxed{D=5/8}$$

Put $x+1=0 \Rightarrow x=-1$ in II.

$$(-1)^2+2(-1)+2=(A(-1)+B)(-1+1)(-1-1)+C((-1)^2+3)(-1-1)+D(-1)^2+3)(-1+1)$$

$$1-2+2=0+C(4)(-2)+0 \Rightarrow 1=-8C \Rightarrow \boxed{C=-1/8}$$

Rearrange II

$$x^2+2x+2=(Ax+B)(x^2-1)+C(x^2+3)(x-1)+D(x^2+3)(x+1)$$

$$x^2+2x+2=Ax^3-Ax+Bx^2-B+Cx^3-Cx^2+3Cx-3C+Dx^3+Dx^2+3Dx+3D$$

Comparing II

$$x^3;\ 0=A+C+D \Rightarrow 0=A-\frac{1}{8}+\frac{5}{8} \Rightarrow 0=A+\frac{4}{8} \Rightarrow 0=A+\frac{1}{2} \Rightarrow \boxed{A=-\frac{1}{2}}$$

$$x^2;1=B-C+D \Rightarrow 1=B-(-\frac{1}{8})+\frac{5}{8} \Rightarrow 1=B+\frac{1}{8}+\frac{5}{8} \Rightarrow 1=B+\frac{6}{8}$$

$$1=B+3/4 \Rightarrow B=1-\frac{3}{4} \Rightarrow \boxed{B=\frac{1}{4}}$$

I become

$$\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}=\frac{-\frac{1}{2}x+\frac{1}{4}}{x^2+3}+\frac{(-1/8)}{x+1}+\frac{5/8}{x-1}$$

$$=\frac{-2x+1}{4(x^2+3)}-\frac{1}{8(x+1)}+\frac{5}{8(x-1)}=\frac{5}{8(x-1)}-\frac{1}{8(x+1)}-\frac{(2x-1)}{4(x^2+3)}$$

8. $$\frac{1}{(x-1)^2(x^2+2)}$$

Sol $$\frac{1}{(x-1)^2(x^2+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+2} \longrightarrow I$$

'X' by $(x-1)^2(x^2+2)$ we get.

$$1 = A(x-1)(x^2+2)+B(x^2+2)+(Cx+D)(x-1)^2 \longrightarrow II$$

Put $x-1=0 \Rightarrow x=1$ in II.

$$1 = A(1-1)((1)^2+2)+B((1)^2+2)+(C(1)+D)(1-1)^2$$

$$1 = 0+B(3)+0 \Rightarrow 1 = 3B \Rightarrow \boxed{B=1/3}$$

Rearrange - II.

$$1 = Ax^3+2Ax-Ax^2-2A+Bx^2+2B+Cx^3-2Cx^2+Cx+Dx^2-2Dx+D$$

Comparing Co-efficients

$$x^3;\ 0 = A+C \quad \rightarrow III$$

$$x^2;\ 0 = -A+B-2C+D \quad \longrightarrow IV$$

$$x;\quad 0 = 2A+C-2D \quad \longrightarrow V$$

$$Constant\ \ 1 = -2A+2B+D \quad \longrightarrow VI$$

$$(VI\times 2+V)\ \ 2 = -4A+4B+\cancel{2D}$$

$$\underline{0 = 2A+C-\cancel{2D}}$$

$$2 = -2A+4B+C$$

$$2 = -2A+4(\frac{1}{3})+C\,(put\ value\ of\ B)$$

$$2 = -2A+\frac{4}{3}+C \Rightarrow -2A+C = 2-\frac{4}{3} = \frac{2}{3} \quad \longrightarrow VII$$

$$VII - III \Rightarrow \quad -2A+\cancel{C} = 2/3$$

$$\underline{A \pm \cancel{C} = 0}$$

$$-3A = 2/3$$

$$-3A = 2/3 \Rightarrow \boxed{\frac{-2}{9} = A}$$

$$From\ III \quad A+C=0 \Rightarrow -\frac{2}{9}+C=0 \Rightarrow \boxed{C=\frac{2}{9}}$$

$$IV \quad \Rightarrow \quad 0 = -A+B-2C+D$$

$$0 = -(-\frac{2}{9})+\frac{1}{3}-2(\frac{2}{9})+D$$

$$0 = \frac{2}{9}+\frac{1}{3}-\frac{4}{9}+D \Rightarrow 0 = \frac{2+3-4}{9}+D \Rightarrow 0 = \frac{1}{9}+D \Rightarrow \boxed{D=-\frac{1}{9}}$$

I become $\dfrac{1}{(x-1)^2(x^2+2)} = \dfrac{-2/9}{x-1}+\dfrac{1/3}{(x-1)^2}+\dfrac{\frac{2x}{9}-\frac{1}{9}}{x^2+2}$

$$= \frac{-2}{9(x-1)} + \frac{1}{3(x-1)^2} + \frac{2x-1}{9(x^2+2)}$$

9. $\frac{x^4}{1-x^4}$ **(Improper)**

$$-x^4+1 \overline{\smash{)}\, x^4} \quad \text{quotient } -1,\quad \underline{x^4 \mp 1},\quad \text{remainder } 1$$

Sol $x^4 = -1 + \frac{1}{1-x^4}$ ——— I

Now $\frac{1}{1-x^4} = \frac{1}{(1-x^2)(1+x^2)} = \frac{1}{(1-x)(1+x)(1+x^2)}$

Now $\frac{1}{(1-x)(1+x)(1+x^2)} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}$ $\longrightarrow II$

'×' by $(1-x)(1+x)(1+x^2)$

$1 = A(1+x)(1+x^2) + B(1-x)(1+x^2) + (Cx+D)(1-x)(1+x)$ $\longrightarrow III$

Put $x+1=0 \Rightarrow x=-1$ in III.

$1 = A(1-1)(1+(-1)^2) + B(1-(-1))(1+(-1)^2) + (Cx+D)(1-(-1))(1-1)$

$1 = 0 + B(1+1)(1+1) + 0 \Rightarrow 1 = 4B \Rightarrow \boxed{B = 1/4}$

Put $x-1=0 \Rightarrow x=1$ in III $\Rightarrow 1 = A(1+1)(1+1)+0+0 \Rightarrow 1 = 4A \Rightarrow \boxed{A = 1/4}$

Rearrange III we get.

$1 = A + Ax^2 + Ax + Ax^3 + B + Bx^2 - Bx - Bx^3 + Cx - Cx^3 + D - Dx^2$

Comparing Co-efficients

$X^3; 0 = A - B - C \Rightarrow 0 = \frac{1}{4} - \frac{1}{4} - C \Rightarrow 0 = -C \Rightarrow \boxed{C = 0}$

$x^2; \quad 0 = A + B - D$

$$0 = \frac{1}{4} + \frac{1}{4} - D \quad \Rightarrow 0 = \frac{1}{2} - D \Rightarrow \boxed{D = \frac{1}{2}}$$

I become

$$\frac{1}{1-x^4} = -1 + \frac{1}{1-x^4} = -1 + \frac{1}{(1-x)(1+x)(1+x^2)}$$

$$= -1 + \frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{0(x)+1/2}{1+x^2}$$

$$= -1 + \frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1+x^2)}$$

10. $$\frac{x^2-2x+3}{x^4+x^2+1}$$

Sol $$\frac{x^2-2x+3}{x^4+x^2+1}=\frac{x^2-2x+3}{x^4+x^2+1+x^2-x^2}=\frac{x^2-2x+3}{x^4+2x^2+1-x^2}$$

$$=\frac{x^2-2x+3}{(x^2+1)^2-x^2}=\frac{x^2-2x+3}{(x^2+1+x)(x^2+1-x)}$$

Suppose $$\frac{x^2-2x+3}{(x^2+x+1)(x^2-x+1)}=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2-x+1} \quad \text{——} I$$

'X' by $(x^2+x-1)(x^2-x+1)$ we get.

$$x^2-2x+3=(Ax+B)(x^2-x+1)+(Cx+D)(x^2+x+1)$$

$$x^2-2x+3=Ax^3-Ax^2+Ax+Bx^2-Bx+B+Cx^3+Cx^2+Cx+Dx^2+Dx+D$$

Comparing Co-efficients

$x^3;\ 0=A+C$ ——II

$x^2;\ 1=-A+B+C+D$ ——III

$x;\ -2=+A-B+C+D$——IV

$Catt;\ 3=B+D$ ——V

$III \Rightarrow 1=-A+B+C+D$

$or\ 1=-A+C+B+D$

$Use\ B+D=3$

$1=-A+C+3 \Rightarrow -A+C=-2$ ——VII

$IV \Rightarrow -2=A-B+C+D$

$or\quad -2=A+C-B+D$

$Put\ A+C=0$

$-2=-B+D$ ——VI

$V+VI \Rightarrow 3=\cancel{B}+D$

$\underline{-2=-\cancel{B}+D}$

$1=2D \Rightarrow \boxed{D=1/2}$

$II+VII$

$V \Rightarrow 3=B+\frac{1}{2} \Rightarrow B=3-\frac{1}{2} \Rightarrow \boxed{B=2/5}$

$0=\cancel{A}+C$

$\underline{-2=-\cancel{A}+C}$

$-2=2C \Rightarrow \boxed{C=-1}$

$II \Rightarrow 0=A+C$

$0=A-1 \Rightarrow \boxed{A=1}$

I become

$$\frac{x^2-2x+3}{x^4+x^2+1}=\frac{x^2-2x+3}{(x^2+x+1)(x^2-x+1)}=\frac{(1)x+5/2}{x^2+x+1}+\frac{(-1)x+1/2}{x^2-x+1}$$

$$\frac{2x+5}{2(x^2+x+1)}+\frac{(-2x+1)}{x^2-x+1}=\frac{-(2x-1)}{(x^2-x+1)}+\frac{2x+5}{2(x^2+x+1)}$$

EXERCISE 5.4

Resolve into Partial Fractions:

1. $$\frac{x^3+2x+2}{(x^2+x+1)^2}$$

Sol. **Suppose**

$$\frac{x^3+2x+2}{(x^2+x+1)^2}=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{(x^2+x+1)^2}$$

'X' both sides by $(x^2+x+1)^2$ we get

$$x^3+2x+2=(Ax+B)(x^2+x+1)+Cx+D$$

$$or\ x^3+2x+2=Ax^3+Ax^2+Ax+Bx^2+Bx+B+Cx+D$$

Comparing Co-efficients

$x^3;\ \boxed{1=A}$

$x^2;\ 0=A+B\Rightarrow 0=1+B\Rightarrow\boxed{B=-1}$

$x;\ 2=A+B+C\Rightarrow 2=1-1+C\Rightarrow\boxed{C=2}$

Constant; $2=B+D$

$2=-1+D\Rightarrow\boxed{D=3}$

Hence

$$\frac{x^3+2x+2}{(x^2+x+1)^2}=\frac{(1)x-1}{x^2+x+1}+\frac{2x+3}{(x^2+x+1)^2}$$

Or

$$\frac{x^3+2x+2}{(x^2+x+1)^2}=\frac{x-1}{x^2+x+1}+\frac{2x+3}{(x^2+x+1)^2}$$

2. $$\frac{x^2}{(x^2+1)^2(x-1)}$$

Sol. **Suppose**

$$\frac{x^2}{(x^2+1)^2(x-1)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}+\frac{E}{x-1}\quad\longrightarrow I$$

'X' both sides by $(x^2+1)^2(x-1)$ we get

$$x^2=(Ax+B)(x-1)(x^2+1)+(Cx+D)(x-1)+E(x^2+1)^2\quad\longrightarrow II$$

Put $x-1=0\Rightarrow x=1$ in II

$$(1)^2=(Ax+B)(1-1)+(1^2+1)+(Cx+D)(1-1)+E(1^2+1)^2$$

$$1=0+0+4E\Rightarrow\boxed{E=1/4}$$

Rearrange II

$$x^2 = (Ax^2 - Ax + Bx - B)(x^2+1) + Cx^2 - Cx + Dx - D + E(x^4 + 2x^2 + 1)$$

$$x^2 = Ax^4 - Ax^3 + Bx^3 - Bx^2 + Ax^2 - Ax + Bx - B + Cx^2 - Cx + Dx - D + Ex^4 + 2Ex^2 + E$$

Comparing Co-efficients

$x^4; 0 = A + E \Rightarrow 0 = A + 1/4 \Rightarrow \boxed{A = -1/4}$

$x^3; 0 = -A + B \Rightarrow 0 = -(-1/4) + B \Rightarrow \boxed{B = -1/4}$

$x^2; 1 = -B + A + C + 2E$

$1 = -(-1/4) - \frac{1}{4} + C + 2(\frac{1}{4})$

$1 = (\cancel{\frac{1}{4}}) - \cancel{\frac{1}{4}} + C + \frac{1}{2} \Rightarrow C = 1 - 1/2 \Rightarrow \boxed{C = 1/2}$

$x; 0 = -A + B - C + D$

$0 = -(-1/4) + (-1/4) - \frac{1}{2} + D$

$0 = 1/4 - 1/4 + D - 1/2$

$0 = D - 1/2 \Rightarrow \boxed{D = 1/2}$

Hence

$$\frac{x^2}{(x^2+1)^2(x-1)} = \frac{-1/4x - 1/4}{x^2+1} + \frac{1/2x + 1/2}{(x^2+1)^2} + \frac{1/4}{x-1}$$

$$= \frac{-x-1}{4(x^2+1)} + \frac{x+1}{2(x^2+1)^2} + \frac{1}{4(x-1)}$$

Or

$$= \frac{1}{4(x-1)} - \frac{(x+1)}{4(x^2+1)} + \frac{x+1}{2(x^2+1)^2}$$

3. $\dfrac{\mathbf{2x-5}}{\mathbf{(x^2+2)^2(x-2)}}$ **Federal**

Sol. **Suppose**

$$\frac{2x-5}{(x^2+2)^2(x-2)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} + \frac{E}{x-2} \quad \rightarrow I$$

'X' both sides by $(x^2+2)^2(x-2)$ we get

$$2x - 5 = (Ax+B)(x^2+2)(x-2) + (Cx+D)(x-2) + E(x^2+2)^2 \quad \rightarrow II$$

Put $x - 2 = 0 \Rightarrow x = 2$ in II

$$2(2) - 5 = (Ax+B)(2^2+2)(2-2) + (Cx+D)(2-2) + E(2^2+2)^2$$

$$4-5=0+0+36E \Rightarrow -1=36E \Rightarrow \boxed{E=-1/36}$$

Rearrange II

$$2x-5=(Ax^3+2Ax+Bx^2+2B)(x-2)+Cx^2-2Cx+Dx-2D+E(x^4+4x^2+4)$$

$$2x-5=(Ax^4+2Ax^2+Bx^3+2Bx-2Ax^3-4Ax-2Bx^2-4B+Cx^2-2Cx+Dx-2D+Ex^4+4Ex^2+4E$$

Comparing Co-efficients.

$$x^4;\ 0=A+E \Rightarrow 0=A-1/36 \Rightarrow \boxed{A=1/36}$$

$$x^3;\ 0=B-2A \Rightarrow 0=B-2(1/36) \Rightarrow 0=B-\frac{1}{18} \Rightarrow \boxed{B=1/18}$$

$$x^2;\ 0=2A-2B+C+4E \Rightarrow 0=2(1/36)-2(1/18)+C+4(-1/36)$$

$$or\ \ 0=\frac{1}{18}-\frac{1}{9}+C-\frac{1}{9} \Rightarrow 0=C+\frac{1-2-2}{18} \Rightarrow 0=C-\frac{3}{18} \Rightarrow \boxed{C=3/18}$$

$$x;\ 2=2B-4A-2C+D \Rightarrow 2=2(\frac{1}{18})-4(\frac{1}{36})-2(\frac{3}{18})+D$$

$$or\ \ 2=\frac{1}{9}-\frac{1}{9}-\frac{3}{9}+D \Rightarrow 2+\frac{3}{9}=D \Rightarrow D=\frac{21}{9}=\boxed{7/3=D}$$

Hence

$$\frac{2x-5}{(x^2+2)^2(x-2)}=\frac{\frac{1}{36}x+\frac{1}{18}}{x^2+2}+\frac{\frac{3}{18}x+\frac{7}{3}}{(x^2+2)^2}+\frac{-1/36}{x-2}$$

$$=\frac{x+2}{36(x^2+2)}+\frac{3x+42}{18(x^2+2)^2}-\frac{1}{36(x-2)}$$

$$=\frac{-1}{36(x-2)}+\frac{x+2}{36(x^2+2)}+\frac{3(x+14)}{18(x^2+2)^2}=-\frac{1}{36(x-2)}+\frac{x+2}{36(x^2+2)}+\frac{x+14}{6(x^2+2)^2}$$

4. $\displaystyle \mathbf{\frac{8x^2}{(x^2+1)^2(1-x^2)}}$

Sol. $$\frac{8x^2}{(x^2+1)^2(1-x^2)}=\frac{8x^2}{(x^2+1)^2(1-x)(x+1)}$$

Suppose

$$\frac{8x^2}{(x^2+1)^2(1-x)(x+1)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}+\frac{E}{x-1}+\frac{F}{x+1} \longrightarrow I$$

'X' both sides by $(x^2+1)^2(1-x)(1+x)$ we get

$$8x^2=(Ax+B)(x^2+1)(1-x)(1+x)+(Cx+D)(1-x)(1+x)+E(x^2+1)^2(1+x)+F(x^2+1)^2(1-x) \longrightarrow II$$

Put $1-x=0 \Rightarrow x=1$ in II.

$$8(1)^2 = 0+0+0+E(1^2+1)^2(1+1)+0$$

$$8 = E(4)(2) \Rightarrow 8 = 8E \Rightarrow \boxed{E=1}$$

Put $1+x=0 \Rightarrow x=-1$ in II.

$$8(-1)^2 = 0+0+0+F((-1^2)+1)^2(1-(-1))$$

$$8 = F(4)(2) \Rightarrow \boxed{F=1}$$

Rearrange II

$$8x^2 = (Ax^3+Ax+Bx^2+B)(1-x^2)+(Cx+D)(1-x^2)+E(x^4+2x^2+1)(1+x)+F(x^4+2x^2+1)(1-x)$$

$$\text{or } 8x^2 = Ax+B-Ax^5-Bx^4+Cx-Cx^3+D-Dx^2+Ex^4+2Ex^2+E$$
$$+Ex^5+2Ex^3+Ex+Fx^4+2Fx^2+F-Fx^5-2Fx^3-Fx$$

Comparing Co-efficient.

$$x^5;\ 0=-A+E-F \longrightarrow III \Rightarrow 0=-A+1-1 \Rightarrow \boxed{A=0}$$

$$x^4;\ 0=-B+E+F \longrightarrow IV \Rightarrow 0=-B+1+1 \Rightarrow 0=-B+2 \Rightarrow \boxed{B=2}$$

$$x^3;\ 0=-C+2E-2F \longrightarrow V \Rightarrow 0=-C+2(1)-2(1) \Rightarrow 0=-C+2-2 \Rightarrow \boxed{C=0}$$

$$x^2;8=2E+2F \longrightarrow VI$$

$$x;0=A+C+E-F \longrightarrow VII$$

Constant; $0=B+D+E+F \Rightarrow 0=2+D+1+1 \Rightarrow \boxed{D=-4}$

Put values in I.

$$\frac{8x^2}{(x^2+1)^2(1-x)(x+1)} = \frac{0+2}{(x^2+1)}+\frac{0-4}{(x^2+1)^2}+\frac{1}{1-x}+\frac{1}{1+x}$$

$$=\frac{2}{(x^2+1)}-\frac{4}{(x^2+1)^2}+\frac{1}{(1-x)}+\frac{1}{(1+x)} = \frac{1}{1-x}+\frac{1}{1+x}+\frac{2}{x^2+1}-\frac{4}{(x^2+1)^2}$$

5. $$\frac{4x^4+3x^3+6x^2+5x}{(x-1)(x^2+x+1)^2}$$

Sol. Suppose

$$\frac{4x^4+3x^3+6x^2+5x}{(x-1)(x^2+x+1)^2} = \frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}+\frac{Dx+E}{(x^2+x+1)^2} \longrightarrow I$$

'X' both sides by $(x-1)(x^2+x+1)^2$ we get

$$4x^4+3x^3+6x^2+5x = A(x^2+x+1)^2+(Bx+C)(x-1)(x^2+x+1)+(Dx+E)(x-1) \longrightarrow II$$

Put $x-1=0 \Rightarrow x=1$ in II.

$$4(1)^4+3(1)^3+6(1)^2+5(1) = A(1^2+1+1)^2+0+0$$

$$4+3+6+5 = A(3)^2 \Rightarrow 18=9A \Rightarrow A=\frac{18}{9} \Rightarrow \boxed{A=2}$$

Rearrange equation II

$4x^4+3x^3+6x^2+5x=A(x^4+x^2+1+2x^3+2x+2x^2)+(Bx+C)(x^3-1)+(Dx+E)(x-1)$

$4x^4+3x^3+6x^2+5x=Ax^4+2Ax^3+3Ax^2+2Ax+A+Bx^4-Bx+Cx^3-C+Dx^2-Dx+Ex-E$

Comparing Co-efficients

$x^4;\ 4=A+B \longrightarrow III \quad \Rightarrow 4=2+B \Rightarrow \boxed{B=2}$

$x^3;\ 3=2A+C \longrightarrow IV \quad \Rightarrow 3=2(2)+C \Rightarrow 3=4+C \Rightarrow \boxed{C=-1}$

$x^2;\ 6=3A+D \longrightarrow V \quad \Rightarrow 6=3(2)+D \Rightarrow 6=6+D \Rightarrow \boxed{D=0}$

$x;\ 5=2A-B-D+E \longrightarrow VI \quad \Rightarrow 5=2(2)-2-0+E \Rightarrow 5=2+E \Rightarrow \boxed{E=3}$

Constant; $\quad 0=A-C-E \longrightarrow VII$

Put values in I

$$\frac{4x^4+3x^3+6x^2+5x}{(x-1)(x^2+x+1)^2}=\frac{2}{x-1}+\frac{2x-1}{x^2+x+1}+\frac{3}{(x^2+x+1)^2}$$

6. $\dfrac{\mathbf{2x^4-3x^3-4x}}{\mathbf{(x^2+2)^2(x+1)^2}}$

Sol. Suppose

$$\frac{2x^4-3x^3-4x}{(x^2+2)^2(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{Cx+D}{x^2+2}-\frac{Ex+F}{(x^2+2)^2} \longrightarrow I$$

'X' both sides by $(x^2+2)^2(x+1)^2$ we get

$2x^4-3x^3-4x=A(x+1)(x^2+2)^2+B(x^2+2)^2+(Cx+D)(x+1)^2(x^2+2)(Ex+F)(x+1)^2 \longrightarrow II$

Put $x+1=0 \Rightarrow x=-1$ in II

$2(-1)^4-3(-1)^3-4(-1)=0+B\left((-1)^2+2\right)^2+0+0$

$2(1)-3(-1)+4=B(1+2)^2 \Rightarrow 2+3+4=B(3)^2 \Rightarrow 9=9B \Rightarrow \boxed{B=1}$

Rearrange II. $2x^4-3x^3-4x=A(x+1)(x^4+4x^2+4)+B(x^4+4+4x^2)$

$+(Cx+D)(x^2+2x+1)(x^2+2)+(Ex+F)(x^2+2x+1)$

$2x^4-3x^3-4x=Ax^5+4Ax+4Ax^3+Ax^4+4A+4Ax^2+Bx^4+4B+4Bx^2+Cx^5+2Cx^4+Cx^3+2Cx^3$

$+4Cx^2+2Cx+Dx^4+2Dx^3+Dx^2+2Dx^2+4Dx+2D+Ex^3+2Ex^2+Ex+Fx^2+2Fx+F$

Comparing Co-efficients

$x^5;\ 0 \quad =A+C \longrightarrow III$

$x^4;\ 2 \quad =A+B+2C+D \longrightarrow IV$

$x^3;\ -3 \ =4A+C+2C+2D+E \Rightarrow -3=4A+3C+2D+E \rightarrow V$

$x^2;0=4A+4B+4C+D+2D+2E+F \Rightarrow 0=4A+4B+4C+3D+2E+F \rightarrow VI$

$x;\ -4 \quad =4A+2C+4D+E+2F \rightarrow VII$

$constant;\ 0 \quad = 4A+4B+2D+F \quad \rightarrow VIII$

From III $\boxed{C=-A}$ Put values of B and C in IV.

$2 = A+1+2(-A)+D \Rightarrow 2=1=A-2A+D \Rightarrow \boxed{1=-A+D} \Rightarrow \boxed{D=1+A} \rightarrow IX$

$Put\ values\ in\ V\ -3 = 4A+3(-A)+2(1+A)+E$

$-3 = 4A-3A+2+2A+E \Rightarrow -3-2 = 3A+E \Rightarrow \boxed{-5=3A+E} \rightarrow X$

Put values in VI.

$0 = 4A+4(1)+4(-A)+3(1+A)+2(-5-3A)+F$

$0 = \cancel{4A}+4-\cancel{4A}+3+3A-10-6A+F$

$0 = -3-3A+F \qquad \Rightarrow 3 = -3A+F \quad \rightarrow XI$

Put values in VII.

$-4 = 4A+2(-A)+4)(1+A)+(-5-3A)+2F$

$-4 = 4A-2A+4+4A-5-3A+2F$

$-4 = 3A-1+2F \Rightarrow -4+1 = 3A+2F \Rightarrow -3 = 3A+2F \qquad \rightarrow XII$

Adding XI & XII $\qquad 3 = -\cancel{3A}+F$

$\underline{-3 = \cancel{3A}+2F}$

$0 = 3F \Rightarrow \boxed{F=0}$

from XI $\qquad 3 = -3A+0 \Rightarrow \boxed{A=-1}$

from III $\qquad 0 = -1+C \Rightarrow \boxed{C=1}$

from IX $\qquad 1 = -(-1)+D \Rightarrow 1 = 1+D \Rightarrow \boxed{D=0}$

from X $\qquad -5 = 3(-1)+E \Rightarrow -5 = -3+E \Rightarrow E-5+3 \Rightarrow \boxed{E=-2}$

Put values in I.

$$\frac{2x^4-3x^3-4x}{(x^2+2)^2(x+1)^2} = \frac{-1}{x+1}+\frac{1}{(x+1)^2}+\frac{x+0}{x^2+2}+\frac{-2x+0}{(x^2+2)^2}$$

$$= \frac{-1}{x+1}+\frac{1}{(x+1)^2}+\frac{x}{x^2+2}+\frac{2x}{(x^2+2)^2}$$

TEST YOUR SKILLS

Marks: 50

Q # 1. Select the Correct Option

i. $x^2+x-6=0$ is:

a) Equation b) Identity
c) Proper fraction d) Improper fraction

ii. Partial fraction of $\frac{1}{(x+1)(x^2-1)}$ will be of the form:

a) $\frac{A}{x+1}+\frac{Bx+C}{x^2-1}$ b) $\frac{A}{x+1}+\frac{B}{x^2-1}$
c) $\frac{A}{x-1}+\frac{B}{x+1}$ d) None of these

iii. The quotient of two polynomials $\frac{P(x)}{Q(x)}, Q(x)\neq 0$ with no common factor is called:

a) Algebraic Relation b) Rational fraction
c) Partial fraction d) Polynomial

iv. An equation which holds good for all values of variable is called:

a) Equation b) Conditional equation
c) Idenity d) None of these

Q # 2. Short Questions:

i. Resolve into partial fraction $\frac{9}{(x+2)^2(x-1)}$

ii. Resolve into partial fraction $\frac{7x+25}{(x+3)(x+4)}$

iii. Define Conditional equation and improper rational fraction:

Long Questions:

Q # 3. (a) Resolve into partial fraction $\frac{4x}{(x+1)^2(x-1)}$

(b) Resolve $\frac{1}{(x-3)^2(x+1)}$ into partial fraction.

Q # 4. (a) Resolve $\frac{1}{(x-1)(2x-1)(3x-1)}$ into partial fraction

(b) Resolve $\frac{3x+7}{(x^2+4)(x+3)}$ into partial fraction.

SEQUENCE AND SERIES 6

Sequence:

Sequence is a function whose domain is subset of the set of natural numbers.

Real Sequence:

If all members of a sequence are real numbers, then it is called a real sequence.

Finite Sequence:

If the domain of a sequence is a finite set, then the sequence is called a finite sequence.

Infinite Sequence:

If the domain of a sequence is an infinite set, then the sequence is called an infinite sequence.

Series:

The sum of an indicated number of terms in a sequence is called a series.

Exercise 6.1

1. Write the first four terms of the following sequences, if

i. $a_n = 2n-3$

Sol. $a_1 = 2(1)-3 = -1$

$a_2 = 2(2)-3 = 1$

$a_3 = 2(3)-3 = 3$

$a_4 = 2(4)-3 = 5$

First four terms are $-1, 1, 3, 5$

ii. $a_n = (-1)^n . n^2$

Sol. $a_1 = (-1)^1.(1)^2 = (-1)(1) = -1$

$a_2 = (-1)^2.(2)^2 = (1)(4) = 4$

$a_3 = (-1)^3.(3)^2 = (-1)(9) = -9$

$a_4 = (-1)^4.(4)^2 = (1)(16) = 16$

First four terms are $-1, 4, -9, 16$

iii. $a_n = (-1)(2n-3)$

Sol. $a_1 = (-1)^1(2(1)-3) \quad = (-1)(-1) = 1$

$a_2 = (-1)^2(2(2)-3) = 1(4-3) = (1)(1) = 1$

$a_3 = (-1)^3(2(3)-3) = (-1)(6-3) = (-1)(3) = -3$

$a_4 = (-1)^4(2(4)-3) = (1)(5) = 5$

First four terms are $1, 1, -3, 5$

iv. $a_n = 3n-5$

Sol. $a_1 = 3(1)-5 = -2$

$a_2 = 3(2)-5 = 1$

$a_3 = 3(3)-5 = 4$

$a_4 = 3(4)-5 = 7$

First four terms are $-2, 1, 4, 7$

v. $a_n = \dfrac{n}{2n+1}$

Sol. $a_1 = \dfrac{1}{2(1)+1} = \dfrac{1}{2+1} = \dfrac{1}{3}$

$a_2 = \dfrac{2}{2(2)+1} = \dfrac{2}{5}$

$a_3 = \dfrac{3}{2(3)+1} = \dfrac{3}{6+1} = \dfrac{3}{7}$

$a_4 = \dfrac{4}{2(4)+1} = \dfrac{4}{8+1} = \dfrac{4}{9}$

First four terms are $\dfrac{1}{3}, \dfrac{2}{5}, \dfrac{3}{7}, \dfrac{4}{9}$

vi. $a_n = \dfrac{1}{2^n}$

Sol. $a_1 = \dfrac{1}{2^1} = \dfrac{1}{2}$, $a_2 = \dfrac{1}{2^2} = \dfrac{1}{4}$, $a_3 = \dfrac{1}{2^3} = \dfrac{1}{8}$, $a_4 = \dfrac{1}{2^4} = \dfrac{1}{16}$

First four term $\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}$

vii. $a_n - a_{n-1} = n+2,\ a_1 = 2$

Sol. Put $n = 2, 3, 4$

$a_2 - a_{2-1} = 2+2 \Rightarrow a_2 - a_1 = 4$

$a_2 - 2 = 4 \Rightarrow a_2 = 6$

$a_3 - a_{3-1} = 3+2 \Rightarrow a_3 - a_2 = 5$

$$a_3 - 6 = 5 \Rightarrow a_3 = 11$$

$$a_4 - a_{4-1} = 4+2 \Rightarrow a_4 - a_3 = 6$$

$$a_4 - 11 = 6 \Rightarrow a_4 = 17$$

First four terms 2,6,11,7

viii. $\boldsymbol{a_n = na_{n-1}, a_1 = 1}$

Sol. **Put** $n = 2,3,4$

$$a_2 = 2a_{2-1} = 2a_1 = 2(1) = 2$$

$$a_3 = 2a_{3-1} = 3a_2 = 3(2) = 6$$

$$a_4 = 4a_{4-1} = 4a_3 = 4(6) = 24$$

First four terms are $1, 2, 6, 24$

ix. $\boldsymbol{a_n = (n+1)a_{n-1}, a_1 = 1}$

Sol. $a_2 = (2+1)a_{2-1} \Rightarrow 3a_1 = 3(1) = 3$

$$a_3 = (3+1)a_{3-1} \Rightarrow 4a_2 = 4(3) = 12$$

$$a_4 = (4+1)a_{4-1} \Rightarrow 5a_3 = 5(12) = 60$$

First four terms are $1, 3, 12, 60$

x. $\boldsymbol{a_n = \dfrac{1}{a+(n-1)d}}$

Sol. $a_1 = \dfrac{1}{a+(1-1)d} = \dfrac{1}{a},\ a_2 = \dfrac{1}{a+(2-1)d} = \dfrac{1}{a+d}$

$$a_3 = \frac{1}{a+(3-1)d} = \frac{1}{a+2d},\ a_4 = \frac{1}{a+(4-1)d} = \frac{1}{a+3d}$$

2. Find the indicated terms of the following sequences:

i. $\boldsymbol{2, 6, 11, 17, \ldots\ldots\ldots a_7 = ?}$

Sol. $a_5 = 17 + 7 = 24$

$$a_6 = 24 + 8 = 32,\ \boxed{a_7 = 32 + 9 = 41}$$

ii. $\boldsymbol{1, 3, 12, 60, \ldots\ldots\ldots a_6}$

Sol. $1, 3, 12, 60, \ldots\ldots\ldots a_6 = ?$

$$a_5 = 60(6) = 360$$

$$a_6 = 360(7) = 2520 \Rightarrow \boxed{a_6 = 2520}$$

iii. $\boldsymbol{1, \frac{3}{2}, \frac{5}{4}, \frac{7}{8}, \ldots\ldots\ldots a_7}$

Sol. Add "2" in Numerator and '×' 2 by Denominator

$\frac{1}{1},\frac{3}{2},\frac{5}{4},\frac{7}{8},\frac{9}{16},\frac{11}{32},\frac{13}{64} \Rightarrow \boxed{a_7=\frac{13}{64}}$

iv. $\mathbf{1,1,-3,5,-7,9.........a_8=?}$

Sol. $1,1,-3,5,-7,9..........a_8=?$

(In even term plus 4, in odd term subtract 4)

$1,1,-3,5,-7,9,-11,13,\Rightarrow \boxed{a_8=13}$

v. $\mathbf{1,-3,5,-7,9,-11,.........a_8}$

Sol. $1,-3,5,-7,9,-11,............a_8=?$

$1,-3,5,-7,9,-11,13,-15\Rightarrow \boxed{a_8=-15}$

3. Find the next two terms:

i. $\mathbf{7,9,12,16,....................}$

Sol. $a_5=16+5=21$

$a_6=21+6=27$

ii. $\mathbf{1,3,7,15,31,.............}$

Sol. $a_6=31+32=63$

$a_7=63+64=127$

iii. $\mathbf{-1.2,12,40}$

Sol. $a_1=-1\times 2^0=-1\times 1=1$

$a_2=1\times 2^1=1\times 2=2$

$a_3=3\times 2^2=3\times 4=12$

$a_4=5\times 2^3=3\times 8=40$

$a_5=7\times 2^4=7\times 16=112$

$a_6=9\times 2^5=9\times 32=288$

Next two terms are 112, 288

iv. $\mathbf{1,-3,5,-7,9,-11}$

Sol. $1,-3,5,-7,9,-11,13,-13$

Next two terms are $13,-13$

Arithmetic Sequence: **Sargodha 2006, Faisalabad 2009**

A sequence $\{a_n\}$ is an arithmetic sequence or Arithmetic progression if $a_n - a_{n-1}$ is the same number for all $n \in N$ and $n > 1$

Example 3: (6.2) Find the number of terms in A.P if $a_1 = 3, d = 7, a_n = 59$.

Sol : $a_n = a_1 + (n-1)d$

$59 = 3 + (n-1)7$

$59 - 3 = (n-1)7$ **Multan 2007, 2008, 2010**

$56 = (n-1)7 \Rightarrow n-1 = 56/7 = 8 \Rightarrow \boxed{n=9}$

Exercise 6.2

Theorem:

$$a_n = a_1 + (n-1)d$$

$$a_2 = a_1 + d = a_1 + (2-1)d$$

$$a_3 = a_2 + d = a_1 + d + d = a_1 + 2d = a_1 + (3-1)d$$

$$a_4 = a_1 + 3d = a_1 + (4-1)d$$

. . .

. . .

. . .

. . .

$$a_n = a_1 + (n-1)d$$

1. **Write the first four terms of the following arithmetic sequences, if**

i. **$a_1 = 5$ and other three consecutive terms are 23,26,29**

Sol. $a_1 = 5$, *and* $23, 26, 29 \Rightarrow d = 3$

$a_1 = 5$

$a_2 = a_1 + d = 5 + 3 = 8$

$a_3 = a_1 + 2d = 5 + 2(3) = 11$

$a_4 = a_1 + 3d = 5 + 3(3) = 14$

First four terms 5,8,11,14

ii. **$a_5 = 17$ *and* $a_9 = 37$**

Sol. $a_5 = a_1 + 4d = 17 \longrightarrow I,$ $\quad a_9 = a_1 + 8d = 37 \longrightarrow II$

$II - I \Rightarrow$

$$a_1+8d=37$$
$$a_1\pm 4d=17$$
$$4d=20 \quad \boxed{d=5}$$

Now

Put in I

$a_1+4(5)=17$

$a_1+20=17\Rightarrow \boxed{a_1=-3}$

$a_2=a_1+d=-3+5=2$

$a_3=a_1+2d=-3+2(5)$

$a_3=-3+10=7$

$a_4=a_1+3d=-3+3(5)$

$a_4=-3+15=12$

First four terms are -3,2,7,12.

iii. $\boldsymbol{3a_7=7a_4 \text{ and } a_{10}=33}$

Sol. $3(a_1+6d)=7(a_1+3d) \text{ and } a_1+9d=33$ ——— I

$3a_1+18d=7a_1+21d$

Or $7a_1+21d-3a_1-18d=0$

Or $4a_1+3d=0$ ——— II

Solve I & II $\quad$ (I '×' by 4)

$$4a_1+36d=132$$
$$4a_1\pm 3d=0$$
$$33d=132 \quad \boxed{d=4}$$

Put in II

$4a_1+3(4)=0\Rightarrow 4a_1=-12=\boxed{a_1=-3}$

Now $\quad a_1=-3$

$a_2=a_1+d=-3+4=1$

$a_3=a_1+2d=-3+2(4)=-3+8=5$

$a_4=a_1+3d=-3+3(4)=-3+12=9$

First four term are $-3,1,5,9$

2. If $a_{n-3}=2n-5$, find the nth term of the sequence.

Sol. $a_{n-3}=2n-5,\ a_n=?$ Sargodha 2009, Faisalabad 2007, 2008, Rawalpindi 2009

Replace n by n + 3

$a_{n+3-3}=2(n+3)-5$

$a_n=2n+6-5$

$a_n=2n+1$

3. If the 5th term of an A.P. is 16 and the 20th term is 46, what is its 12th term?

Sol. $a_5=16,\ a_{20}=46,\ a_{12}=?$

$a_5=a_1+4d=16$ ——— I, $a_{20}=a_1+19d=46$ ——— II

$a_1+19d=46\quad (II-I)$

$a_1=4d=16$

$-\quad -\quad -$

$15d=30\Rightarrow \boxed{d=2}$

Put $d=2\ in\ I$ $a_1+4(2)=16$

$a_1+8=16\Rightarrow \boxed{a_1=8}$

$a_{12}=a_1+11d$

$a_{12}=8+11(2)=8+22=30$

4. Find the 13th term of the sequence $x,1,2,-x,3-2x,$............

Sol. $a_{13}=?\ \ x,1,2-x,3-2x,$..............

$\boxed{a_1=x}$ $\boxed{d=1-x}$

$a_n=a_1+(n-1)d$

$a_{13}=a_1+12d$

$=x+12(1-x)=x+12-12x$

$\boxed{a_{13}=12-11x}$

5. Find the 18th term of the A.P. if its 6th term is 19 and the 9th term is 31.

Sol. $a_{18}=?\ a_6=19,\ a_9=31$

$a_9=a_1+8d=31$ ——— I

$a_6=a_1+5d=19$ ——— II

$I - II \Rightarrow$

$$a_1 + 8d = 31$$
$$a_1 + 5d = 19$$
$$- \quad - \quad -$$
$$3d = 12 \Rightarrow \boxed{d = 4}$$

Put in II $a_1 + 5(4) = 19 \Rightarrow \boxed{a_1 = -1}$

$$a_{18} = a_1 + 17d$$
$$a_{18} = -1 + 17(4) = -1 + 68 = 67$$

6. Which term of the A.P.5, 2, – 1,....... is –85? **Faisalabad 2008**

Sol. $5, 2, -1, ... \; is \; -85?$ (Which term)

$$a_1 = 5, d = 2 - 5 = -3, a_n = -85$$
$$a_n = a_1 + (n-1)d$$
$$-85 = 5 + (n-1)(-3) \Rightarrow -85 - 5 = (n-1)(-3)$$
$$-90 = (n-1)(-3) \Rightarrow n - 1 = \frac{-90}{-3} = 30$$
$$n = 1 + 30 \Rightarrow n = 31$$

So -85 is 31st term.

7. Which term of the A.P. – 2, 4, 10,....... Is 148? **Sargodha 2011**

Sol. $a_1 = -2, d = 6, a_n = 148, n = ?$

$$a_n = a_1 + (n-1)d$$
$$148 = -2 + (n-1)6 \Rightarrow 148 + 2 = (n-1)6$$
$$150 = (n-1)6$$
$$n - 1 = \frac{150}{6} = 25 \Rightarrow n = 26$$

So 148 is 26th term.

8. How many terms are there in the A.P. in which $a_1 = 11, a_n = 68, d = 3$?

Sol. $a_1 = 11, a_n = 68, d = 3, n = ?$

$$a_n = a_1 + (n-1)d$$
$$68 = 11 + (n-1)3 \Rightarrow 68 - 11 = 3(n-1)$$
$$3(n-1) = 57 \Rightarrow n - 1 = 19 \Rightarrow \boxed{n = 20}$$

9. If the nth term of the A.P. is $3n - 1$, Find the A.P.

Sol. $a_n = 3n - 1$

$a_1 = 3(1) - 1 = 2$

$a_2 = 3(2) - 1 = 5$

$a_3 = 3(3) - 1 = 8$

$a_4 = 3(4) - 1 = 11$

Sequence is 2, 5, 8, 11, ... $3n-1$,

10. **Determine whether**

i. -19

Sol. **Determine -19 is term of 17, 13, 9, ... or 17, 13, 9, ...,** -19 *in A.P*

$a_1 = 17$ $\quad a_n = a_1 + (n-1)d$

$d = -4$ $\quad -19 = 17 + (n-1)(-4) \Rightarrow -19 - 17 = (n-1)(-4)$

$a_n = -19$ $\quad -36 = (n-1)(-4) \Rightarrow n-1 = 9 \Rightarrow \boxed{n=10}$

Yes (-19) is 10th term of the sequence.

ii. **2 is the terms of the A.P. 17,13,9,...... or not.**

Sol. $a_1 = 17, d = -4, a_n = 2$

$a_n = a_1 + (n-1)d$

$2 = 17 + (n-1)(-4) \Rightarrow 2 = 17 - 4n + 4$

$2n = 21 - 4 \Rightarrow n = \dfrac{19}{4}$ Not possible.

2 is Not term of this sequence.

11. **If l, m, n are the pth, qth and rth terms of an A.P., show that**

$(i)\ l(q-r) + m(r-p) + n(p-q) = 0 \qquad (ii)\ p(m-n) + q(n-l) + r(l-m) = 0$

Sargodha 2008, Multan 2009

Sol. **(Method-I)** $l = a_1 + (p-1)d - I$

$m = a_1 + (q-1, \ - II$

$n = a_1 + (r-1)d - III$

$I - II \quad l = a_1 + pd - d \qquad II - III \ m = a_1 + qd - d$

$\quad\ \underline{m} = \underline{a_1} \pm qd \mp d \qquad\qquad \underline{n} = \underline{a_1} \pm rd \mp d$

$(l-m) = (p-1)d$ —— $IV \qquad m - n = (q-r)d$ —— V

Divide IV by V $\dfrac{l-m}{m-n} = \dfrac{(p-q)d}{(q-r)d}$

$(l-m)(q-r) = (p-q)(m-n)$

$lq - lr - mq + mr = pm - pn - qm + qn$ —— VI

$$lq - lr - mq + mr - pm + pn + qm - qn = 0$$

$$\boxed{l(q-r) + m(r-p) + n(p-q) = 0}$$

Shift L.H.S to R.H.S in VI.

$$pm - pn - \cancel{qm} + qn - lq + lr + \cancel{mq} - mr = 0$$

$$\boxed{p(m-n) + q(n-l) + r(l-m) = 0}$$

(Method-II) $l = a_1 + (p-1)d,\ m = a_1 + (q-1)d,\ n = a_1 + (r-1)d$

L.H.S. $= l(p-r) + m(r-p) + n(p-q) = (a_1 + (p-1)d)(q-r) + (a_1 + (q-1)d)$

$(r-p) + (a_1 + (r-1)d)(p-q)$

$= (a_1 + pd - d)(q-r) + (a_1 + qd - d)(r-p) + (a_1 + rd - d)(p-q)$

$= \cancel{a_1q} - \cancel{a_1r} + \cancel{pqd} - \cancel{pdr} - \cancel{dq} + \cancel{dr} + \cancel{a_1r} - \cancel{a_1p} + \cancel{qdr} - \cancel{qdp} - \cancel{dr} + \cancel{dp}$

$+ \cancel{a_1p} - \cancel{a_1q} + \cancel{rdp} - \cancel{rdq} - \cancel{dp} + \cancel{dq}$

= 0 = R.H.S Hence Proved.

12. **Find the nth term of the sequence,** $\left(\frac{4}{3}\right)^2, \left(\frac{7}{3}\right)^2, \left(\frac{10}{3}\right)^2$ **,.........** **Faisalabad 2007**

Sol. $\left(\frac{4}{3}\right)^2, \left(\frac{7}{3}\right)^2, \left(\frac{10}{3}\right)^2, ...a_n = ?$

Take 4, 7, 10, ...,n; $a_n = a_1 + (n-1)d$

$a_1 = 4, d = 3, n = n$

Then $a_n = a_1 + (n-1)d$

$a_n = 4 + (n-1)3$

$a_n = 4 + 3n - 3 \Rightarrow a_n = 3n + 1$

So $\boxed{a_n = \left(\frac{3n+1}{3}\right)^2}$

13. If $\frac{1}{a}, \frac{1}{b}$ *and* $\frac{1}{c}$ **are in A.P., show that** $b = \frac{2ac}{a+c}$ **Faisalabad 2007**

Given $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P

Sol. So $\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}$

Or $\frac{1}{c}+\frac{1}{a}=\frac{1}{b}+\frac{1}{b}$

$\frac{a+c}{ac}=\frac{2}{b}$

$\frac{ac}{a+c}=\frac{b}{2}$ (Take reciprocal) $\Rightarrow \boxed{b=\frac{2ac}{a+c}}$

14. If $\frac{1}{a},\frac{1}{b}$ *and* $\frac{1}{c}$ are in A.P., show that the common difference is $\frac{a-c}{2ac}$

Sol. Given $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in A.P

So $d=\frac{1}{c}-\frac{1}{b}(3rd-2nd)\longrightarrow I$

$d=\frac{1}{b}-\frac{1}{a}(2nd-1st)\longrightarrow II$

$I+II \Rightarrow 2d=\frac{1}{c}-\frac{1}{a}$

$2d=\frac{a-c}{ac}$

$$\boxed{d=Common\ difference=\frac{a-c}{2ac}}$$

Exercise 6.3

Theorem: **A.M:** $= A = \dfrac{a+b}{2}$

Proof: If A is A.M. between two numbers a & b then a, A, b are in A.P then

$$A-a=b-A \Rightarrow A+A=a+b \Rightarrow 2A=a+b \qquad \Rightarrow \boxed{A=\frac{a+b}{2}}$$

1. Find A.M. between

i. $3\sqrt{5}$ **and** $5\sqrt{5}$ **Faisalabad 2008**

Sol. Here $a=3\sqrt{5}$ & $b=5\sqrt{5}$

Then A.M $=\dfrac{a+b}{2}=\dfrac{3\sqrt{5}+5\sqrt{5}}{2} \Rightarrow A.M=\dfrac{8\sqrt{5}}{2}=4\sqrt{5}$

ii. $x-3$ **and** $x+5$

Sol. $a=x-3$ and $b=x+5$ then A.M$=\dfrac{a+b}{2}=\dfrac{x-3+x+5}{2}$

$$A.M=\frac{2x+2}{2}=\frac{2(x+1)}{2}=x+1$$

iii. $1-x+x^2$ **and** $1+x+x^2$ **Sargodha 2008**

Sol. $a=1-x+x^2$ & $b=1+x+x^2$

$$A.M=\frac{a+b}{2}=\frac{1-x+x^2+1+x+x^2}{2}=\frac{2+2x^2}{2}=\frac{2(1+x^2)}{2}=1+x^2$$

2. If 5, 8 are two A.Ms, between a & b, find a and b.

Sol. $a,5,8,b$ **are in A.P** **Sargodha 2010, Lahore 2009, Multan 2010**

$$\Rightarrow 8-5=5-a \Rightarrow 3=5-a \Rightarrow 3-5=-a \Rightarrow \boxed{a=2}$$

$$\& \ b-8=8-5 \Rightarrow b=8+3 \Rightarrow \boxed{b=11}$$

3. Find 6 A.Ms. between 2 and 5.

Sol. Suppose $A_1, A_2, A_3, A_4, A_5, \& A_6$, *are* $6AMs$ *between* $2 \ \& \ 5$

then $2, A_1, A_2, A_3, A_4, A_5, A_6,$ 5 are in **A.P.**

$a_1=2$ —— I & $a_8=a_1+7d=5$ —— II

(Put I in II) $2+7d=5 \Rightarrow 7d=3 \Rightarrow d=\dfrac{3}{7}$

$A_1=a_1+d=2+\dfrac{3}{7}=\dfrac{17}{7}$ $\qquad (A_1 \text{ is } a_2, A_2 \text{ is } a_3 \text{ so on})$

$$A_2 = a_1 + 2d = 2 + 2\left(\frac{3}{7}\right) = 2 + \frac{6}{7} = \frac{14+6}{7} = \frac{20}{7}$$

$$A_3 = a_1 + 3d = 2 + 3\left(\frac{3}{7}\right) = 2 + \frac{9}{7} = \frac{14+9}{7} = \frac{23}{7}$$

$$A_4 = a_1 + 4d = 2 + 4\left(\frac{3}{7}\right) = 2 + \frac{12}{7} = \frac{14+12}{7} = \frac{26}{7}$$

$$A_5 = a_1 + 5d = 2 + 5\left(\frac{3}{7}\right) = 2 + \frac{15}{7} = 2 + \frac{15}{7} = \frac{14+15}{7} = \frac{29}{7}$$

$$A_6 = a_1 + 6d = 2 + 6\left(\frac{3}{7}\right) = 2 + \frac{18}{7} = \frac{14+18}{7} = \frac{32}{7}$$

Hence 6 A.Ms are $\frac{17}{7}, \frac{20}{7}, \frac{23}{7}, \frac{26}{7}, \frac{29}{7}, \frac{32}{7}$

4. Find four A.Ms between $\sqrt{2}$ & $\frac{12}{\sqrt{2}}$ **Sargodha 2011**

Sol. Let $A_1, A_2, A_3, A_4,$ are four A.M between $\sqrt{2}$ and $\frac{12}{\sqrt{2}}$ then

$\sqrt{2}\ , A_1, A_2, A_3, A_4, \frac{12}{\sqrt{2}}$ are in A.P

$$a_1 = \sqrt{2} \text{——} I \quad \& \quad a_6 = a_1 + 5d = \frac{12}{\sqrt{2}} \Rightarrow \sqrt{2} + 5d = \frac{12}{\sqrt{2}}\ value\ of\ a_1$$

$$5d = \frac{12}{\sqrt{2}} - \sqrt{2} = \frac{12-2}{\sqrt{2}} = \frac{10}{\sqrt{2}} \Rightarrow d = \frac{10}{\sqrt{2}} \times \frac{1}{5} = \frac{2}{\sqrt{2}}$$

$$d = \frac{\sqrt{2}\times\sqrt{2}}{\sqrt{2}} = \sqrt{2} \Rightarrow d = \sqrt{2}$$

$$A_1 = a_1 + d = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$$

$$A_2 = a_1 + 2d = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$$

$$A_3 = a_1 + 3d = \sqrt{2} + 3\sqrt{2} = 4\sqrt{2}$$

$$A_4 = a_1 + 4d = \sqrt{2} + 4\sqrt{2} = 5\sqrt{2}$$

Hence 4 A.Ms are $2\sqrt{2}, 3\sqrt{2}, 4\sqrt{2}, 5\sqrt{2}$

5. Insert 7 A.Ms between 4 and 8.

Sol. Let $A_1, A_2, A_3, A_4, A_5, A_6, A_7,$ are 7 A.Ms between 4 & 8.

Then . $4, A_1, A_2, A_3, A_4, A_5, A_6, A_7, 8$ are A.P.

$$a_1 = 4 - I,\ a_9 = a_1 + 8d = 8 \Rightarrow 4 + 8d = 8 \Rightarrow 8d = 8 - 4 = 4$$

$$8d = 4 \Rightarrow d = \frac{4}{8} = \frac{1}{2}$$

$$A_1 = a_1 + d = 4 + \frac{1}{2} = \frac{8+1}{2} = \frac{9}{2}$$

$$A_2 = a_1 + 2d = 4 + 2\left(\frac{1}{2}\right) = 4 + 1 = 5$$

$$A_3 = a_1 + 3d = 4 + 3\left(\frac{1}{2}\right) = 4 + \frac{3}{2} = \frac{8+3}{2} = \frac{11}{2}$$

$$A_4 = a_1 + 4d = 4 + 4\left(\frac{1}{2}\right) = 4 + 2 = 6$$

$$A_5 = a_1 + 5d = 4 + 5\left(\frac{1}{2}\right) = 4 + \frac{5}{2} = \frac{8+5}{2} = \frac{13}{2}$$

$$A_6 = a_1 + 6d = 4 + 6\left(\frac{1}{2}\right) = 4 + 3 = 7$$

$$A_7 = a_1 + 7d = 4 + \frac{7}{2} = \frac{8+7}{2} = \frac{15}{2}$$

Hence 7 A.Ms are $\frac{9}{2}, 5, \frac{11}{2}, 6, \frac{13}{2}, 7, \frac{15}{2}$

6. Find three A.Ms between 3 and 11.

Sol. Let A_1, A_2, A_3 are three AMs between 3 & 11.

Then . $3, A_1, A_2, A_3, 11$ are in A.P.

$$a_1 = 3 \,\&\, a_5 = a_1 + 4d = 11 \Rightarrow 3 + 4d = 11 \Rightarrow 4d = 8 \Rightarrow \boxed{d = 2}$$

$$A_1 = a_1 + d = 3 + 2 = 5$$

$$A_2 = a_1 + 2d = 3 + 2(2) = 3 + 4 = 7$$

$$A_3 = a_1 + 3d = 3 + 3(2) = 3 + 6 = 9$$

$3\ AMs\ are\ 5, 7, 9$

7. Find n so that $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ may be A.M. between a and b. Rawalpindi 2009

Sol. If $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ be A.M between a & b then we have $\frac{a^n + b^n}{a^{n-1} + b^{n-1}} = \frac{a+b}{2}$

$$\Rightarrow 2(a^n+b^n)=(a+b)(a^{n-1}+b^{n-1})$$

$$2a^n+2b^n=aa^{n-1}+ab^{n-1}+ba^{n-1}+bb^{n-1}=a^n+ab^{n-1}+ba^{n-1}+.b^n$$

$$2a^n-a^n+2b^n-b^n=ab^{n-1}+a^{n-1}b$$

$$a^n+b^n=a^{n-1}b+ab^{n-1}\Rightarrow a^n-a^{n-1}b=ab^{n-1}-b^n$$

$$a^{n-1}.a-a^{n-1}b=ab^{n-1}-b^{n-1}.b$$

$$a^{n-1}\cancel{(a-b)}=b^{n-1}\cancel{(a-b)}\Rightarrow a^{n-1}=b^{n-1}\Rightarrow\frac{a^{n-1}}{b^{n-1}}=\frac{b^{n-1}}{b^{n-1}}\ (\div\ by\ b^{n-1})$$

$$\Rightarrow\left(\frac{a}{b}\right)^{n-1}=1=\left(\frac{a}{b}\right)^{0}\Rightarrow n-1=0\Rightarrow n=1$$

8. Show that the sum of n A.Ms between a and b is equal to n times their A.M.

Sol. Let $A_1, A_2, A_3 \ldots\ldots\ldots\ldots\ldots, A_n$ be n A.Ms between a & b.

Then . $a, A_1, A_2, A_3, \ldots\ldots\ldots\ldots, A_n, b$ are in A.P. **Faisalabad 2008, Multan 2008**

Here $a_1=a\ \&\ n=n+2, a_{n+2}=b, d=?$

$$a_n=a_1+(n-1)d\ \ put\ \ n=n+2$$

$$a_{n+2}=a_1+(n+2-1)d=a_1+(n+1)d$$

$$\Rightarrow a_{n+2}=a_1+(n+1)d\Rightarrow b=a_1+(n+1)d\Rightarrow d=\frac{b-a}{n+1}$$

Now $A_1+A_2+A_3\ldots\ldots\ldots\ldots A_n=\frac{n}{2}[A_1+A_n]$

$$=\frac{n}{2}[a_1+d+a_1+nd]$$

$$=\frac{n}{2}[2a_1+(n+1)d]$$

$$=\frac{n}{2}\left[2a_1+(n+1).\frac{(b-a_1)}{(n+1)}\right]$$

$$=\frac{n}{2}[2a_1+b-a_1]$$

$$=\frac{n}{2}[a_1+b]\Rightarrow n\left(\frac{a_1+b}{2}\right)=n\left(\frac{a+b}{2}\right)$$

$$=n(A.M\ between\ \ a\,and\,b)$$

Hence $A_1+A_2\ldots\ldots\ldots\ldots\ldots A_n=n$ (A.M)

Exercise 6.4

1. Find the sum of all the integral multiples of 3 between 4 and 97.

Sol. Integral multiple of 3 between 4 & 97 are is series $6+9+12+15+\ldots\ldots\ldots\ldots+96$

$a_1=6, d=9-6=3, a_n=96, n=?$

$a_n=a_1+(n-1)d \Rightarrow 96=6+(n-1)3 \Rightarrow 96=6+3n-3$

$\Rightarrow 96=3n+3 \Rightarrow 3n=96-3 \Rightarrow 3n=96-3=93 \Rightarrow n=31$

$S_n=\frac{n}{2}(a_1+a_n) \Rightarrow S_{31}=\frac{31}{2}(6+96)=\frac{31}{2}(102)=31(55)=1705$

2. Sum the series **Sargodha 2008**

i. $-3+(-1)+1+3+5+\ldots\ldots\ldots\ldots+a_{16}$

Sol. $a_1=-3, d=-1-(-3)=-1+3=2, n=16$

$S_n=\frac{n}{2}[2a_1+(n-1)d] \Rightarrow S_{16}=\frac{16}{2}[2(-3)+(16-1)2]=8(-6+30)=8(24)=192$

ii. $\frac{3}{\sqrt{2}}+2\sqrt{2}+\frac{5}{\sqrt{2}}+\ldots\ldots\ldots+a_{13}$ **Multan 20088**

Sol. $a_1=\frac{3}{\sqrt{2}}, d=2\sqrt{2}-\frac{3}{\sqrt{2}}=\frac{2(2)-3}{\sqrt{2}}=\frac{4-3}{\sqrt{2}}=\frac{1}{\sqrt{2}}, n=13$

$S_n=\frac{n}{2}[2a_1+(n-1)d] \Rightarrow S_{13}=\frac{13}{2}\left[2\left(\frac{3}{\sqrt{2}}\right)+(13-1)\frac{1}{\sqrt{2}}\right]$

$=\frac{13}{2}\left[\frac{6}{\sqrt{2}}+\frac{12}{\sqrt{2}}\right]=\frac{13}{2}\left[\frac{18}{\sqrt{2}}\right]=\frac{117}{\sqrt{2}}$

iii. $1.11+1.41+1.71+\ldots\ldots\ldots\ldots\ldots\ldots+a_{10}$

Sol. $a_1=1.11, d=1.41-1.11=0.30, n=10$

$S_n=\frac{n}{2}[2a_1+(n-1)d] \Rightarrow S_{10}=\frac{10}{2}[2(1.11)+(10-1)0.30]$

$S_{10}=5(2.22+9(0.30))=5(2.22+27)=5(4.92)=24.60$

iv. $-8-3\frac{1}{2}+1+\ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{11}$ **Multan 2009**

Sol. $or\ -8-\frac{7}{2}+1\ldots\ldots\ldots\ldots\ldots\ldots a_{11}$

$$a_1=-8, d=1-\left(\frac{-7}{2}\right)=1+\frac{7}{2}=\frac{9}{2}, n=11$$

$$S_n=\frac{n}{2}[2a_1+(n-1)d] \Rightarrow S_{11}=\frac{11}{2}\left[2(-8)+(11-1)\frac{9}{2}\right]$$

$$S_{11}=\frac{11}{2}[-16+45]=\frac{11}{2}[29]=\frac{319}{2}=159.5$$

v. $(x-a)+(x+a)+(x+3a)$**..............to n terms.**

Sol. $a_1=(x-a),\ d=x+a-(x-a)=x+a-x+a=2a,\ n=n$

$$S_n=\frac{n}{2}[2a_1+(n-1)d]=\frac{n}{2}[2(x-a)+(n-1)2a]$$

$$=\frac{n}{2}[2x-2a+2na-2a]=\frac{2n}{2}[x-a+na-a]$$

$$=n(x+na-2a)=n(x+(n-2)a)$$

vi. $\frac{1}{1-\sqrt{x}}+\frac{1}{1-x}+\frac{1}{1+\sqrt{x}}+$**.......to n terms.** **Multan 2010**

Sol. $a_1=\frac{1}{1-\sqrt{x}}, d=\frac{1}{1-x}-\frac{1}{1-\sqrt{x}}=\frac{1}{(1-\sqrt{x})(1+\sqrt{x})}-\frac{1}{1-\sqrt{x}}$

$$=\frac{1-(1+\sqrt{x})}{(1-\sqrt{x})(1+\sqrt{x})}=\frac{1-1-\sqrt{x}}{1-x}=\frac{-\sqrt{x}}{1-x}, n=n$$

$$S_n=\frac{n}{2}[2a_1+(n-1)d]=\frac{n}{2}\left[2\left(\frac{1}{1-\sqrt{x}}\right)+(n-1)\left(\frac{-\sqrt{x}}{1-x}\right)\right]$$

$$=\frac{n}{2}\left[\frac{2}{1-\sqrt{x}}-\frac{(n-1)\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right]=\frac{n}{2}\left[\frac{2(1+\sqrt{x})-(n-1)\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right]$$

$$=\frac{n}{2}\left[\frac{2+2\sqrt{x}-n\sqrt{x}+\sqrt{x}}{1-x}\right]=\frac{n}{2}\left[\frac{2+3\sqrt{x}-n\sqrt{x}}{1-x}\right]$$

$$=\frac{n}{2}\left[\frac{2+(3-n)\sqrt{x}}{1+x}\right]$$

vii. $\frac{1}{1+\sqrt{x}}+\frac{1}{1-x}+\frac{1}{1-\sqrt{x}}+$**.......to n terms.**

Sol.

3. How many terms of the series

i. $-7+(-5)+(-3)+$ ………… **amount to 65?**

Sol. $a_1=-7, d=-5-(-7)=-5+7=2,\ n=?\ S_n=65$

$$S_n=\frac{n}{2}[2a_1+(n-1)d]\Rightarrow 65=\frac{n}{2}[2(-7)+(n-1)2]$$

$$\Rightarrow 65=\frac{n}{2}[-14+2n-2]\Rightarrow 65=\frac{n}{2}[2n-16]\Rightarrow 65=n(n-8)$$

$$\Rightarrow 65=n^2-8n\Rightarrow n^2-8n-65=0\Rightarrow n^2-13n+5n-65=0$$
$$n(n-13)+5(n-13)=0\Rightarrow(n-13)(n+5)\Rightarrow n-13=0\ or\ n+5=0$$

$\Rightarrow n=13 \quad or \quad n=-5$ (Not Possible) Hence $\boxed{n=13}$

ii. $-7+(-4)+(-1)+$ ………… **amount to 114?**

Sol. $a_1=-7, d=-4-(-7)=-4+7=3,\ n=?\ S_n=114$

$$S_n=\frac{n}{2}[2a_1+(n-1)d]\Rightarrow 114=\frac{n}{2}[2(-7)+(n-1)3]$$

$$\Rightarrow 114=\frac{n}{2}[-14+3n-3]\Rightarrow 228=n(3n-17)$$

$$\Rightarrow 228=3n^2-17n\Rightarrow 3n^2-17n-228=0$$
$$\Rightarrow 3n^2-36n+19n-228=0\Rightarrow 3n(n-12)+19(n-12)=0$$
$$\Rightarrow(n-12)(3n+19)=0\Rightarrow n-12=0\ or\ 3n+19=0$$

$\Rightarrow n=12 \quad or \quad n=-\frac{19}{3}$ (Not Possible) Hence $\boxed{n=12}$

4. Sum the Series

i. $3+5-7+9+13+15+17-19+$ ……. **to 3n terms?** **Lahore 2009**

Sol. **By adding three terms we get.**

$1+7+13+$ ………….. to n terms

$a_1=1, d=7-1=6, n=n$

$$S_n=\frac{n}{2}[2a_1+(n-1)d]=\frac{n}{2}[2(1)+(n-1)6]=\frac{n}{2}[2+6n-6]$$

$$=\frac{n}{2}[6n-4]=n(3n-2)$$

ii. $1+4-7+10+13-16+19+22-25+$ …….. **to 3n term?**

Sol. **Adding three terms we have**

$-2+7+16+$ ………… to n terms

$a_1=-2, d=7-(-2)=7+2=9, n=n$

$$S_n = \frac{n}{2}[2a_1 + (n-1)d] = \frac{n}{2}[2(-2) + (n-1)9]$$

$$= \frac{n}{2}[-4 + 9n - 9] = \frac{n}{2}[9n - 13]$$

5. **Find the sum of 20 terms of the series whose rth term is 3r+1**

Sol. $a_r = 3r + 1, S_{20} = ?$

Put $r = 1, 2, 3, 4, ...$

$a_1 = 3(1) + 1 = 3 + 1 = 4$

$a_2 = 3(2) + 1 = 6 + 1 = 7$

$a_3 = 3(3) + 1 = 1 + 9 = 10$

$a_4 = 3(4) + 1 = 12 + 1 = 13$

6. **If $S_n = n(2n-1)$, then find the series.** Multan 2007

Sol. $S_n = n(2n-1)$

Put $n = 1, 2, 3, 4, ...$

$S_1 = a_1 = 1(2(1) - 1) = 1(2-1) = 1 \Rightarrow \boxed{a_1 = 1}$

$S_2 = a_1 + a_2 = 2(2(2) - 1)$

$or\ a_1 + a_2 = 2(4-1) = 2(3)$

$or\ 1 + a_2 = 6 = \boxed{a_2 = 5}$

$S_3 = a_1 + a_2 + a_3 = 3(2(3) - 1) = 3(6-1)$

$or\ 1 + 5 + a_3 = 3(5) \Rightarrow 6 + a_3 = 15 \Rightarrow a_3 = 15 - 6 = 9$

Required Series is

$1 + 5 + 9 + ...$

7. **The Ratio of the sums of n terms of two series in A.P. is $3n+2 : n+1$. Find the ratio of their 8^th terms.**

Sol. $S_n = \frac{n}{2}[2a + (n-1)d] \,\&\, S'n = \frac{n}{2}[2a' + (n-1)d']$

According to the given condition

$$S_n : S'_n = 3n + 2 : n + 1 \Rightarrow \frac{S_n}{S'_n} = \frac{3n+2}{n+1}$$

$$\frac{\frac{n}{2}[2a+(n-1)d]}{\frac{n}{2}[2a'+(n-1)d']}=\frac{3n+2}{n+1}$$

Dividing numerator and denominator on R.H.S by 2.

$$\frac{a+\left(\frac{n-1}{2}\right)d}{a'+\left(\frac{n-1}{2}\right)d'}=\frac{3n+2}{n+1} \text{ ——} I \quad \textit{Compare with } a+7d \textit{ with } a+\left(\frac{n-1}{2}\right)d$$

$$\Rightarrow \frac{n-1}{2}=7 \Rightarrow n-1=14 \Rightarrow n=15$$

$$\textit{Put } n=15 \textit{ in } I \frac{a+\left(\frac{15-1}{2}\right)d}{a'+\left(\frac{15-1}{2}\right)d'}=\frac{3(15)+2}{15+1}$$

$$\Rightarrow \frac{a+\frac{14}{2}d}{a'+\frac{14}{2}d'}=\frac{45+2}{15+1} \Rightarrow \frac{a+7d}{a'+7d'}=\frac{47}{16} \Rightarrow \frac{a_8}{a_8'}=\frac{47}{16} \Rightarrow a_8 : a_8' = 47:16$$

Hence ratio of 8th term is 47 : 16

8. If S_2, S_3, S_5, are the sums of 2n, 3n, 5n, terms of an A.P. show that $S_5=5(S_3-S_2)$.

Sol. $S_2=\frac{2n}{2}[2a_1+(2n-1)d]$ **Federal**

$$S_3=\frac{3n}{2}[2a_1+(3n-1)d]$$

$$S_5=\frac{5n}{2}[2a_1+(5n-1)d]$$

Now $S_3-S_2=\frac{3n}{2}[2a_1+(3n-1)d]-\frac{2n}{2}[2a_1+(2n-1)d]$

$$=\frac{n}{2}\left[3\{2a_1+(3n-1)d\}-2\{2a_1+(2n-1)d\}\right]$$

$$=\frac{n}{2}[3(2a_1+3nd-d)-2(2a_1+2nd-d)]$$

$$=\frac{n}{2}[6a_1+9nd-3d-4a_1-4nd+2d]$$

$$=\frac{n}{2}[2a_1+5nd-d]=\frac{n}{2}[2a_1+(5n-1)d]$$

'x' by 5

$$5(S_3-S_2)=\frac{5n}{2}[2a+(5n-1)d]$$

$$=S_5 \text{ Hence } S_5=5(S_3-S_2)$$

9. Obtain the sum of all integers in the first 1000 integers which are neigher divisible by 5 nor by 2.

Sol. First thousand (1000) integers which are neither divisible by 5 nor by 2 are

$$1+3+7+9+11+13+17+19+21+23+27+29+.....+991+993+1997+9999$$

Adding four, four numbers $20+60+100+........+3980$

To find n, $a_n=a_1+(n-1)d$

$$3980=20+(n-1)40$$

$$n-1=\frac{3980-20}{40}=99 \Rightarrow n-1=99 \Rightarrow n=100$$

$$a_1=20,\ d=60-20=40,\ n=100$$

$$S_n=\frac{n}{2}(a_1+a_2) \Rightarrow S_{100}=\frac{100}{2}[20+3980]$$

$$\Rightarrow S_{100}=50[20+3980]=50(4000)=200000$$

10. S_8 and S_9 are the sums of the first eight and nine terms of an A.P., find S, if $50S_9=63S_8$ and $a_1=2$

Sol. $50S_9=63S_8$ $\qquad S_9=?$

$$50.\frac{9}{2}[2a_1+(9-1)d]=63.\frac{8}{2}[2a_1+(8-1)d] \,\&\, a_1=2$$

Put value of a_1 in this equation

$$\cancel{50}^{25}.\frac{9}{\cancel{2}}[2(2)+8d]=63.\frac{\cancel{8}^{4}}{\cancel{2}}[2(2)+7d]$$

$$225(4+8d)=252(4+7d)$$

$$\Rightarrow 900+1800d=1008+1764d \Rightarrow 1800d-1764d=1008-900$$

$$\Rightarrow 36d = 108 \Rightarrow d = \frac{108}{36} \Rightarrow d = 3$$

$$S_n = \frac{n}{2}[2a+(n-1)d] \; Put \; a = 2, d = 3, n = 9$$

$$S_9 = \frac{9}{2}[2(2)+(9-1)3] \Rightarrow S_9 = \frac{9}{2}[4+24] = \frac{9}{2}(28) = 9(14) \Rightarrow S_9 = 126$$

11. The sum of 9 terms of an A.P. is 171 and its eighth term is 31. Find the series.

Sol. $S_9 = 171 \; \& \; a_8 = a_1 + 7d = 31$ **Multan 2008**

$$\Rightarrow S_9 = \frac{9}{2}[2a_1 + 8d] = 171 \; \& \; a_8 = a_1 + 7d = 31 \longrightarrow I$$

$$\Rightarrow S_9 = 9(a_1 + 4d) = 171 \Rightarrow 9a_1 + 36d = 171 \longrightarrow II$$

'×' I by 9 we get $9a_1 + 63d = 279 \longrightarrow III$

$III - II$ $\qquad$ $Put \; d = 4 \; in \; I$

$$9a_1 + 63d = 279$$

$$\underline{9a_1 \pm 36d = 171}$$

$$27d = 108 \Rightarrow \boxed{d = 4}$$

$$a_1 + 7(4) = 31 \Rightarrow a_1 + 28 = 31$$

$$a_1 = 31 - 28 \Rightarrow \boxed{a = 3}$$

$a_1 = 3,$ $\qquad a_2 = a_1 + d = 3 + 4 = 7$, $\qquad a_3 = a_1 + 2d$

$$= 3 + 2(4) = 3 + 8 = 11$$

So series is 3+7+11+............

12. The sums of S_7, and S_9 is 203 and $S_9 - S_7 = 49$, S_7 *and* S_9 being the sums of the first 7 and 9 terms of an A.P. respectively. Determine the series.

Sol. Given $S_9 + S_7 = 203 \longrightarrow I \; \& \; S_9 + S_7 = 49 \longrightarrow II$ Series = ?

We know that $S_9 = \frac{9}{2}[2a_1 + 8d] \; \& \; S_7 = \frac{7}{2}[2a_1 + 6d]$

Put value in I.

$$\frac{9}{2}[2a_1 + 8d] + \frac{7}{2}[2a_1 + 6d] = 203$$

$$\Rightarrow 9(a_1 + 4d) + 7(a_1 + 3d) = 203 \Rightarrow 9a_1 + 36d + 7a_1 + 21d = 203$$

$$\Rightarrow 16a_1 + 57d = 203 \longrightarrow III$$

Now solving II $\qquad \frac{9}{2}[2a_1 + 8d] - \frac{7}{2}[2a_1 + 6_d] = 49$

$\Rightarrow 9(a_1+4d)-7(a_1+3d)=49 \Rightarrow 9a_1+36d-7a_1-21d=49$

$\Rightarrow 2a_1+15d=49 \longrightarrow IV$

'×'by 8 $16a_1+120d=392 \longrightarrow V$

$V-III \Rightarrow$

$\cancel{16a_1}+120d=392$

$\underline{\cancel{16a_1} \pm 57 = 203}$

$63d=189 \Rightarrow \boxed{d=3}$

Put in IV $2a_1+15(3)=49$

$\Rightarrow 2a_1+45=49 \Rightarrow 2a_1=49-45=4\ \boxed{a_1=2}$

$a_1=2, a_2=a_1+d=2+3=5$

$a_3=a_1+2d=2+2(3)=2+6=8$

So series is 2 +5+8+............

13. **S_7 and S_9 are the sums of the first 7 and 9 terms of an A.P. respectively. If $\dfrac{S_9}{S_7}=\dfrac{18}{11}$ *and* $a_7=20$, Find the series.**

Sol. Given $\dfrac{S_9}{S_7}=\dfrac{18}{11} \Rightarrow \dfrac{\frac{9}{2}[2a_1+8d]}{\frac{7}{2}[2a_1+6d]}=\dfrac{18}{11}$

and $a_1=a_1+6d=20 \longrightarrow I$

and $3a_1+18d=60 \longrightarrow II$

$\Rightarrow \dfrac{9(a_1+4d)}{7(a_1+3d)}=\dfrac{18}{11} \Rightarrow \dfrac{9a_1+36d}{7a_1+21d}=\dfrac{18}{11}$

$\Rightarrow 11(9a_1+36d)=18(7a_1+21d)$

$\Rightarrow 99a_1+396d=126a_1+378d$

$\Rightarrow 396d-378d=126a_1-99a_1$

$18d=27a_1 \Rightarrow 27a_1-18d=0 \longrightarrow III$

Adding II & III

$27a_1-\cancel{18d}=0$

$\underline{3a_1+\cancel{18d}=60}$

Put value of a_1 in I $30a_1 = 60 \Rightarrow \boxed{a_1 = 2}$

$2 + 6d = 20 \Rightarrow 6d = 18 \Rightarrow \boxed{d = 3}$

$a_1 = 2, \quad a_2 = a_1 + d = 2 + 3 = 5$

$a_3 = a_1 + 2d = 2 + 2(3) = 2 + 6 = 8$

Hence $2 + 5 + 8 + ...$ is required series .

14. The Sum of three numbers in an A.P. is 24 and their product is 440. Find the numbers.

Sol. Suppose the numbers are $a_1 - d,\ a_1, a_1 + d$ in **A.P.**

Then $a_1 - d + a_1 + a_1 + d = 24 \Rightarrow 3a_1 = 24 \Rightarrow \boxed{a_1 = 8}$

& $(a_1 - d)(a_1)(a_1 + d) = 440 \Rightarrow a_1(a_1^2 - d^2) = 440$

$\Rightarrow 8(64 - d^2) = 440 \Rightarrow 64 - d^2 = 55 \Rightarrow -d^2 = 55 - 64 = -9$

$\Rightarrow d^2 = 9 \Rightarrow d = \pm 3$

When $d = 3$ then

$a_1 - d = 8 - (-3) = 8 + 3 = 11, a_1 = 8, a_1 + d = 8 + (-3) = 8 - 3 = 5$

Hence $5, 8, 11$ When $d = 3$ and $a_1 - d = 8 - 3 = 5, a_1 = 8, \quad a_1 + d = 8 + 3 = 11$

15. Find four numbers in A.P. whose sum is 32 and the sum of whose squares is 276.

Sol. Suppose four numbers in A.P. are $a_1 - 3d,\ a_1 - d, a_1 + d, a_1 + 3d$ in **A.P.**

I condition $\Rightarrow a_1 - 3d + a_1 - 3d + a_1 + d + a_1 + 3d = 32$

$\Rightarrow 4a_1 = 32 \Rightarrow \boxed{a_1 = 8}$ **Multan 2010, Faisalabad 2009**

II condition $\Rightarrow (a_1 - 3d)^2 + (a_1 - d)^2 + (a_1 + d)^2 + (a_1 + 3d)^2 = 276$

$a_1 - \cancel{6a_1d} + 9d^2 + a_1^{\ 2} - \cancel{2a_1d} + d^2 + a_1^{\ 2} + \cancel{2a_1d} + d^2 + a_1^{\ 2} + \cancel{6a_1d} + d^2 = 276$

$4a_1^2 + 20d^2 = 276 '\div' \ by\ 4 \Rightarrow a_1^2 + 5d^2 = 69$

Put value of $a_1 \Rightarrow 8^2 + 5d^2 = 69 \Rightarrow 64 + 5d^2 = 69$

$\Rightarrow 5d^2 = 69 - 64 = 5 \Rightarrow d^2 = 1 \Rightarrow d = \pm 1$

When $d = 1$ then

$a_1 - 3d = 8 - 3(1) = 8 - 3 = 5; \ a_1 - d = 8 - 1 = 7$

$a_1 + d = 8 + 1 = 9 \qquad , \qquad a_1 + 3d = 8 + 3(1) = 8 + 3 = 11$

When $d = -1$ then

$a_1 - 3d = 8 - 3(-1) = 8 + 3 = 11, \ a_1 - d = 8 - (-1) = 9$

$a_1 + d = 8 + (-1) = 8 - 1 = 7 \qquad a_1 + 3d = 8 + 3(-1) = 8 - 3 = 5$

Hence numbers are 5, 7,9,11

16. Find the five numbers in A.P. whose sum is 25 and the sum of whose squares is 135.

Sol. Suppose five numbers in A.P. are $a_1-2d,\ a_1-d, a_1, a_1+d, a_1+2d$ in **A.P.**

I condition $\Rightarrow a_1-2d+a_1-d+a_1+a_1+d+a_1+2d=25$ **Multan 2010**

$$\Rightarrow 5a_1=25 \Rightarrow \boxed{a_1=5}$$

II condition $\Rightarrow (a_1-2d)^2+(a_1-d)^2+a_1^2+(a_1+d)^2+(a_1+2d)^2=135$

$$\Rightarrow a_1-4a_1d+4d^2+a_1^2-2a_1d+d^2+a_1^2+2a_1d+d^2+a_1^2+4a_1d+4d^2=135$$

$$5a_1^2+10d^2=135 \Rightarrow 5(5)^2+10d^2=135$$

$$\Rightarrow 125+10d^2=135 \Rightarrow 10d^2=135-125=10 \Rightarrow d^2=1$$

$$\Rightarrow d=\pm 1$$

When $d=1$ then

$a_1-2d=5-2(1)=5-2=3$

$a_1-d=5-1 \quad =3$

$a_1=5$

$a_1+d=5+1 \quad =6$

$a_1+2d=5+2(1)=5+2=7$

When $d=-1$ then

$a_1-2d=5-2(-1)=5+2=7$

$a_1-d=5-(-1)=5+1 \quad =6$

$a_1 \quad =5$

$a_1+d=5+(-1)=5-1 \quad =4$

$a_1+2d=5+2(-1)=5-2=3$

Hence Fine numbers are 3, 4, 5, 6, 7.

17. The sum of the 6th and 8th terms of an A.P. is 40 and the product of 4th and 7th terms is 220. Find the A.P.

Sol. Given $a_6+a_8=40 \longrightarrow I \ \&\ (a_4)(a_7)=220 \longrightarrow II$

$$I \Rightarrow a_1+5d+a_1+7d=40 \Rightarrow 2a_1+12d=40('\div'\ by\ 2)\ a_1+6d=20 \longrightarrow III$$

$$II \Rightarrow (a_1+3d)(a_1+6d)=220$$

$$(use\ II) \Rightarrow (a_1+3d)(20)=220 \Rightarrow a_1+3d=\frac{220}{20}=11$$

$$\Rightarrow a_1+3d=11 \longrightarrow IV$$

$III - IV$

$$\cancel{a_1} + 6d = 20$$
$$\underline{-\cancel{a_1} \pm 3d = -11}$$

$$3d = 9 \Rightarrow \boxed{d = 3}$$

Put in III

$$a_1 + 6(3) = 20$$
$$a_1 + 18 = 20$$
$$a_1 = 20 - 18 = 2 \Rightarrow \boxed{a_1 = 2}$$

$$a_1 = 2,\ a_2 = a_1 + d = 2 + 3 = 5$$
$$a_3 = a_1 + 2d = 2 + 2(3) = 2 + 6 = 8$$

Hence A.P. is 2, 5, 8,

18. **If a^2, b^2 and c^2 are in A.P., show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.**

Sol. *If a^2, b^2, c^2 are in A.P then*

$$c^2 - b^2 = b^2 - a^2 \longrightarrow I$$

Now If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P then

$$\frac{1}{a+b} - \frac{1}{c+a} = \frac{1}{c+a} - \frac{1}{b+c} \Rightarrow \frac{c+\cancel{a}-\cancel{a}-b}{\cancel{(c+a)}(a+b)} = \frac{b+\cancel{c}-\cancel{c}-a}{\cancel{(c+a)}(b+c)}$$

$$\Rightarrow \frac{c-b}{a+b} = \frac{b-a}{b+c} \Rightarrow (b+c)(c-b) = (b-a)(b+a)$$

$$\Rightarrow c^2 - b^2 = b^2 - a^2$$

$$\Rightarrow c^2 - b^2 = c^2 - b^2 \longrightarrow use\ I\ (Common\ difference\ is\ same)$$

Hence Proved that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P

Exercise 6.5

1. A man deposits in a bank Rs.10 in the first month; Rs.15 in the second month; Rs.20 in the third month and so on. Find how much he will have deposited in the bank by 9th month.

Sol. Given $10+15+20+..........+a_9$

$a_1=10, d=15-10=5, n=9$

$$S_n=\frac{n}{2}[2a_1+(n-1)d]$$

$$S_9=\frac{9}{2}[2(10)+(9-1)5]$$

$$=\frac{9}{2}[20+8(5)]=\frac{9}{2}[20+40]=\frac{9}{2}(60)=9(30)=270$$

2. 378 trees are planted in rows in the shapes of an isosceles triangle, the numbers in successive rows decreasing by one from the base to the top. How many trees are there in the row which forms the base of the triangle?

Sol. In first row we have 1 tree in second 2 trees in third 3 and so on, so we have

$1+2+3................a_n=378$

Here $S_n=378,\ a_1=1,\ d=2-1=1, n=?$

$$S_n=\frac{n}{2}[2a_1+(n-1)d]$$

$$378=\frac{n}{2}[2(1)+(n-1)1]$$

$$378=\frac{n}{2}(2+n-1)\Rightarrow 2\times 378=n(n+1)$$

$$\Rightarrow n^2+n=756\Rightarrow n^2+n-756=0$$

$$n^2+28n-27n-756=0\Rightarrow n(n+28)-27(n+28)=0$$

$$(n+28)(n-27)=0$$

$$n+28=0 \ \ or \ \ n-27=0$$

$n=-28$ not possible or $n=27$

So $n=27$

Hence in isosceles triangle total rows are 27.

In first row we have 1 tree in second 2 So on in 27 row we have 27 trees.

3. A man borrows Rs.1100 and agree to repay with a total interest of Rs.230 in 14 installments, each installment being less than the preceding by Rs.10. What should be his first installment?

Sol. Total amount to repay = 1100 + 230 = 1330

So $S_n = 1330, n = 14, d = -10, a_1 = ?$

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

$$1330 = \frac{14}{2}[2a_1 + (14-1)(-10)]$$

$$1330 = 7[2a_1 + (13)(-10)]$$

$$1330 = 7[2a_1 - 130]$$

$$\frac{1330}{7} = 2a_1 - 130 \Rightarrow 2a_1 - 130 = 190 \Rightarrow a_1 = \frac{320}{2} = 160 \Rightarrow a_1 = 160$$

So first installment = 160

4. **A clock strikes once when its hour hand is at one, twice when it is at two and so on. How many times does the clock strike in twelve hours?**

Sol. According to the statement $1 + 2 + 3 + \ldots\ldots\ldots\ldots + a_{12} = ?$

$a = 1, d = 2 - 1 = 1, n = 12, S_n = ?$

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

$$S_{12} = \frac{12}{2}[2(1) + (12-1)1] = 6[2 + 11] = 6(13) = 78$$

5. **A student save Rs.12 at the end of the first week and goes on increasing his saving Rs.4 weekly. After how many weeks will he be able to save Rs.2100?**

Sol. In 1st week = 12
In 2nd week = 12 + 4=16
In 3rd week = 16 + 4 =20
So series is $12 + 16 + 20 + \ldots\ldots\ldots\ldots a_n = 2100$

$a_1 = 12, d = 16 - 12 = 4, n = ?, S_n = 2100$

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

$$2100 = \frac{n}{2}[2(12) + (n-1)4] \Rightarrow 2100 = \frac{n}{2}[24 + 4n - 4]$$

$$2100 = \frac{n}{2}[4n + 20]$$

$$2100 = n[2n + 10]$$

$$\Rightarrow 2100 = 2n^2 - 10n$$

$$\Rightarrow 2n^2 + 10n - 2100 = 0 \text{ '}\div\text{' by } 2$$

$n^2+5n-1050=0 \Rightarrow n^2+35n-30n-1050=0$

$n(n+35)-30(n+35)=0$

$(n+35)(n-30)=0$

$n+35=0$ *or* $n-30=0$

$n=-35$ not possible or $n=30$

He will have 2100 in 30 weeks.

6. **An object falling from rest, falls 9 meter during the first second, 27 meter during the next second, 45 meter during the third second and so on.**
   (i) **How far will it fall during the fifth second?**
   (ii) **How far will it fall up to the fifth second?**

Sol. Given $9, 27, 45 + \ldots\ldots\ldots$

For (i) $a_1 = 9, d = 27-9 = 18, n = 5$

$a_n = a_1 + (n-1)d$

$a_5 = 9 + (5-1)18 = 9 + 4(18)$

$= 9 + 72 = 81\, meters$

For (ii) $a_1 = 9, d = 18, n = 5, S_n = ?$

$$S_n = \frac{5}{2}[2a_1 + (n-1)d]$$

$$S_5 = \frac{5}{2}[2(9) + (5-1)18] = \frac{5}{2}[18 + 4(18)]$$

$$S_5 = \frac{5}{2}[18+72] = \frac{5}{2}[90] = 5(45) = 225\, meters$$

7. **An investor earned Rs.6000 for year 1980 and Rs.12000 for year 1990 on the same investment. If his earning have increased by the same amount each year, how much income he has received from the investment over the past eleven years?**

Sol. $a_1 = 600,\ n = 11\ ,\ a_n = 12000,\ S_n = ?$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{11} = \frac{11}{2}(6000+12000) = \frac{11}{2}(18000) = 99000$$

8. **The sum of interior angles of polygons having sides 3, 4, 5, ... Etc form an A.P. Find the sum of the interior angles for a 16 sides polygon.**

Sol. Sum of angle of 3 sides Polygon $= \pi$

Sum of angle of 4 sides Polygon $= 2\pi$

Sum of angle of 5 sides Polygon $= 3\pi$

A.P. is $\pi, 2\pi, 3\pi \ldots\ldots\ldots\ldots\ldots a_{14}$

$a_n = a + (n-1)d \quad (a_1 = \pi, d = 2\pi - \pi = \pi, n = 14)$

$a_{14} = \pi + (14-1)\pi$

$= \pi + 13\pi = 14\pi$

9. The prize money Rs.60,000 will be distributed among the eight teams according to their positions determined in the match series. The award increases by the same amount for each higher position. If the last place than is given Rs.4000, how much will be awarded to the first place team?

Sol. Given $S_n = 60,000$

$n = 8,\ a_8 = 4000,\ a_1 = ?$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$60,000 = \frac{8}{2}(a_1 + 4000) \Rightarrow 60,000 = 4(a_1 + 4000)$$

$$15000 = a_1 + 4000 \Rightarrow \boxed{a_1 = 11000}$$

10. An equilateral triangular base is filled by placing eight balls in the first row, 7 balls in the second row and so on with one ball in the last row. After this base layer, second layer is formed by placing 7 balls in its first row, 6 balls in its second row and so on with one ball in its last row. Continuing this process a pyramid of balls is formed with one ball on top. How many balls are there in the pyramid?

Sol. Let $S_1, S_2, \ldots\ldots S_8$ denote sum of $1, 2, \ldots\ldots\ldots\ldots 8$ layer so

$S_8 = 8 + 7 + 6 + \ldots\ldots\ldots\ldots + 1$

$= \frac{n}{2}(a_1 + a_n) = \frac{8}{2}(8+1) = 36$

$S_7 = 7 + 6 + 5 + \ldots\ldots\ldots\ldots + 1$

$= \frac{7}{2}(7+1) = \frac{7}{2}(8) = 28$

$S_6 = 6 + 5 + \ldots\ldots\ldots\ldots + 1$

$S_6 = \frac{6}{2}(6+1) = 3(7) = 21$

$S_5 = 5 + 4 + 3 + 2 + 1 = 15$

$S_4 = 4 + 3 + 2 + 1 = 10$

$S_3 = 3 + 2 + 1 = 6$

$S_2 = 2 + 1 = 3$

$S_1 = 1 = 1$

Total balls $= 36 + 28 + 21 + 15 + 10 + 6 + 3 + 1 = 128$

Exercise 6.6

Geometric Progression (G.P):

A sequence $\{a_n\}$ is geometric sequence or geometric progression

if $\dfrac{a_n}{a_{n-1}}$ is the same non zero number for all $n \in N$ and $n > 1$.

Theorem:

$$a_n = a_1 r^{n-1}$$

Proof:

We have geometric sequence $a_1,\ a_1r, a_1r^2$ ....Where

$$a_1 = a_1 = a_1r^{1-1}$$

$$a_2 = a_1r = a_1r^{2-1}$$

$$a_3 = a_1r^2 = a_1r^{3-1}$$

$$\vdots \quad \vdots \quad \vdots$$

$$a_n = a_1r^{n-1}$$

1. **Find the 5th term of the G.P: 3, 6, 12, ... Sgd 2008, Fsd 2007, Multan 2008**

Sol. $3, 6, 12, \ldots\ldots\ldots\ldots a_5 = ?$

$$a_1 = 3, r = \frac{6}{3} = 2,\ n = 5$$

$$a_n = a_1r^{n-1}$$

$$a_5 = 3(2)^{5-1} = 3(2)^4 = 3(16) = 48$$

2. **Find the 11th term of the sequence, $1+i,\ 2, \dfrac{4}{1+i}$**

Sol. $1+i,\ 2, \dfrac{4}{1+i} \ldots\ldots\ldots\ldots, a_{11} = ?$ **Sargodha 2008, 2010 Multan 2009**

$a_1 = 1+i, \qquad r = \dfrac{2}{1+i}, n = 11$

Note: $\boxed{\begin{aligned}&(1+i)^2 = 1^2 + i^2 + 2i\\ &= 1 - 1 + 2i = 2i\end{aligned}}$

$$a_n = a_1r^{n-1}$$

$$a_{11} = (1+i)\left(\frac{2}{1+i}\right)^{11-1} = (1+i)\left(\frac{2}{1+i}\right)^{10} = (1+i)\frac{2^{10}}{(1+i)^{10}} = (1+i).\frac{1024}{\left[(1+i)^2\right]^5} = (1+i)\frac{1024}{(2i)^5}.$$

$$=(1+i).\frac{1024}{32i^5}=(1+i)\frac{32}{i^2.i^2.i}=(1+i)\frac{32}{(-1)(-1)i}$$

$$=(1+i).32(-i)=32(-i-i^2)=32(-i+1)=32(1-i)$$

3. Find the 12^{th} term of $1+i, 2i, -2+2i, \ldots\ldots$ Sargodha 2011

Sol. $1+i,\ 2i, -2+2i, \ldots\ldots a_{12}=?$

$$a_1=1+i, \qquad r=\frac{2i}{1+i}, n=12$$

$$a_n=a_1r^{n-1}$$

$$a_{12}=(1+i)\left(\frac{2i}{1+i}\right)^{12-1}$$

$$a_{12}=(1+i)\frac{(2i)^{11}}{(1+i)^{11}}$$

$$=\frac{2^{11}\times i^{11}}{(1+i)^{10}}=\frac{2048\times i\times(i^2)^5}{\left[(1+i)^2\right]^5}=\frac{2048\times i\times(-1)^5}{(2i)^5}=\frac{-2048i}{32i^5}=\frac{-64i}{i^2.i^2.i}=\frac{-64i}{(-1)(-1)i}$$

$$=-64$$

4. Find the 11^{th} term of the sequence $1+i, 2, 2(1-i)$

Sol. $1+i,\ 2, (2-i), \ldots\ldots a_{11}=?$

$$a_1=1+i, \qquad r=\frac{2}{1+i}, n=11$$

$$a_n=a_1r^{n-1}$$

$$a_{11}=(1+i)\left(\frac{2}{1+i}\right)^{11-1}=(1+i)\left(\frac{2^{10}}{(1+i)^{10}}\right)=(1+i)\frac{1024}{\left[(1+i)^2\right]^5}=(1+i)\left(\frac{1024}{(2i)^5}\right)=(1+i)\frac{1024}{32i}$$

$$=(1+i)(32(-i))=32(-i-i^2)=32(-i+1)=32(1-i)$$

5. If an automobile depreciates in values 5% every year, at the end of 4 years what is the value of the automobile purchased for Rs.12,000?

Sol. $r=1-5\%=1-\frac{5}{100}=1-0.05=0.095$

$$a_1=12000, n=5$$

$$a_n=a_1r^{n-1}$$

$$a_5=(12000)(0.95)^{5-1}$$

$=(12000)(0.95)^4$
$=(12000)(0.8145)=9774\ Rs.$

6. **Which term of the sequence:** $x^2-y^2, x+y, \dfrac{x+y}{x-y}, \ldots\ldots\ldots$ **is** $\dfrac{x+y}{(x-y)^9}$**?**

Sol. $x^2-y^2, x+y, \dfrac{x+y}{x-y}, \ldots\ldots\ldots, \dfrac{x+y}{(x-y)^9}$

Federal

$a_1=x^2-y^2 \quad , \quad a_n=\dfrac{x+y}{(x-y)^9}$

$r=\dfrac{x+y}{x^2-y^2}=\dfrac{x+y}{(x-y)(x+y)}=\dfrac{1}{x-y}, n=?$

$a_n=a_1r^{n-1}$

$$\frac{x+y}{(x-y)^9}=(x^2-y^2)\left(\frac{1}{x-y}\right)^{n-1} \Rightarrow \frac{x+y}{(x-y)^9}=(x+y)(x-y).\frac{1}{(x-y)^{n-1}}$$

$$\frac{x+y}{(x-y)^9}=\frac{x+y}{(x-y)^{n-2}} \Rightarrow n-2=9 \Rightarrow n=11$$

7. **If a,b,c,d are in G.P, prove that**

i. **$a-b, b-c, c-d$ are in G.P**

Sol. Given a, b, c, d are in G.P then

$\dfrac{b}{a}=\dfrac{c}{b}=\dfrac{d}{c}$ Or $\dfrac{b}{a}=\dfrac{c}{b}, \dfrac{c}{b}=\dfrac{d}{c}, \dfrac{d}{c}=\dfrac{b}{a}$

$\Rightarrow b^2=ac \,\&\, c^2=bd \,\&\, bc=ad$ ——— I

Now if $a-b, b-c, c-d$ are in G.P then

$\dfrac{c-d}{b-c}=\dfrac{b-c}{a-b}$

$\Rightarrow (a-b)(c-d)=(b-c)(b-c)=(b-c)^2$

L.H.S $=(a-b)(c-d)$

$=ac-ad-bc+bd$

$=b^2-bc-bc+c^2$ (use I)

$=b^2-2bc+c^2$

$=(b-c)^2$ So L.H.S = R.H.S

Hence $(a-b),(b-c),(c-d)$ are in G.P

ii. $a^2-b^2, b^2-c^2, c^2-d^2$ **are in G.P** **Federal**

Sol. If $a^2-b^2, b^2-c^2, c^2-d^2$ are in G.P

Then $\dfrac{c^2-d^2}{b^2-c^2}=\dfrac{b^2-c^2}{a^2-b^2}$

$\Rightarrow (b^2-c^2)^2=(a^2-b^2)(c^2-d^2)$

$\text{R.H.S}=a^2c^2-a^2d^2-b^2c^2+b^2d^2$

$=(ac)^2-(ad)^2-b^2c^2+(bd)^2$

Use I $=(b^2)^2-(bc)^2-b^2c^2+(c^2)^2$

$=(b^2)^2-b^2c^2-b^2c^2+(c^2)^2$

$=(b^2)^2-2b^2c^2+(c^2)^2$

$=(b^2-c^2)^2=\text{L.H.S}$

iii. $a^2+b^2, b^2+c^2, c^2+d^2$ **are in G.P**

Sol. Then $\dfrac{c^2+d^2}{b^2+c^2}=\dfrac{b^2+c^2}{a^2+b^2}$

$\Rightarrow \quad (b^2+c^2)^2=(a^2+b^2)(c^2+d^2)$

$\text{R.H.S} \quad =a^2c^2+a^2d^2+b^2c^2+b^2d^2$

$=(ac)^2+(ad)^2+b^2c^2+(bd)^2$

$=(b^2)^2+b^2c^2+b^2c^2+(c^2)^2 \quad (use-I)$

$=(b^2)^2+2b^2c^2+(c^2)^2$

$=(b^2+c^2)^2=\text{L.H.S}$

Hence $a^2+b^2, b^2+c^2, c^2+d^2$ are in G.P.

8. Show that the reciprocals of the terms of the geometric sequence a_1, a_1r^2, a_1r^4 .... form another geometric sequence. **Multan 2008**

Sol. We have to prove that $\dfrac{1}{a_1}, \dfrac{1}{a_1r^2}, \dfrac{1}{a_1r^4}$ are in G.P

So $\quad r=\dfrac{third}{second}=\dfrac{\dfrac{1}{a_1r^4}}{\dfrac{1}{a_1r^2}}$

So $\quad r = \frac{1}{a_1r^4} \times \frac{a_1r^2}{1} = \frac{1}{r^2}$

Also $\quad r = \frac{second}{First} = \frac{\frac{1}{a_1r^2}}{\frac{1}{a_1}}$

$= \frac{1}{a_1r^2} \times \frac{a_1}{1} = \frac{1}{r^2}$

Ratio are same Hence $\frac{1}{a_1}, \frac{1}{a_1r^2}, \frac{1}{a_1r^4}$ are in G.P

9. Find the nth of the geometric sequence if ; $\frac{a_5}{a_3} = \frac{4}{9}$ and $a_3 = \frac{4}{9}$

Sol. Given $\frac{a_5}{a_3} = \frac{4}{9} \Rightarrow \frac{a_1r^4}{a_1r^2} = \frac{4}{9}$

$\Rightarrow r^2 = \frac{4}{9} \Rightarrow r = \pm\frac{2}{3}$ Now when $r = \frac{2}{3}$

Also given $a_2 = \frac{4}{9} \Rightarrow a_1r = \frac{4}{9} \Rightarrow a_1\left(\frac{2}{3}\right) = \frac{4}{9} \Rightarrow a_1 = \frac{4}{9} \times \frac{3}{2} = \frac{2}{3}$

When $r = \frac{-2}{3}$ $then\ a_1\left(\frac{-2}{3}\right) = \frac{4}{9}$

$\Rightarrow a_1 = \frac{4}{9}\left(-\frac{3}{2}\right) \Rightarrow a_1 = \frac{-2}{3}$

$a_n = a_1r^{n-1} = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^{n-1} = \left(\frac{2}{3}\right)^{n-1+1} = \left(\frac{2}{3}\right)^n \left(if\ a_1\ and\ r = \frac{2}{3}\right)$

$a_n = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)^{n-1} = \left(-\frac{2}{3}\right)^n = (-1)^n.\left(\frac{2}{3}\right)^n \left(if\ a_1 = r = -\frac{2}{3}\right)$

10. Find three, consecutive numbers in G.P whose sum is 26 and their product is 216.

Sol. Suppose three numbers in G.P. are $\frac{a_1}{r}, a_1, a_1r$

Condition II $\Rightarrow \left(\frac{a_1}{r}\right)(a_1)(a_1r) = 216$

$a_1^3 = 216 \Rightarrow a_1^3 = (6)^3 \Rightarrow \boxed{a_1 = 6}$

Condition I $\Rightarrow \frac{a_1}{r} + a_1 + a_1 r = 26$

$$a_1\left(\frac{1}{r} + 1 + r\right) = 26$$

$$a_1\left(\frac{1 + r + r^2}{r}\right) = 26 \Rightarrow 6(r^2 + r + 1) = 26r$$

$\Rightarrow 6r^2 + 6r + 6 - 26r = 0$

$\Rightarrow 6r^2 - 20r + 6 = 0$

'$\div$' $by\ 2 \Rightarrow 3r^2 - 10r + 3 = 0$

$3r^2 - 9r - r + 3 = 0 \Rightarrow 3r(r-3) - 1(r-3) = 0$

$(r-3)(3r-1) = 0$

$r - 3 = 0\ or\ 3r - 1 = 0$

$r = 3\ or\ r = \frac{1}{3}$

When $r = \frac{1}{3}$ & $a_1 = 6$

$\frac{a}{r} = \frac{6}{1/3} = 6 \times 3 = 18$

$a_1 = 6\ and\ a_1 r = 6\left(\frac{1}{3}\right) = 2$

Hence three numbers in G.P are 2, 6, 18

11. If the sum of the four consecutive terms of a G.P is 80 and A.M of the second and the fourth of them is 30. Find the terms

Sol. Suppose four numbers in G.P. are $a_1, a_1 r, a_1 r^2, a_1 r^3$

Condition I $\Rightarrow a_1 + a_1 r + a_1 r^2 + a_1 r^3 = 80$

$or\ a_1 + a_1 r^2 + a_1 r + a_1 r^3 = 80 \longrightarrow I$

Condition II $\Rightarrow \frac{a_1 r + a_1 r^3}{2} = 30$

$\Rightarrow a_1 r + a_1 r^3 = 60 \longrightarrow II$

$use\ II\ in\ I \qquad a_1 + a_1 r^2 + 60 = 80$

$\Rightarrow \quad a_1 + a_1 r^2 = 80 - 60 = 20$

$\Rightarrow a_1 + a_1 r^2 = 20 \longrightarrow III$

'×' by r we get.

$a_1 r + a_1 r^3 = 20r$

Using II

$$60 = 20r \Rightarrow r = \frac{20}{60} = 3, \boxed{r = 3}$$

Put in III $a + a(3)^2 = 20$

$a_1 + 9a_1 = 20 \Rightarrow 10a_1 = 20 \Rightarrow a_1 = \boxed{2}$

So $a_1 = 2$

$a_1 r = (2)(3) = 6$

$a_1 r^2 = (2)(3)^2 = (2)(9) = 18$

$a_1 r^3 = (2)(3)^2 = 2(27) = 54$

Hence required four terms are 2,6, 18, 54

12. **If $\frac{1}{a}, \frac{1}{b}$ *and* $\frac{1}{c}$ are in G.P show that the common ration is $\pm\sqrt{\frac{a}{c}}$**

Fsd 2007, 2008 Lahore 2009, Rawalpindi 2009, Multan 2007, 2009, 2010

Sol. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in G.P

$$\text{Then } r = \frac{Second}{First} = \frac{\frac{1}{b}}{\frac{1}{a}} = \frac{1}{b} \times \frac{a}{1} = \frac{a}{b} \longrightarrow I$$

$$r = \frac{Third}{Second} = \frac{\frac{1}{c}}{\frac{1}{b}} = \frac{1}{c} \times \frac{b}{1} = \frac{b}{c} \longrightarrow II$$

$I \times II$

$$r^2 = \frac{a}{b} \times \frac{b}{c} = \frac{a}{c} \Rightarrow r = \pm\sqrt{\frac{a}{c}}$$

13. **If the numbers 1, 4 and 3 are subtracted from three consecutive terms of an A.P., the resulting numbers are in G.P. Find the numbers if their sum is 21.**

Sol. Suppose three now in A.P are $a_1 - d, a_1, a_1 + d$

Condition II $\Rightarrow a_1 - d + a_1 + a_1 + d = 21$

$$3a = 21 \Rightarrow a_1 = \frac{21}{3} = 7 \Rightarrow \boxed{a = 7}$$

Condition I $\Rightarrow a_1 - d - 1, a_1 - 4, a_1 + d - 3$ are in G.P

$7 - d - 1, 7 - 4, 7 + d - 3$ are in G.P

$6 - d, 3, 4 + d$ are in G.P

$$\Rightarrow \frac{4+d}{3} = \frac{3}{6-d} \Rightarrow (4+d)(6-d) = 9$$

$\Rightarrow 24 - 4d + 6d - d^2 = 9$

$\Rightarrow 24 + 2d - d^2 - 9 = 0$

$\Rightarrow -d^2 + 2d + 15 = 0$

$\Rightarrow d^2 - 2d - 15 = 0 \qquad (\text{'}\times\text{' } by - 1)$

$\Rightarrow d^2 - 5d + 3d - 15 = 0$

$d(d-5) + 3(d-5) = 0 \Rightarrow (d-5)(d+3) = 0 \Rightarrow d - 5 = 0 \; or \; d + 3 = 0$

$d = 5 \; or \; d = -3$

When $\quad d = 5 \; and \; a_1 = 7$

$a_1 - d = 7 - 5 = 2$

$a_1 = 7$

$a_1 + d = 7 + 5 = 12$

When $\quad d = -3 \; and \; a_1 = 7$

$a_1 - d = 7 - (-3) = 7 + 3 = 10$

$a_1 = 7$

$a_1 + d = 7 + (-3) = 7 - 3 = 4$

Required numbers are 4, 7, 10, or 2, 7, 12

14. If three consecutive numbers in A.P are increased by 1, 4, 15 respectively, the resulting numbers are in G.P. Find the original numbers if their sum is 4.

Sol. Suppose three numbers in A.P are $a_1 - d, a_1, a_1 + d$

Condition II $\Rightarrow a_1 - d + a_1 + a_1 + d = 6$

$3a_1 = 6 \Rightarrow \boxed{a_1 = 2}$

Condition I $\Rightarrow a_1 - d + 1, a_1 + 4, a_1 + d + 15$ are in G.P

$or \; 2 - d + 1, 2 + 4, 2 + d + 15$ are in G.P (put a = 2)

$3 - d, 6, 17 + d$ are in G.P

$$\Rightarrow \frac{17+d}{6}=\frac{6}{3-d}$$
$$\Rightarrow (17+d)(3-d)=36$$
$$51-17d+3d-d^2=36$$
$$\Rightarrow 51-14d-d^2-36=0$$
$$\Rightarrow -d^2-14d+15=0$$
$$\Rightarrow d^2+14d-15=0 \ ('\times' \ by -1)$$
$$\Rightarrow d^2+15d-d-15=0$$
$$\Rightarrow d(d+15)-1(d+15)=0$$
$$\Rightarrow (d+15)(d-1)=0$$
$$d+15=0 \ or \ d-1=0 \Rightarrow d=-15 \ or \ d=1$$

When $d=-15$

then $a_1=2$

$a_1+d=2+(-15)=2-15=-13$ and $a_1-d=2-(-15)=2+15=17$

When $d=1$ then $a_1-d=2-1=1$

$a_1=2$

$a_1+d=2+1=3$

Required numbers are 1,2,3 or -13, 2, 17

Exercise 6.7

Theorem: G.M $=\pm\sqrt{ab}$

If G is geometric mean between a & b then a, G, b are in G.P

$$\Rightarrow \frac{b}{G}=\frac{G}{a} \Rightarrow G^2=ab$$
$$\Rightarrow \boxed{G=\pm\sqrt{ab}}$$

1. Find G.M between

i. -2 and 8 **Multan 2008**

Sol. Here $a=-2$ & $b=8$

$$G=\pm\sqrt{ab}=\pm\sqrt{(-2)(8)}=\pm\sqrt{-16}$$
$$=\pm\sqrt{16}=\pm 4i$$

ii. $a=-2i, b=8i$ **Faisalabad 2007**

Sol. $G=\pm\sqrt{ab}=\pm\sqrt{(-2i)(8i)}=\pm\sqrt{-16i^2}=\pm\sqrt{-16(-1)}=\pm\sqrt{16}=4$

2. Insert two G.Ms between

i. 1 and 8 **Lahore 2009**

Sol. Suppose G_1, G_2 are two G.Ms between 1 & 8 then

$1, G_1, G_2, 8$ are in G.P

$\boxed{a_1 = 1}$ & $a_4 = a_1 r^3 = 8 \Rightarrow (1)r^3 = 8 \Rightarrow r^3 = (2)^3 \Rightarrow \boxed{r = 2}$

$G_1 = a_2 = a_1 r = (1)(2) = 2$

$G_2 = a_3 = a_1 r^2 = (1)(2)^2 = 4$

So two G.Ms are 2,4

ii. 2 and 16 **Sargodha 2006, Fsd 2009, Gujranwala 2009, Multan 2008**

Sol. Suppose G_1, G_2 are two G.Ms between 2 & 16 then

$2, G_1, G_2, 16,$ are in G.P

$\boxed{a_1 = 2}$ & $a_4 = a_1 r^3 = 16 \Rightarrow 2r^3 = 16 \Rightarrow r^3 = 8 = 2^3 \Rightarrow \boxed{r = 2}$

$G_1 = a_2 = a_1 r = 2(2) = 4$

$G_2 = a_3 = a_1 r^2 = (2)(2)^2 = (2)(4) = 8$ **two G.M,s are 4, 8.**

3. Insert three G.Ms between

i. 1 and 16

Sol. Suppose G_1, G_2, G_3 are three G.M,s between 1 & 16 then

$1, G_1, G_2, G_3, 16$ are in G.P

$\boxed{a_1 = 1}$ & $a_5 = a_1 r^4 = 16(1)r^4 = 16 = r^4 = (2)^4 \Rightarrow r = 2$

$G_1 = a_2 = a_1 r = (1)(2) = 2$ and $G_2 = a_3 = a_1 r^2 = (1)(2)^2 = 4$

$G_3 = a_4 = a_1 r^3 = (1)(2)^3 = 8$

So three G.M,s are 2, 4, 8

ii. 2 and 32

Sol. Suppose G_1, G_2, G_3 are three G.M,s between 2 & 32 then

$2, G_1, G_2, G_3, 32$ are in G.P

$\boxed{a_1 = 2}$ & $a^5 = a_1 r^4 = 32$

$2r^4 = 32 = r^4 = 16 = (2)^4 \Rightarrow \boxed{r = 2}$

$G_1 = a_1 r = (2)(2) = 4$

$G_2 = a_1 r^2 = (2)(2)^2 = 2(4) = 8$

$G_3 = a_1 r^3 = (2)(2)^3 = 2(8) = 16$

So three G.M,s are 4, 8, 16

4. Insert four real geometric means between 3 and 96. **Gujranwala 2009**

Sol. Let G_1, G_2, G_3, G_4 are four G.M,s between 3 & 96 then

$3, G_1, G_2, G_3, G_4, 96$ are in G.P

$\boxed{a_1 = 3}$ & $a^6 = a_1 r^5 = 96 \Rightarrow 3r^5 = 96$

$\Rightarrow r^5 = \frac{96}{3} = 32 = 2^5 \Rightarrow \boxed{r = 2}$ then

$G_1 = a_1 r = 3(2) = 6$

$G_2 = a_1 r^2 = 3(2)^2 = 3(4) = 12$

$G_3 = a_1 r^3 = 3(2)^2 = 3(8) = 24$

$G_4 = a_1 r^4 = 3(2)^2 = 3(16) = 48$ So four G.M,s are $b = 6, 12, 24, 48$

5. If both x and y are positive distinct real numbers, show that the geometric mean between x and y is less than their arithmetic mean.

Sol. Given $x > 0$ & $y > 0$ then

Here $a = x,$ & $b = y$

G.M$= \sqrt{xy}$ & $A.M = \frac{a+b}{2} = \frac{x+y}{2}$

Now $A.M - G.M = \frac{x+y}{2} - \sqrt{xy} = \frac{x+y-2\sqrt{xy}}{2}$

$$= \frac{(\sqrt{x})^2 + (\sqrt{y})^2 - 2\sqrt{xy}}{2} = \frac{(\sqrt{x}-\sqrt{y})^2}{2} = \frac{(\sqrt{x}-\sqrt{y})^2}{2} > 0$$

$\Rightarrow \quad A.M - G.M > 0 \Rightarrow A.M > G.M$ or $\boxed{G.M < A.M}$

6. For what value of n, $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the positive geometric mean between a and b?

Sol. If $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ be $G.M$ between a & b. **Multan 2007, 2009, Federal**

Then $\frac{a^n + b^n}{a^{n-1} + b^{n-1}} = \sqrt{ab} = (ab)^{1/2}$

$\Rightarrow a^n + b^n = (a^{n-1} + b^{n-1}) a^{1/2} b^{1/2}$

$a^n + b^n = a^{n-1+1/2} b^{1/2} + a^{1/2} b^{n-1+1/2}$

$a^n + b^n = a^{n-1/2} b^{1/2} + a^{1/2} b^{n-1/2}$

$a^n + b^n = a^{n-1/2} b^{1/2} + a^{1/2} b^{n-1/2}$

$$a^n - a^{n-1/2}b^{1/2} = a^{1/2}b^{n-1/2} - b^n \Rightarrow a^{n-1/2}(a^{1/2} - b^{1/2}) = b^{n-1/2}(a^{1/2} - b^{1/2})$$

$$\Rightarrow \frac{a^{n-1/2}}{b^{n-1/2}} = 1 \Rightarrow \left(\frac{a}{b}\right)^{n-1/2} = 1 = \left(\frac{a}{b}\right)^0 \Rightarrow n - \frac{1}{2} = 0 \Rightarrow n = \frac{1}{2}$$

7. The A.M of two positive integral numbers exceeds their (positive) G.M. by 2 and their sum is 20, find the numbers.

Sol. Suppose two number are a & b then.

Condition I $\Rightarrow \frac{a+b}{2} = \sqrt{ab} + 2$

$'\times'$ $by\ 2 \Rightarrow a + b = 2\sqrt{ab} + 4 \longrightarrow I$

Condition II $a + b = 20 \Rightarrow a = 20 - b \longrightarrow II$

$20 - b + b = 2\sqrt{(20-b)b} + 4$ (Put II in I)

$\Rightarrow 20 - 4 = 2\sqrt{20b - b^2}$

$\Rightarrow \quad 16 = 2\sqrt{20b - b^2} \Rightarrow 8 = \sqrt{20b - b^2}$ squaring both side.

$64 = 20b - b^2 \Rightarrow b^2 - 20b + 64 = 0$

$\Rightarrow \quad b^2 - 16b - 4b + 64 = 0$

$b(b - 16) - 4(b - 4) = 0$

$\Rightarrow \quad (b-16)(b-4) = 0 \Rightarrow b - 16 = 0\ or\ b - 4 = 0 \Rightarrow b = 16\ or\ b = 4$

When $b = 16$ then $a = 20 - 16 = 4$

When $b = 4$ then $a = 20 - 4 = 16$

Hence two numbers are 4, 16, or 16, 4.

8. The A.M between two numbers is 5 and their (positive) G.M is 4. Find the numbers.

Sol. Suppose two number are a & b then

Condition I $\Rightarrow \frac{a+b}{2} = 5$

$\Rightarrow a + b = 10 \longrightarrow I$

Condition II $\Rightarrow \sqrt{ab} = 4 \Rightarrow ab = 16 \longrightarrow II$ (from I) $a = 10 - b$ Put in II.

$(10 - b)b = 16 \Rightarrow 10b - b^2 = 16$

$\Rightarrow b^2 - 10b + 16 = 0 \Rightarrow b^2 - 8b - 2b + 16 = 0 \Rightarrow b(b-8) - 2(b-8) = 0$

$(b-8)(b-2) = 0 \Rightarrow b - 8 = 0\ or\ b - 2 = 0 \Rightarrow b = 8\ or\ b = 2$

When $b = 8$ then $a = 10 - 8 = 2$

When $b = 2$ then $a = 10 - 2 = 8$

Hence two numbers are 2, 8, or 8, 2.

Exercise 6.8

Theorem:

$$S_n = \frac{a_1(r^n - 1)}{r-1}, |r| > 1 \text{ and } S_n = \frac{a_1(1-r^n)}{1-r}, |r| < 1$$

Proof: We know that

$$S_n = a_1 + a_1 r + a_1 r^2 + \dots\dots\dots + a_1 r^{n-1}$$

'$\times$' both sides by $(1-r)$

$$(1-r)S_n = (1-r)(a_1 + a_1 r + a_1 r^2 + \dots\dots\dots a_1 r^{n-1})$$

$$= a_1 + a_1 r + a_1 r^2 + \dots\dots\dots + a_1 r^{n-1} - a_1 r - a_1 r^2 - a_1 r^3 \dots\dots\dots a_1 r^{n-1} - a_1 r^n$$

$$(1-r)S_n = a_1 - a_1 r^n = a_1(1-r^n)$$

$$\Rightarrow S_n = \frac{a_1(1-r^n)}{1-r} \text{ if } |r| < 1$$

and $S_n = \dfrac{a_1(r^n - 1)}{r-1} \text{ if } |r| > 1$

Theorem:

$$S_\infty = \frac{a}{r-1}$$

Proof: We know that

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a_1(1-r^n)}{1-r} = a_1\left[\lim_{n\to\infty}\frac{1}{1-r} - \lim_{n\to\infty}\frac{r^n}{1-r}\right]$$

$$= a_1\left[\frac{1}{1-r} - 0\right] \Rightarrow \boxed{S_\infty = \frac{a}{1-r}}$$

1. **Find the sum of first 15 terms of the geometric sequence $1, \frac{1}{3}, \frac{1}{9} \dots\dots\dots$**

Sol. $1, \frac{1}{3}, \frac{1}{9} \dots\dots\dots$

$$S_n = 1 + \frac{1}{3} + \frac{1}{9} + \dots\dots\dots + a_{15}$$

$$a_1 = 1, r = \frac{\frac{1}{3}}{1} = \frac{1}{3} < 1, n = 15$$

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{(1-\left(\frac{1}{3}\right)^{15})}{1-\frac{1}{3}} = \frac{1-\frac{1}{14348907}}{\frac{3}{2}} = \frac{14348907-1}{14348907} \times \frac{3}{2}$$

$$= \frac{14348906}{14348907} \times \frac{3}{2} = \frac{7172453}{4782969}$$

2. Sum to n terms, the series

i. .2 + .22 + .222 + ... **Faisalabad 2007, Multan 2007**

Sol. $S_n = .2 + .22 + .222 + \ldots\ldots\ldots\ldots + n$ terms

$$= 2(.1 + .11 + .111 + \ldots\ldots\ldots + n \text{ terms})$$

$$= \frac{2}{9}(.9 + .99 + .999 + \ldots\ldots\ldots\ldots + n \text{ terms}) = \frac{2}{9}\left(\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots\ldots + n \text{ term}\right)$$

$$\text{So} = \frac{2}{9}\left[\left(1-\frac{1}{10}\right) + \left(1-\frac{1}{100}\right) + \left(1-\frac{1}{1000}\right) + \ldots\ldots\ldots \text{ to } n \text{ terms}\right]$$

$$= \frac{2}{9}\left[(1+1+1+\ldots\ldots\ldots \text{ to } n \text{ terms}) - \left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots\ldots n \text{ term}\right)\right]$$

$$= \frac{2}{9}\left[n - \left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots\ldots n \text{ term}\right)\right]$$

$$a_1 = \frac{1}{10}, r = \frac{\frac{1}{100}}{\frac{1}{10}} = \frac{1}{100} \times \frac{10}{1} = \frac{1}{10} < 1, n = n$$

$$= \frac{2}{9}\left[n - \frac{a_1(1-r^n)}{1-r}\right] = \frac{2}{9}\left[n - \frac{\frac{1}{10}\left(1-\frac{1}{10}\right)^n}{1-\frac{1}{10}}\right]$$

$$=\frac{2}{9}\left[n-\frac{\frac{1}{10}\left(1-\frac{1}{10^n}\right)}{\frac{9}{10}}\right]=\frac{2}{9}\left[n-\frac{10}{9}\times\frac{1}{10}\left[1-\frac{1}{10^n}\right]\right]$$

$$=\frac{2}{9}\left[n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right]=\frac{2}{9}\left[n-\frac{1}{9}\left(\frac{10^n-1}{10^n}\right)\right]$$

ii. 3 + 33 + 333 + ... **Sargodha 2008, 2010, 2011**

Sol. $3+33+333+..............+to\ n\ term$

$3(1+11+111+..........\ to\ n\ term)$

$$=\frac{3}{9}(9+99+999+..........\ to\ n\ term)$$

$$=\frac{1}{3}[(10-1)+(100-1)+(1000-1)+..........\ to\ n\ term)]$$

$$=\frac{1}{3}[(10+100+1000+..............+n\,term)-(1+1+1+.....n\,term)]$$

$$=\frac{1}{3}[10+100+1000+..........\ to\ n\ term-n]$$

$$a=10, r=\frac{100}{10}=10>1, n=n$$

$$=\frac{1}{3}\left[10\left(\frac{10^n-1}{10-1}\right)-n\right]=\frac{1}{3}\left[\frac{10}{9}(10^n-1)-n\right]$$

3. Sum to n terms, the series

i. $1+(a+b)+(a^2+ab+b^2)+(a^3+a^2b+ab^2+b^3)+..........$

Sol. $1+(a+b)+(a^2+ab+b^2)+(a^3+a^2b+ab^2+b^3)+..........$to n terms

'×' & '÷' by $(a-b)$

$$=\frac{1}{(a-b)}\left[(a-b)+(a^2-b^2)+(a^3-b^3)+............n\ terms\right]$$

$$=\frac{1}{(a-b)}\left[a+a^2+a^3+.........\ n\ terms-(b+b^2+b^3+........\ n\ terms)\right]$$

$$=\frac{1}{(a-b)}\left[\frac{a_1(a^n-1)}{a-1}-\frac{b_1(b^n-1)}{b-1}\right]$$

$$=\frac{1}{(a-b)}\left[\frac{a(b-1)(a^n-1)-b(a-1)(b^n-1)}{(a-1)(b-1)}\right]$$

$$=\frac{a(b-1)(a^n-1)-b(a-1)(b^n-1)}{(a-b)(a-1)(b-1)}$$

ii. $r+(1+k)r^2+(1+k+k^2)r^3+\ldots\ldots\ldots$ **Sargodha 2006, Multan 2007**

Sol. $r+(1+k)r^2+(1+k+k^2)r^3+\ldots\ldots\ldots$ to n terms

'×' & '÷' by $(1-k)$ we get.

$$=\frac{1}{(1-k)}\left[(1-k)r+(1-k^2)r^2+(1-k^3)r^3+\ldots\ldots\ldots n\ terms\right]$$

$$=\frac{1}{(1-k)}\left[r+r^2+r^3+\ldots\ldots\ldots n\ terms\right]-\left(kr+k^2r^2+\ldots\ldots\ldots n\ term\right]$$

First series $a_1=r, r=\frac{r^2}{r}=r, n=n$, Second series $a_1=kr, r=\frac{k^2r^2}{kr}=kr$

$$S_n=\frac{1}{(1-k)}\left[\frac{r(r^n-1)}{r-1}-\frac{kr((kr)^n-1)}{kr-1}\right]$$

4. **Sum the series** $2+(1-i)+\left(\frac{1}{i}\right)+\ldots\ldots$ **to 8 terms.**

Sol. $2+(1-i)+\frac{1}{i}+\ldots\ldots\ldots$ to 8 terms $a_1=2, r=\frac{1-i}{2}, n=8 |r|<1$ clearly.

$$S_n=\frac{a_1(1-r^n)}{1-r}=\frac{2\left(1-\left(\frac{1-i}{2}\right)^8\right)}{1-\frac{1-i}{2}}$$

$$Note\ (1-i)^8=\left[(1-i)^2\right]^4$$
$$=(1+i^2-2i)^4=\left(1-1-2i\right)^4=(-2i)^4$$
$$=16i^4=16i^2.i^2=16(-1)(-1)=16$$

$$=\frac{2(2^8-(1-i)^8)}{2^8\left(\frac{2-1+i}{2}\right)}=\frac{256-(1-i)^8}{64(1+i)}$$

$$S_8=\frac{256-16}{64(1+i)}=\frac{240}{64(1+i)}=\frac{15}{4(1+i)}$$

5. Find the sum of the following infinite geometric series:

i. $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\ldots\ldots\ldots$

Sol. $a_1=\frac{1}{5}, r=\frac{\frac{1}{25}}{\frac{1}{5}}=\frac{1}{25}\times\frac{25}{1}=\frac{1}{5}$

$$S_\infty=\frac{a_1}{1-r}=\frac{\frac{1}{5}}{1-\frac{1}{5}}=\frac{\frac{1}{5}}{\frac{4}{5}}=\frac{1}{5}\times\frac{5}{4}=\frac{1}{4}$$

ii. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots\ldots$ **Sargodha 2009**

Sol. $a_1=\frac{1}{2}, r=\frac{\frac{1}{4}}{\frac{1}{2}}=\frac{1}{4}\times\frac{2}{1}=\frac{1}{2}$

$$S_\infty=\frac{a_1}{1-r}=\frac{\frac{1}{2}}{1-\frac{1}{2}}=\frac{\frac{1}{2}}{\frac{1}{2}}=1$$

iii. $\frac{9}{4}+\frac{3}{2}+1+\frac{2}{3}+\ldots\ldots\ldots$

Sol. $a_1=\frac{9}{4}, r=\frac{\frac{3}{2}}{1}=\frac{3}{2}$

$$S_\infty=\frac{a_1}{1-r}=\frac{\frac{9}{4}}{1-\frac{3}{2}}=\frac{\frac{9}{4}}{\frac{3-2}{3}}$$

$$S_\infty=\frac{\frac{9}{4}}{\frac{1}{3}}=\frac{9}{4}\times\frac{3}{1}=\frac{27}{4}$$

iv. $2+1+0.5+\ldots\ldots\ldots\ldots$ **Sargodha 2009, Faisalabad 2008**

Sol. $a_1=2, r=\dfrac{1}{2}$

$$S_\infty=\frac{a_1}{1-r}=\frac{2}{1-\frac{1}{2}}=\frac{2}{\frac{1}{2}}=2\times\frac{2}{1}=4$$

v. $4+2\sqrt{2}+2+\sqrt{2}+1+\ldots\ldots\ldots\ldots$ **Multan 2007, 2008**

Sol. $a_1=4, r=\dfrac{1}{\sqrt{2}}$

$$S_\infty=\frac{a_1}{1-r}=\frac{4}{1-\frac{1}{\sqrt{2}}}=\frac{4}{\frac{\sqrt{2}-1}{\sqrt{2}}}$$

$$=\frac{4\times\sqrt{2}}{\sqrt{2}-1}=\frac{4\sqrt{2}}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}$$

$$=\frac{4\sqrt{2}(\sqrt{2}+1)}{2-1}=4(2+\sqrt{2})$$

vi. $0.1+0.05+0.005+\ldots\ldots\ldots\ldots$

Sol. $a_1=0.1, r=\dfrac{0.05}{0.1}=0.5$

$$S_\infty=\frac{a_1}{1-r}=\frac{0.01}{1-0.5}=\frac{0.1}{0.5}=0.2$$

6. Find vulgar fractions equivalent to the following recurring decimals.

i. $1.\dot{3}\dot{4}$ **Multan 2009**

Sol. $=1.343434\ldots\ldots\ldots\ldots$

$=1+0.343434\ldots\ldots\ldots\ldots$

$=1+(0.34+0.0034+\ldots\ldots)$

$a_1=0.34,\ r=\dfrac{0.0034}{0.34}=0.01$

$$=1+\frac{a_1}{1-r}=1+\frac{0.034}{1-0.01}$$

$$=1+\frac{0.34}{0.99}=1+\frac{34}{99}=\frac{99+34}{99}=\frac{133}{99}$$

ii. **0.7** **Multan 2010**

Sol. $0.7777.................$

$= 0.7 + 0.07 + 0.007 + ...$

$$a_1 = 0.7,\ r = \frac{0.07}{0.7} = 0.1$$

$$S_\infty = \frac{a_1}{1-r} = \frac{0.7}{1-0.01} = \frac{0.7}{0.9} = \frac{\frac{7}{10}}{\frac{9}{10}} = \frac{7}{9}$$

iii. $\mathbf{0.\dot{2}\dot{5}\dot{9}}$

Sol. $= 0.259259259259..........$

$= 0.259 + 0.000259 + ...$

$$a_1 = 0.259,\ r = \frac{0.000259}{0.259} = 0.001$$

$$S_\infty = \frac{a_1}{1-r} = \frac{0.259}{1-0.001}$$

$$= \frac{0.259}{0.999} = \frac{\frac{259}{1000}}{\frac{999}{1000}} = \frac{259}{999}$$

iv. $\mathbf{1.\dot{5}\dot{3}}$

Sol. $= 1.535353...........$

$= 1 + .535353...............$

$= 1 + (0.53 + 0.0053 + ...)$

$$a_1 = 0.53,\ r = \frac{0.0053}{0.53} = 0.01$$

we know that $S_\infty = \dfrac{a_1}{1-r}$

$$1.\dot{5}\dot{3} = 1 + \frac{0.53}{1-0.01} = 1 + \frac{0.53}{0.99}$$

$$= 1 + \frac{\frac{53}{100}}{\frac{99}{100}} = 1 + \frac{53}{99} \qquad = \frac{99+53}{99} = \frac{152}{99}$$

v. $0.\dot{1}\dot{5}\dot{9}$ **Federal**

Sol. $=0.159159159............$

$=0.159+0.000159+...........$

$a_1=0.159,\ r=\dfrac{0.000159}{0.159}=0.01$

$0.\dot{1}\dot{5}\dot{9}=\dfrac{a_1}{1-r}=\dfrac{0.159}{1-0.001}$

$=\dfrac{0.159}{0.999}=\dfrac{\frac{159}{1000}}{\frac{999}{1000}}=\dfrac{159}{999}$

vi. $1.\dot{1}\dot{4}\dot{7}$

Sol. $=1.147147147...........$

$=1+(0.147+0.000147+.......)$

$a_1=0.147,\ r=\dfrac{0.000147}{0.147}=0.0001$

$S_\infty=\dfrac{a_1}{1-r}=\dfrac{0.147}{1-0.001}$

$=\dfrac{0.147}{0.999}=\dfrac{\frac{147}{1000}}{\frac{999}{1000}}=\dfrac{147}{999}$

$1.\dot{1}\dot{4}\dot{7}=1+\dfrac{a}{1-r}$

$=1+\dfrac{147}{999}=\dfrac{999+147}{999}\quad=\dfrac{1147}{999}$

7. **Find the sum to infinity of the series;**

$r+(1+k)r^2+(1+k+k^2)r^3+.......r$ *and* k being proper fractions

$r+(1+k)r^2+(1+k+k^2)r^3+..........$

Sol. '$\times$' & '$\div$' *by* $(1-k)$

$=\dfrac{1}{(1-k)}\left[(1-k)r+(1-k^2)r^2+(1-k^3)r^3+..............\right]$

$$=\frac{1}{(1-k)}\left[r+r^2+r^3+........-(kr+k^2r^2+.........)\right]$$

For First series $a_1=r,\ r=r$

For second series $a_1=kr,\ r=k^2r^2$

$$=\frac{1}{(1-k)}\left[\frac{a}{1-r}-\frac{a}{1-r}\right]$$

$$=\frac{1}{(1-k)}\left[\frac{r}{1-r}-\frac{kr}{1-kr}\right]$$

$$=\frac{1}{(1-k)}\left[\frac{r(1-kr)-kr(1-r)}{(1-r)(1-kr)}\right]$$

$$=\frac{1}{(1-k)}\left[\frac{r-\cancel{kr^2}-kr+\cancel{kr^2})}{(1-r)(1-kr)}\right]$$

$$=\frac{1}{\cancel{(1-k)}}\left[\frac{r\cancel{(1-k)}}{(1-r)(1-kr)}\right]=\frac{r}{(1-r)(1-kr)}$$

8. **If $y=\frac{x}{2}+\frac{1}{4}x^2+\frac{1}{8}x^3+.....$ and if $0<x<2$, then prove that $x=\frac{2y}{1+y}$**

Sol. $y=\frac{x}{2}+\frac{1}{4}x^2+\frac{1}{8}x^3+...........$ **Faisalabad 2007**

$$a_1=\frac{x}{2},r=\frac{\frac{1}{4}x^2}{\frac{1}{2}x}=\frac{x^2}{4}\times\frac{2}{x}=\frac{x}{2}$$

$$y=S_\infty=\frac{a}{1-r}=\frac{\frac{x}{2}}{1-\frac{x}{2}}=\frac{\frac{x}{2}}{\frac{2-x}{2}}$$

$$=\frac{x}{2}\times\frac{2}{2-x}=\frac{x}{2-x}$$

$$y=\frac{x}{2-x}\Rightarrow(2-x)y=x\Rightarrow 2y-xy-x=0$$

$$2y=xy+x=x(1+y)$$

$$\Rightarrow x=2y/(1+y)$$

9. **If** $y=\frac{2}{3}x+\frac{4}{9}x^2+\frac{8}{27}x^3$ **....... and if** $0<x<\frac{3}{2}$**, then show that** $x=\frac{3y}{2(1+y)}$

Sol. $y=\frac{2}{3}x+\frac{4}{9}x^2+\frac{8}{27}x^3+..........$

$$a_1=\frac{2}{3}x, r=\frac{\frac{4}{9}x^2}{\frac{2}{3}x}=\frac{4x^2}{9}\times\frac{3}{2x}=\frac{2x}{3}$$

$$y=S_\infty=\frac{a_1}{1-r}=\frac{\frac{2}{3}x}{1-\frac{2x}{3}}=\frac{\frac{2x}{3}}{\frac{3-2x}{3}}$$

$$=\frac{2x}{3}\times\frac{3}{3-2x}=\frac{2x}{3-2x}\Rightarrow y=\frac{2x}{3-2x}$$

$$y(3-2x)=2x\Rightarrow 3y-2xy=2x\Rightarrow 3y=2xy+2x\Rightarrow 3y=2x(y+1)$$

$$2x=\frac{3y}{y+1}\Rightarrow \boxed{x=\frac{3y}{2(y+1)}}$$

10. **A ball is dropped from a height of 27 meters and it rebounds two third of the distance it falls. If it continues to fall in the same way what distance will it travel before coming to rest?** **Sargodha 2009**

Sol. According to given Condition we have $27, 2\times27\times\frac{2}{3}, 2\times27\times\frac{2}{3}\times\frac{2}{3},........$

$$S_\infty=27+2\times27\times\frac{2}{3}+2\times27\times\frac{2}{3}\times\frac{2}{3}+........$$

$$=27+2(18+12+.............),\left(a_1=18, r=\frac{12}{18}=\frac{2}{3}\right)$$

$$=27+2\left(\frac{a_1}{1-r}\right)$$

$$=27+2\left(\frac{18}{1-\frac{2}{3}}\right)=27+2\left(\frac{18}{\frac{3-2}{3}}\right)=27+2\left(\frac{18}{\frac{1}{3}}\right)=27+2\times18\times\frac{3}{1}$$

$$=27+108=135m$$

11. What distance will a ball travel before coming to rest if it is dropped from a height of 75 meters and after each fall it rebounds 2/5of distance it fell? Multan 2007

Sol. According to the given condition $75, 2\times75\times\frac{2}{5}, 2\times75\times\frac{2}{5}\times\frac{2}{5}, \ldots\ldots$

$$S_\infty = 75+2\times75\times\frac{2}{5}+2\times75\times\frac{2}{5}\times\frac{2}{5}+\ldots\ldots$$

$$=75+2(30+12+\ldots\ldots\ldots\ldots)\left(a_1=30, r=\frac{12}{30}=\frac{2}{5}\right)$$

$$S_\infty = 75+2\left(\frac{a_1}{1+r}\right)=75+2\left(\frac{30}{1-\frac{2}{5}}\right)=75+2\left(\frac{30}{\frac{5-2}{5}}\right)=75+2\left(\frac{30}{\frac{3}{5}}\right)=75+2\times30\times\frac{5}{3}$$

$$=75+100=175 \text{ meters.}$$

12. If $y=1+2x+4x^2+8x^3+\ldots\ldots$

(i) Show that $x=\frac{y-1}{2y}$

(ii) Find the interval in which the series is convergent

Sol. $y=1+2x+4x^2+8x^3+\ldots\ldots\ldots$

$a_1=1, r=\frac{2x}{1}=2x$

$y=S_\infty=\frac{a_1}{1-r}=\frac{1}{1-2x}$

$\Rightarrow \quad y(1-2x)=1$

$y-(2xy)=1\Rightarrow y-1=2xy \Rightarrow \quad \boxed{x=\frac{y-1}{2y}}$

For interval series will be convergent if

$|r|<1\Rightarrow|2x|<1\Rightarrow|x|<\frac{1}{2} \quad \Rightarrow \boxed{-\frac{1}{2}<x<\frac{1}{2}}$

13. If $y=1+\frac{x}{2}+\frac{x^2}{4}+\ldots\ldots\ldots\ldots$

(i) Show that $x=2\left(\frac{y-1}{y}\right)$

(ii) Find the interval in which the series is convergent

Sol. $y = 1 + \frac{x}{2} + \frac{x^2}{4} + \ldots\ldots\ldots\ldots$

$$a_1 = 1, r = \frac{\frac{x}{2}}{1} = \frac{x}{2}$$

$$y = S_\infty = \frac{a_1}{1-r} = \frac{1}{1-\frac{x}{2}}$$

$$y = \frac{1}{\frac{2-x}{2}} = \frac{2}{2-x}$$

$\Rightarrow \quad y(2-x) = 2 \Rightarrow \quad 2y - xy = 2 \Rightarrow \quad 2y - 2 = xy$

$\Rightarrow \quad \frac{2(y-1)}{y} = x \Rightarrow \quad \boxed{x = \frac{2(y-1)}{y}}$

Series will convergent if

$|r| < 1 \Rightarrow |x/2| < 1 \Rightarrow |x| < 2 \Rightarrow \boxed{-2 < x < 2}$

14. The sum of an infinite geometric series is 9 and the sum of the squares of its terms is 81/5. Find the series ?

Sol. Suppose in finite series $a_1 + a_1 r + a_1 r^2 + \ldots\ldots\ldots\ldots$

Condition I $\Rightarrow S_\infty = \frac{a_1}{1-r} = 9 \Rightarrow a_1 = 9(1-r) \longrightarrow I$

Condition II $\Rightarrow a_1^2 + a_1^2 r^2 + a_1^2 r^4 + \ldots\ldots\ldots = \frac{81}{5} \Rightarrow \frac{a_1^2}{1-r^2} = \frac{81}{5} \Rightarrow 5a_1^2 = 81(1-r^2)$

$5.81(1-r)^2 = 81(1-r)(1+r)$ $(use\, I)$

$5(1-r) = 1 + r \Rightarrow 5 - 5r = 1 + r \Rightarrow 5 - 1 = 5r + r \Rightarrow 4 = 6r \Rightarrow \boxed{r = 2/3}$

$\left(put\, r = \frac{2}{3}\ in\ I\right)$ $a_1 = 9\left(1 - \frac{2}{3}\right) = 9\left(\frac{1}{3}\right) \Rightarrow \boxed{a_1 = 3}$

$a_1 r = (3)\left(\frac{2}{3}\right) = 2$ *and* $a_1 r^2 = (3)\left(\frac{2}{3}\right)^2 = 3\left(\frac{4}{9}\right) = \frac{4}{3}$

So infinite series is $3 + 2 + \frac{4}{3} + \ldots\ldots\ldots\ldots$

Exercise 6.9

1. **A man deposits in a bank Rs.8 in the first year, Rs.24 in the second year Rs.72 in the third year and so on. Find the amount he will have deposited in the bank by the fifth year.**

Sol. Given $8+24+72+\dots\dots\dots+a_5$

$$a_1=8,\ r=\frac{24}{8}=3,\ n=5$$

$$S_n=\frac{a_1(r^n-1)}{r-1},|r|>1$$

$$S_5=\frac{8(3^5-1))}{3-1}=\frac{8(243-1)}{2}=4(242)=968$$

2. **A man borrows Rs.32760 without interest and agrees to repay the loan in installments, each installment being twice the preceding one. Find the amount of the last installment, if the amount of the first installment is Rs.8,**

Sol. Given $S_n=32760, r=2,\ a_1=8,\ a_n=?$

$$S_n=\frac{a_1(r^n-1)}{r-1}$$

$$32760=\frac{8(2^n-1)}{2-1}$$

$$\Rightarrow 32760=8(2^n-1)$$

$$\Rightarrow 2^n-1=\frac{32760}{8}=4095$$

$$2^n=4095+1=4096$$

$$\Rightarrow 2^n=2^{12}\Rightarrow n=12$$

Now $a_n=ar^{n-1}$

$$a_{12}=8(2)^{12-1}=8(2)^{11}$$

$$=8(2048)=16384$$

3. **The population of a certain village is 62500. What will be its population after 3 years if it increases geometrically at the rate of 4% annually?**

Sol. $a_1=62500,\ n=4$

$$r=1+4\%=1+\frac{4}{100}=1+0.04=1.04$$

$$a_n=ar^{n-1}\Rightarrow a_4=62500(1.04)^{4-1}$$

$$a_4=62500(1.04)^3=62500(1.1249)=70304$$

4. The enrolment of a famous school doubled after every eight years from 1970 to 1994. If the enrolment was 6000 in 1994. What was its enrolment in 1970?

Sol. According to the given condition $1970, 1978, 1986, 1994$ *are* a_1, a_2, a_3, a_4

$n = 4, r = 2, a_4 = 6000, a_1 = ?$

$a_n = a_1 r^{n-1}$

$a_4 = a_1 r^{4-1} \Rightarrow 6000 = a_1 (2)^3$

$\Rightarrow 6000 = 8a_1 \Rightarrow a_1 = \dfrac{6000}{8} \Rightarrow \boxed{a_1 = 750}$

5. A Singular cholera bacteria produces two complete bacteria in $1/2$ hours. If we start with a colony of a bacteria, How many bacteria will have in n hours?

Sol. Given in $1/2$ hours = 2 bacteria

$\dfrac{1}{2} + \dfrac{1}{2} = 1$ hour = 4A

$1 + \dfrac{1}{2} = \dfrac{3}{4}$ hour = 8A

$\dfrac{3}{2} + \dfrac{1}{2} = 2$ hour = 16A

$2 + \dfrac{1}{2} = \dfrac{5}{2}$ hour = 32A

$\dfrac{5}{2} + \dfrac{1}{2} = 3$ hour = 64A

So in 1, 2, 3, hours 4A, 16A, 64A, ... $n = n$

$a_n = ar^{n-1}$

$a_n = 4A(4)^{n-1}$

$= A4^n = A2^{2n}$ bacteria

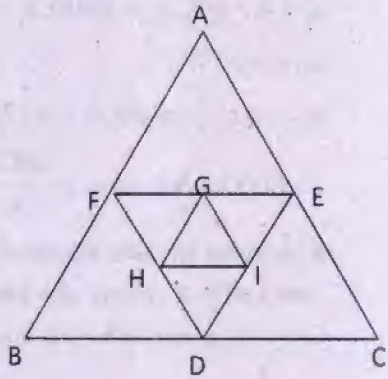
6. Joining the mid points of the sides of an equilateral triangle, an equilateral triangle having half the perimeter of the original triangle is obtained. We form a sequence of nested equilateral triangles in the same manner described above with the original triangle having perimeter $3/2$ What will be the total perimeter of all the triangles formed in this way?

Sol. According to the given condition perimeter of $\Delta ABC = 3/2$

Perimeter of triangle DEF $= \dfrac{1}{2}\left(\dfrac{3}{2}\right) = \dfrac{3}{4}$

Perimeter of triangle GHI $= \frac{1}{2}$ (perimeter of DEF)

$$= \frac{1}{2}\left(\frac{3}{4}\right) = \frac{3}{8}$$



So series is

$$= \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \ldots\ldots\ldots\ldots$$

$$a_1 = \frac{3}{2}, r = \frac{\frac{3}{4}}{\frac{3}{2}} = \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$$

$$S_\infty = \frac{a_1}{1-r} = \frac{\frac{3}{2}}{1-\frac{1}{2}} = \frac{\frac{3}{2}}{\frac{1}{2}} = \frac{3}{2} \times \frac{2}{1} = 3$$

Exercise 6.10

Theorem: **Prove that A, G, H are in G.P or** $G^2 = A \times H$ *or* $\frac{G}{A} = \frac{H}{G}$

Proof: **We know that** **Multan 2008**

$$A = \frac{a+b}{2}, G = \sqrt{ab}, H = \frac{2ab}{a+b}$$

Then $G^2 = (\sqrt{ab})^2 = ab \longrightarrow I$

$$A \times H = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab \longrightarrow II$$

Comparing I & II

$\boxed{G^2 = A \times H}$ *or* $G \times G = A \times H$

$\Rightarrow \boxed{\frac{G}{A} = \frac{H}{G}}$ It is clear that

A, G, H Here in G.P

Theorem: **Prove that** $A > G > H$

Proof: $A > G$ *if* $\frac{a+b}{2} > \sqrt{ab}$

Squaring both sides

$$\Rightarrow \frac{(a+b)^2}{4} > ab$$

$$\Rightarrow a^2 + b^2 + 2ab > 4ab$$

$$\Rightarrow a^2 + b^2 + 2ab - 4ab > 0$$

$$\Rightarrow a^2 + b^2 - 2ab > 0 \Rightarrow (a-b)^2 > 0$$

Which is True if a & b are distinct real Therefore $A > G \longrightarrow I$

Now $G > H$ *if* $\sqrt{ab} > \dfrac{2ab}{a+b}$

$$\Rightarrow ab > \frac{4a^2b^2}{(a+b)^2}$$

$$\Rightarrow ab > \frac{4(ab)(ab)}{a^2 + 2ab + b^2}$$

$$\Rightarrow (\cancel{ab})(a^2 + 2ab + b^2) > 4ab(\cancel{ab})$$

$$\Rightarrow a^2 + 2ab + b^2 - 4ab > 0$$

$$\Rightarrow a^2 + b^2 - 2ab > 0 \Rightarrow (a-b)^2 > 0$$

Which is True if a & b are different, therefore $G > H \text{———} II$

Combining I & II

$\boxed{A > G > H}$

1. Find the 9th term of the harmonic sequence:

i. $\dfrac{1}{3}, \dfrac{1}{5}, \dfrac{1}{7}, \ldots\ldots\ldots$ **Faisalabad 2008, Sargodha 2009, Multan 2010**

Sol. Given $\dfrac{1}{3}, \dfrac{1}{5}, \dfrac{1}{7}, \ldots\ldots\ldots$ in H.P, $a_9 = ?$

Then $\dfrac{3}{1}, \dfrac{5}{1}, \dfrac{7}{1}, \ldots\ldots\ldots$ are in A.P

or $3, 5, 7 \ldots\ldots\ldots\ldots$ are in A.P

$a_1 = 3, d = 5 - 3 = 2, n = 9$

$a_n = a_1 + (n-1)d$

$a_9 = 3 + (9-1)2$

$$= 3 + (8)(2) = 3 + 16 = 19 \, in \, A.P \Rightarrow a_9 = \frac{1}{19} \, in \, H.P.$$

ii. $\dfrac{-1}{5}, \dfrac{-1}{3}, -1, \ldots\ldots\ldots$ **Multan 2008**

Sol. $\frac{-1}{5}, \frac{-1}{3}, -1, \ldots\ldots\ldots$ in H.P, $a_9 = ?$

$-\frac{5}{1}, \frac{-3}{1}, \frac{-1}{1}, \ldots\ldots\ldots$ *are in A.P or* $-5, -3, -1\ldots..$ *are in A.P*

$a_1 = 5, d = -3-(-5) = -3+5 = 2, n = 9$

$a_n = a_1 + (n-1)d$

$a_9 = 5 + (9-1)(2)$

$= -5 + (8)(-2) = -5 - 16$

$a_9 = -21$ *in A.P* $\Rightarrow a_9 = -\frac{1}{21}$ *in H.P*

2. Find the 12th term of the harmonic sequence:

i. $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots\ldots\ldots$ **Faisalabad 2007, Multan 2009, Sargodha 2008, 2011**

Sol. $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots\ldots\ldots$ in H.P, $a_{12} = ?$

$\Rightarrow 2, 5, 8, \ldots\ldots\ldots$ are in A.P

$\Rightarrow a_1 = 2, d = 5-2 = 3, n = 12$

$a_n = a_1 + (n-1)(d)$

$a_{12} = 2 + (12-1)(3)$

$= 2 + (11)(3) = 2 + 33 + 35$

$a_{12} = 35$ in A.P $\Rightarrow a_{12} = \frac{1}{35}$ in H.P

ii. $\frac{1}{3}, \frac{2}{9}, \frac{1}{6}, \ldots\ldots\ldots$

Sol. $\frac{1}{3}, \frac{2}{9}, \frac{1}{6}, \ldots\ldots\ldots$ H.P, $a_{12} = ?$

$\frac{3}{1}, \frac{9}{2}, 6 \ldots\ldots\ldots\ldots$ are in A.P

$a_1 = 3, d = \frac{9}{2} - 3 = \frac{9-6}{2} = \frac{3}{2}, n = 12$

$a_n = a_1 + (n-1)(d)$

$a_{12} = 2 + (12-1)\left(\frac{3}{2}\right)$

$$=3+11\left(\frac{3}{2}\right)=3+\frac{33}{2}$$

$$a_{12}=\frac{6+33}{2}=\frac{39}{2}\text{ in A.P}\Rightarrow a_{12}=\frac{2}{39}\text{ in H.P}$$

3. Insert five harmonic means between the following given numbers,

i. $\frac{-2}{5}$ ***and*** $\frac{2}{13}$

Sol. Let H_1, H_2, H_3, H_4, H_5, be five

H.M between $-\frac{2}{5}$ & $\frac{2}{13}$ then

$\frac{-2}{5}, H_1, H_2, H_3, H_4, H_5, \frac{2}{13}$ are in H.P

$\Rightarrow \frac{-5}{2}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{1}{H_5}, \frac{13}{2}$ are in A.P

$$a_1=\frac{-5}{2}\ \&\ a_7=a_1+6d=\frac{13}{2}$$

$$\Rightarrow \frac{-5}{2}+6d=\frac{13}{2}$$

$$\Rightarrow 6d=\frac{13}{2}+\frac{5}{2}=\frac{18}{2}\Rightarrow d=\frac{18}{2}\times\frac{1}{6}\Rightarrow \boxed{d=\frac{3}{2}}$$

$$\frac{1}{H_1}=a_2=a_1+d=\frac{-5}{2}+\frac{3}{2}=\frac{-5+3}{2}=\frac{-2}{2}=-1$$

$$\frac{1}{H_2}=a_3=a_1+2d=\frac{-5}{2}+2\left(\frac{3}{2}\right)=\frac{-5}{2}+\frac{6}{2}=\frac{-5+6}{2}=\frac{1}{2}$$

$$\frac{1}{H_3}=a_4=a_1+3d=\frac{-5}{2}+3\left(\frac{3}{2}\right)=\frac{-5}{2}+\frac{9}{2}=\frac{-5+9}{2}=\frac{4}{2}=2$$

$$\frac{1}{H_4}=a_5=a_1+4d=\frac{-5}{2}+4\left(\frac{3}{2}\right)=\frac{-5}{2}+\frac{12}{2}=\frac{-5+12}{2}=\frac{7}{2}$$

$$\frac{1}{H_5}=a_6=a_1+5d=\frac{-5}{2}+5\left(\frac{3}{2}\right)=\frac{-5}{2}+\frac{15}{2}=\frac{-5+15}{2}=\frac{10}{2}=5$$

Hence $H_1=-1, H_2=2, H_3=\frac{1}{2}, H_4=\frac{2}{7}, H_5=\frac{1}{5}$

ii. $\frac{1}{4}$ ***and*** $\frac{1}{24}$

Sol. Let H_1, H_2, H_3, H_4, H_5, H.M,s between $\frac{1}{4}$ *and* $\frac{1}{24}$ then

$\frac{1}{4}, H_1, H_2, H_3, H_4, H_5, \frac{1}{24}$ are in H.P

$\Rightarrow 4, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{1}{H_5}, 24$ are in A.P

$\boxed{a_1 = 4}$ $a_7 = a_1 + 6d = 24$

$4 + 6d = 24 \Rightarrow 6d = 24 - 4$

$\Rightarrow 6d = 24 - 4 = 20 \Rightarrow d = 20/6 = 10/3$

Now

$$\frac{1}{H_1} = a_2 = a_1 + d = 4 + \frac{10}{3} = \frac{12+10}{3} = \frac{22}{3}$$

$$\frac{1}{H_2} = a_3 = a_1 + 2d = 4 + 2\left(\frac{10}{3}\right) = \frac{12+20}{3} = \frac{32}{3}$$

$$\frac{1}{H_3} = a_4 = a_1 + 3d = 4 + 3\left(\frac{10}{3}\right) = \frac{12+30}{3} = \frac{42}{3}$$

$$\frac{1}{H_4} = a_5 = a_1 + 4d = 4 + 4\left(\frac{10}{3}\right) = \frac{12+40}{3} = \frac{52}{3}$$

$$\frac{1}{H_5} = a_6 = a_1 + 5d = 4 + 5\left(\frac{10}{3}\right) = \frac{12+50}{3} = \frac{62}{3}$$

Therefore

$H_1 = \frac{3}{22}$, $H_2 = \frac{3}{32}$, $H_3 = \frac{3}{42}$

$H_4 = \frac{3}{52}$, $H_5 = \frac{3}{62}$

4. Insert four harmonic means between the following given numbers,

i. $\frac{1}{3}$ ***and*** $\frac{1}{23}$

Sol. Suppose H_1, H_2, H_3, H_4, are Four H.M,s between $\frac{1}{3}$ *and* $\frac{1}{23}$

Then $\frac{1}{3}, H_1, H_2, H_3, H_4, \frac{1}{23}$ are in H.P

$\Rightarrow 3, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, 23$ are in A.P

$a_1 = 3, a_6 = a_1 + 5d = 23$

$3 + 5d = 23$

$\Rightarrow 5d = 23 - 3 = 20 \Rightarrow \boxed{d = 4}$

$\frac{1}{H_1} = a_2 = a_1 + d = 3 + 4 = 7$

$\frac{1}{H_2} = a_3 = a_1 + 2d = 3 + 2(4) = 3 + 8 = 11$

$\frac{1}{H_3} = a_4 = a_1 + 3d = 3 + 3(4) = 3 + 12 = 15$

$\frac{1}{H_4} = a_5 = a_1 + 4d = 3 + 4(4) = 3 + 16 = 19$

Hence four H.M,s are

$H_1 = \frac{1}{7}, H_2 = \frac{1}{11}, H_3 = \frac{1}{15}, H_4 = \frac{1}{19}$

ii. $\frac{7}{3}$ ***and*** $\frac{7}{11}$

Sol. Let H_1, H_2, H_3, H_4, are Four H.M,s between $\frac{7}{3}$ *and* $\frac{7}{11}$

Then $\Rightarrow \frac{3}{7}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{11}{7}$ are in A.P

$a_1 = \frac{3}{7}, a_6 = a_1 + 5d = \frac{11}{7} - \frac{3}{7}$

$\frac{3}{7} + 5d = \frac{11}{7} \Rightarrow 5d = \frac{11}{7} - \frac{3}{7}$

$5d = \frac{8}{7} \Rightarrow d = \frac{8}{7} \times \frac{1}{5} = \frac{8}{35}$

$\frac{1}{H_1} = a_2 = a_1 + d = \frac{3}{7} + \frac{8}{35} = \frac{15+8}{35} = \frac{23}{35}$

$\frac{1}{H_2} = a_3 = a_1 + 2d = \frac{3}{7} + 2\left(\frac{8}{35}\right) = \frac{3}{7} + \frac{16}{35} = \frac{15+16}{35} = \frac{31}{35}$

$$\frac{1}{H_3}=a_4=a_1+3d=\frac{3}{7}+3\left(\frac{8}{35}\right)=\frac{3}{7}+\frac{34}{35}=\frac{15+24}{35}=\frac{39}{35}$$

$$\frac{1}{H_4}=a_5=a_1+4d=\frac{3}{7}+4\left(\frac{8}{35}\right)=\frac{3}{7}+\frac{32}{35}=\frac{15+32}{35}=\frac{47}{35}$$

Hence required H.M,s are

$$H_1=\frac{35}{23},H_2=\frac{35}{31},H_3=\frac{35}{39},H_4=\frac{35}{47}$$

iii. $4\,and\,20$ **Sargodha 2010**

Sol. Suppose H_1,H_2,H_3,H_4, are Four H.M,s between $4\,and\,20$

Then 4, H_1,H_2,H_3,H_4, 20 are H.P

$\Rightarrow\frac{1}{4},\frac{1}{H_1},\frac{1}{H_2},\frac{1}{H_3},\frac{1}{H_4},\frac{1}{20}$ are in A.P

$$a_1=\frac{1}{4},a_6=a_1+5d=\frac{1}{20}$$

$$\frac{1}{4}+5d=\frac{1}{20}\Rightarrow 5d=\frac{1}{20}-\frac{1}{4}$$

$$5d=\frac{1-5}{20}=\frac{-4}{20}\Rightarrow d=\frac{-4}{20}\times\frac{1}{5}=\frac{-2}{50}=\frac{-1}{25}$$

$$\frac{1}{H_1}=a_2=a_1+d=\frac{1}{4}-\frac{1}{25}=\frac{25-4}{100}=\frac{21}{100}$$

$$\frac{1}{H_2}=a_3=a_1+2d=\frac{1}{4}+2\left(\frac{-1}{25}\right)=\frac{1}{4}-\frac{2}{25}=\frac{25-8}{100}=\frac{17}{100}$$

$$\frac{1}{H_3}=a_4=a_1+3d=\frac{1}{4}+3\left(\frac{-1}{25}\right)=\frac{1}{4}-\frac{3}{25}=\frac{25-12}{100}=\frac{13}{100}$$

$$\frac{1}{H_4}=a_5=a_1+4d=\frac{1}{4}+4\left(\frac{-1}{25}\right)=\frac{1}{4}-\frac{4}{25}=\frac{25-16}{100}=\frac{9}{100}$$

Hence required H.M,s are

$$H_1=\frac{100}{21},H_2=\frac{100}{17},H_3=\frac{100}{13},H_4=\frac{100}{9}$$

5. **If the 7th and 10th terms of an H.P are $\frac{1}{3}$ *and* $\frac{5}{21}$ respectively, find its 14th term.**

Sol. $a_7=\frac{1}{3}$ & $a_{10}=\frac{5}{21}$, $a_{14}=?$ in H.P

$a_7 = 3$ & $a_{10} = \frac{21}{5}$ in A.P

$\Rightarrow a_7 = a_1 + 6d = 3 \longrightarrow I$

$a_{10} = a_1 + 9d = \frac{21}{5} \longrightarrow II$

$II - I$

$$a_1 + 9d = \frac{21}{5}$$

$$\underline{{}_{-}a_1 \pm 6d = 3}$$

$$3d = \frac{21}{5} - 3 = \frac{21-15}{5} = \frac{6}{5}$$

$$\Rightarrow d = \frac{6}{5} \times \frac{1}{3} \Rightarrow \boxed{d = \frac{2}{5}}$$

Put in I

$$a_1 + 6\left(\frac{2}{5}\right) = 3 \Rightarrow a_1 + \frac{12}{5} = 3$$

$$a_1 = 3 - \frac{12}{5} = \frac{15-12}{5} = \frac{3}{5} \Rightarrow \boxed{a_1 = \frac{3}{5}}$$

$$a_n = a_1 + (n-1)d$$

$$a_{14} = \frac{3}{5} + (14-1)\left(\frac{2}{5}\right) = \frac{3}{5} + 13\left(\frac{2}{5}\right) = \frac{3}{5} + \frac{26}{5} = \frac{3+26}{5} = \frac{29}{5}$$

$$a_{14} = \frac{29}{5} \text{ in A.P} \Rightarrow \boxed{a_{14} = \frac{5}{29} \; in \; H.P}$$

6. If the First term of an H.P is $-1/3$ and the fifth term is $1/5$ Find its 9[th] term?

Sol. $a_1 = -\frac{1}{3}$, $a_5 = \frac{1}{5}$, $a_9 = ?$ in H.P **Multan 2008**

$a_1 = -3 \longrightarrow I$, $a_5 = 5$ in A.P

$a_5 = a_1 + 4d = 5 \Rightarrow -3 + 4d = 5 \Rightarrow 4d = 8 \Rightarrow \boxed{d = 2}$ $use - I$

$$a_n = a_1 + (n-1)d$$

$$a_9 = -3 + (9-1)(2)$$

$$a_9 = -3 + 8(2) = -3 + 16 = 13$$

$$a_9 = 13 \text{ in A.P} \Rightarrow \boxed{a_9 = 1/13 \; in \; H.P}$$

7. If 5 is the harmonic mean between 2 and b, find b?

Sol. Given $a=2, b=b,\ H.M=5$

$$H.M = \frac{2ab}{a+b}$$

Sgd, 2010, Fsd 2008, 2009 Multan 2007, Lahore 2009

Put values

$$5=\frac{2(2)b}{2+b} \Rightarrow 5=\frac{4b}{2+b}$$

$$\Rightarrow 5(2+b)=4b$$

$$\Rightarrow 10+5b=4b$$

$$\Rightarrow 10+5b-4b=0$$

$$\Rightarrow 10+b=0 \Rightarrow \boxed{b=-10}$$

8. If the numbers $\frac{1}{k}, \frac{1}{2k+1}$ *and* $\frac{1}{4k-1}$ are in Harmonic sequence, Find k.

Sol. $\frac{1}{k}, \frac{1}{2k+1}, \frac{1}{4k-1}$ are in H.P

Sargodha 2008, Fsd 2009, Multan 2008

$k, 2k+1, 4k-1$ are in A.P

$$\Rightarrow 4k-1-(2k+1)=2k+1-k$$

$$4k-1-2k-1=k+1$$

$$2k-2-k-1=0$$

$$\Rightarrow k-3=0 \Rightarrow \boxed{k=3}$$

9. Find n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be H.M between a and b.

Sol. $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ be H.M between a & b then

Faisalabad 2008, Lahore 2009

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\frac{2ab}{a+b}$$

$$(a^{n+1}+b^{n+1})(a+b)=(a^n+b^n)(2ab)$$

$$a^{n+2}+a^{n+1}b+ab^{n+1}+b^{n+2}=2a^{n+1}b+2ab^{n+1}$$

$$a^{n+2}+b^{n+2}=2ab^{n+1}+2ab^{n+1}-a^{n+1}b-ab^{n+1}$$

$$a^{n+2}+b^{n+2}=a^{n+1}b+ab^{n+1}$$

$$a^{n+2}-a^{n+1}b=ab^{n+1}-b^{n+2}$$

$$a^{n+1}(a-b)=b^{n+1}(a-b) \qquad \div \textit{both side by} (a-b)$$

$$\frac{a^{n+1}}{b^{n+1}}=1 \Rightarrow \left(\frac{a}{b}\right)^{n+1}=\left(\frac{a}{b}\right)^{0} \Rightarrow n+1=0 \Rightarrow \boxed{n=-1} \qquad \textit{Note} \ \boxed{1=\left(\frac{a}{b}\right)^{0}}$$

10. **If a^2, b^2 and c^2 are in A.P. Show that $a+b, c+a$ and $b+c$ are in H.P.**

Sol. a^2, b^2, c^2 are in A.P

Then $b^2-a^2=c^2-b^2$ ——— I

Now $a+b, c+a, b+c$ are in H.P

If $\frac{1}{a+b}, \frac{1}{c+a}, \frac{1}{b+c}$ are in A.P $\Rightarrow \frac{1}{b+c}-\frac{1}{c+a}=\frac{1}{c+a}-\frac{1}{a+b}$

$\Rightarrow \frac{(c+a)-(b+c)}{(b+c)(c+a)}=\frac{a+b-c-a}{(a+b)(c+a)}$

$\frac{\cancel{c}+a-b-\cancel{c}}{(b+c)(c+a)}=\frac{\cancel{a}+b-c-\cancel{a}}{(a+b)(c+a)}$

'×' both sides by $(c+a)$

$\cancel{(c+a)}\frac{(a-b)}{(b+c)\cancel{(c+a)}}=\cancel{(c+a)}\frac{(b-c)}{(a+b)\cancel{(c+a)}} \Rightarrow \frac{a-b}{b+c}=\frac{b-c}{a+b}$

By cross multiplication

$(a+b)(a-b)=(b+c)(b-c)$

$a^2-b^2=b^2-c^2$

'×' both sides by (-1)

$b^2-a^2=c^2-b^2 \Rightarrow b^2-a^2=b^2-a^2$ $(use\ I)$

Hence proved.

11. **The sum of the first and fifth terms of the harmonic sequence is $4/7$ if the first term is $1/2$ Find the sequence.**

Sol. $a_1+a_5=\frac{4}{7}, a_1=\frac{1}{2}$ in H.P

$\Rightarrow \frac{1}{2}+a_5=\frac{4}{7} \Rightarrow a_5=\frac{4}{7}-\frac{1}{2}=\frac{8-7}{14}=\frac{1}{14}$

$a_5=1/14, a_1=1/2\ in\ H.P.$

$a_5=14, a_1=2\ in\ A.P$

$a_5=a_1+4d=14$

$a_5=2+4d=14 \Rightarrow 4d=14-2=12 \Rightarrow \boxed{d=3}$

$Now\ a_1=2, a_2=a_1+d=2+3=5$

$a_3=a_1+2d=2+2(3)=8$

$2, 5, 8, \ldots\ldots\ldots\ldots in\ A.P\ and\ 1/2, 1/5, 1/8, \ldots\ldots in\ H.P$

12. **Find A, G, H and show that $G^2 = A.H$. If**

i. $a=-2, b=-6$ **Multan 2010**

Sol. $A=\dfrac{a+b}{2}=\dfrac{-2-6}{2}=\dfrac{-8}{2}=-4$

$G=\pm\sqrt{ab}=\pm\sqrt{(-2)(-6)}=\pm\sqrt{12}$

$=\pm\sqrt{2\times2\times3}=\pm2\sqrt{3}$

$H=\dfrac{2ab}{a+b}=\dfrac{2(-2)(-6)}{-2-6}=\dfrac{24}{-8}=-3$

$G^2=(\pm\sqrt{12})^2=12 \text{———} I$

$A\times H=(-4)(-3)=12 \text{———} II$

From I and II $\boxed{G^2=AH}$

ii. $a=2i, b=4i$

Sol. $A=\dfrac{a+b}{2}=\dfrac{2i+4i}{2}=\dfrac{6i}{2}=3i$

$G=\pm\sqrt{ab}=\pm\sqrt{(2i)(4i)}=\pm\sqrt{-8}$

$H=\dfrac{2ab}{a+b}=\dfrac{2(2i)(4i)}{2i+4i}=\dfrac{16i^2}{6i}=\dfrac{-8}{3i}$

$G^2=(\pm\sqrt{-8})^2=-8 \text{———} I$

$AH=(3i)\left(\dfrac{-8}{3i}\right)=-8 \text{———} II$

From I and II $\boxed{G^2=AH}$

iii. $a=9, b=4$

Sol. $A=\dfrac{a+b}{2}=\dfrac{9+4}{2}=\dfrac{13}{2}$

$G=\pm\sqrt{ab}=\pm\sqrt{9\times4}=\pm\sqrt{36}=\pm6$

$H=\dfrac{2ab}{a+b}=\dfrac{2(9)(4)}{9+4}=\dfrac{72}{13}$

$G^2=(\pm6)^2=36$ *and* $AH=\dfrac{13}{2}\times\dfrac{72}{13}=36$ *and* $\boxed{G^2=AH}$

13. **Find A, G, H and verify that $A > G > H (G > 0)$, if**

i. $a = 2, b = 8$ **Federal**

Sol. $A = \frac{a+b}{2} = \frac{2+8}{2} = \frac{10}{2} = 5$

$G = \pm\sqrt{ab} = \pm\sqrt{(2)(8)} = \pm\sqrt{16} = \pm 4 = 4 (G > 0)$

$H = \frac{2ab}{a+b} = \frac{2(2)(8)}{2+8} = \frac{32}{10} = \frac{16}{5}$

Hence $A > G > H$ because $5 > 4 > \frac{16}{5}$

ii. $a = \frac{2}{5}, b = \frac{8}{5}$

Sol. $A = \frac{\frac{2}{5}+\frac{8}{5}}{2} = \frac{\frac{2+8}{5}}{2} = \frac{\frac{10}{5}}{2} = \frac{2}{2} = 1$

$G = \pm\sqrt{ab} = \pm\sqrt{\frac{2}{5}\times\frac{8}{5}} = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5} = \frac{4}{5} (G > 0)$

$H = \frac{2ab}{a+b} = \frac{2\left(\frac{2}{5}\right)\left(\frac{8}{5}\right)}{\frac{2}{5}+\frac{8}{5}} = \frac{\frac{32}{25}}{\frac{8+2}{5}} = \frac{\frac{32}{25}}{\frac{10}{5}} = \frac{32}{25}\times\frac{5}{10} = \frac{16}{25}$

$A = 1, G = \frac{4}{5}, H = \frac{16}{25}$ Therefore $\boxed{A > G > H}$

14. **Find A, G, H and verify that $A < G < H (G < 0)$, if**

i. $a = -2, b = -8$ **Sargodha 2009**

Sol. $A = \frac{a+b}{2} = \frac{-2-8}{2} = \frac{-10}{2} = -5$

$G = \pm\sqrt{ab} = \pm\sqrt{(-2)(-8)} = \pm\sqrt{16} = -4 (G < 0)$

$H = \frac{2ab}{a+b} = \frac{2(-2)(-8)}{-2-8} = \frac{32}{-10} = \frac{-16}{5} = -3.2$

$-5 < -4 < -3.2$ *or* $\boxed{A < G < H}$

ii. $a=\dfrac{-2}{5}, b=\dfrac{-8}{5}$

Sol. $A=\dfrac{a+b}{2}=\dfrac{\frac{-2}{5}+\frac{-8}{5}}{2}=\dfrac{\frac{-2-8}{5}}{2}=\dfrac{\frac{-10}{5}}{2}=\dfrac{-2}{2}=-1$

$G=\pm\sqrt{ab}=\pm\sqrt{\dfrac{-2}{5}\left(\dfrac{-8}{5}\right)}=\pm\sqrt{\dfrac{16}{25}}=\dfrac{-4}{5}(G<0)$

$H=\dfrac{2ab}{a+b}=\dfrac{2\left(-\frac{2}{5}\right)\left(-\frac{8}{5}\right)}{\frac{-2}{5}-\frac{8}{5}}=\dfrac{\frac{32}{25}}{\frac{-2-8}{5}}=\dfrac{\frac{32}{25}}{\frac{-10}{5}}=\dfrac{32}{25}\times\dfrac{-5}{10}=-\dfrac{16}{25}=-0.6$

$A=-1, G=-0.8, H=-0.6 \Rightarrow \boxed{A<G<H}$

15. If the H.M and A.M between two numbers are 4 and $9/2$ respectively, find the numbers. Multan 2009

Sol. $H.M=4, A.M=\dfrac{9}{2} \quad a,b=?$

$H.M=\dfrac{2ab}{a+b}\Rightarrow 4=\dfrac{2ab}{a+b}\longrightarrow I$

$A.M=\dfrac{a+b}{2}\Rightarrow\dfrac{9}{2}=\dfrac{a+b}{2}$

$\Rightarrow a+b=9\longrightarrow II$

(Put II in I) $\Rightarrow 4=\dfrac{2ab}{9}\Rightarrow 2ab=36$

$\Rightarrow ab=\dfrac{36}{2}=18\Rightarrow ab=18$ ——— III *from II* $a+b=9\Rightarrow a=9-b$

Put value in III

$(9-b)b=18\Rightarrow 9b-b^2-18=0$

$\Rightarrow b^2-9b+18=0$

$\Rightarrow b^2-3b-6b+18=0$

$\Rightarrow b^2(b-3)-6(b-3)=0$

$\Rightarrow b-3=0 \ or \ b-6=0\Rightarrow b=3 \ or \ b=6$

When $b=3$ then $a=9-3=6$

When $b=6$ then $a=9-6=3$

Numbers are 6, 3 or 3,6

16. If the (positive) G.M and H.M between two numbers are 4 and $16/5$, find the numbers. **Sargodha 2008**

Sol. $G.M = 4,\ H.M = \frac{16}{5} \quad a, b = ?$

$G.M = \sqrt{ab} \Rightarrow 4 = \sqrt{ab} \Rightarrow ab = 16 \text{———} I$

$H.M = \frac{2ab}{a+b} \Rightarrow \frac{16}{5} = \frac{2ab}{a+b}$

$\Rightarrow \frac{16}{5} = \frac{2(16)}{a+b} \Rightarrow \frac{1}{5} = \frac{2}{a+b}$

By cross multiplication

$a + b = 10 \longrightarrow II$

From II $a = 10 - b \longrightarrow III$

I $become$ $(10-b)b = 16 \Rightarrow 10b - b^2 - 16 = 0 \Rightarrow b^2 - 10b + 16 = 0$

$\Rightarrow b^2 - 2b - 8b + 16 = 0$

$\Rightarrow b(b-2) - 8(b-2) = 0$

$\Rightarrow (b-2)(b-8) = 0$

$\Rightarrow b - 2 = 0 \quad or \quad b - 8 = 0$

$\Rightarrow b = 2 \quad or \quad b = 8$

When $b = 2$ then $a = 10 - 2 = 8$

When $b = 8$ then $a = 10 - 8 = 2$

Numbers are 8,2 or 2,8

17. If the numbers $\frac{1}{2}, \frac{4}{21}$ *and* $\frac{1}{36}$, are subtracted from the three consecutive terms of G.P, the resulting numbers are in H.P. Find the numbers if their product is $\frac{1}{27}$

Sol. Suppose three numbers in G.P are $\frac{a_1}{r}, a_1, a_1 r$ then

Condition II $\Rightarrow \left(\frac{a_1}{r}\right)(a_1)(a_1 r) = \frac{1}{27}$ **Faisalabad 2008, Sargodha 2009**

$\Rightarrow a_1^3 = \frac{1}{27} = \left(\frac{1}{3}\right)^3 \Rightarrow a_1 = \frac{1}{3}$

Condition I $\Rightarrow \frac{a_1}{r} - \frac{1}{2}, a_1 - \frac{4}{21}, a_1 r - \frac{1}{36}$ are in H.P

$\frac{1}{3r}-\frac{1}{2},\frac{1}{3}-\frac{4}{21},\frac{1}{3}r-\frac{1}{36}$ in H.P

$\frac{1}{3r}-\frac{1}{2},\frac{7-4}{21}-\frac{12r-1}{36}$ in H.P

$\frac{2-3r}{6r},\frac{3}{21},\frac{12r-1}{36}$ in H.P

$\Rightarrow\frac{2-3r}{6r},\frac{1}{7},\frac{12r-1}{36}$ in H.P

$\Rightarrow\frac{6r}{2-3r},7,\frac{36}{12r-1}$ in A.P

$\Rightarrow\frac{36}{12r-1}-7=7-\frac{6r}{2-3r}$

$\Rightarrow\frac{36}{12r-1}+\frac{6r}{2-3r}=7+7=14$

'×' both sides by $(12r-1)(2-3r)$

$36(2-3r)+6r(12r-1)=14(12r-1)(2-3r)$

$72-108r+72r^2-6r=14(24r-36r^2-2+3r)$

$72r^2-114r+72=14(-36r^2+27r-2)$

$72r^2-144r+72=-504r^2+378r^2-28$

$72r^2-144r+72+504r^2-378r^2+28=0$

$576r^2-492r+100=0$

$144r^2-123r+25=0$ ('÷' by 4)

$144r^2-75r-48r+25=0$

$3r(48r-25)-1(48r-25)=0$

$(48r-25)(3r-1)=0$

$48r-25=0$ *or* $3r-1=0$

$r=\frac{25}{48}$ *or* $r=\frac{1}{3}$

When $r=\frac{25}{48}$ & $a_1=\frac{1}{3}$

$\frac{a_1}{r}=\frac{\frac{1}{3}}{\frac{25}{48}}=\frac{1}{3}\times\frac{48}{25}=\frac{16}{25}, a_1=\frac{1}{3}, a_1r=\frac{1}{3}\left(\frac{25}{48}\right)=\frac{25}{144}$

When $a_1 = \frac{1}{3}, \quad r = \frac{1}{3}$

$$\frac{a_1}{r} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1, a_1 = \frac{1}{3}, a_1 r = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{9}$$

Required numbers are

$$\frac{16}{25}, \frac{1}{3}, \frac{25}{144} \quad or\ 1, \frac{1}{3}, \frac{1}{9}$$

Sigma Notations

$$1+2+3+\ldots\ldots\ldots\ldots\ldots+n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$ **Multan 2008, Faisalabad 2007, sgd 2008**

$$1^2+2^2+3^2+\ldots\ldots.+n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3+2^3+3^3\ldots\ldots\ldots\ldots+n^3 = \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$and\ S_n = \sum_{k=1}^{n} T_k$$

Exercise 6.11

Sum the following series upto n terms.

1. $\mathbf{1\times 1+2\times 3\times 7+\ldots\ldots\ldots\ldots}$

Sol. $1\times 1+2\times 3\times 7+\ldots\ldots\ldots\ldots n\ term$

$$T_k=[1+(k-1)(1)]\times[1+(k-1)(3)]$$

$$T_k=(1+k-1)(1+3k-3)=k(3k-2)=3k^2-2k$$

$$S_n=\sum_{k=1}^{n}T_k=\sum_{k=1}^{n}(3k^2-2k)=3\sum_{k=1}^{n}k^2-2\sum_{k=1}^{n}k$$

$$=3\frac{n(n+1)(2n+1)}{6}-\frac{2.2(n+1)}{2}=\frac{n(n+1)(2n+1)}{2}-\frac{2n(n+1)}{2}$$

$$=\frac{n(n+1)}{2}[2n+1-2]=\frac{n(n+1)(2n-1)}{2}$$

2. $\mathbf{1\times 3+3\times 6+5\times 9\ldots\ldots\ldots\ldots}$

Sol. $1\times 3+3\times 6+5\times 9\ldots\ldots\ldots\ldots n\ term$

$$T_k=[1+(k-1)2]\times[3+(k-1)3]=(1+2k-2)(3+3k-3)$$

$$=(2k-1)(3k)=6k^2-3k$$

$$S_n=\sum_{k=1}^{n}T_k=\sum_{k=1}^{n}(6k^2-3k)=6\sum_{k=1}^{n}K^2-3\sum_{k=1}^{n}K$$

$$=\cancel{6}\frac{n(n+1)(2n+1)}{\cancel{6}}-\frac{3.n(n+1)}{2}=n(n+1)\left[2n+1-\frac{3}{2}\right]$$

$$=n(n+1)\left[\frac{4n+2-3}{2}\right]=\frac{n(n+1)(4n-1)}{2}$$

3. $\mathbf{1\times 4+2\times 7+3\times 10+\ldots\ldots\ldots\ldots}$

Sol. $1\times 4+2\times 7+3\times 10+\ldots\ldots\ldots\ldots nterm$

$$T_k=[1+(k-1)1]\times[4+(k-1)3]=(1+k-1)(4+3k-3)=k(3k+1)=3k^2+k$$

$$S_n=\sum_{k=1}^{n}T_k=3\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k$$

$$=3\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}=\frac{n(n+1)(2n+1)}{2}+\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{2}[2n+1+1]=\frac{n(n+1)(2n+2)}{2}=\frac{2n(n+1)(n+1)}{2}=n(n+1)^2$$

4. $\mathbf{3\times 5+5\times 9+7\times 13+\ldots\ldots\ldots\ldots}$

Sol. $3\times 5+5\times 9+7\times 13+\ldots\ldots\ldots\ldots n\ term$

$$T_k = [3+(k-1)2]\times[5+(k-1)4] = (3+2k-2)(5+4k-4) = (2k+1)(4k+1)$$
$$= 8k^2+2k+4k+1 = 8k^2+6k+1$$
$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(8k^2+6k+1)$$
$$= 8\sum_{k=1}^{n} k^2 + 6\sum_{k=1}^{n} k + n$$
$$= \frac{8n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2} + n$$
$$= n\left[\frac{4(n+1)(2n+1)}{3} + 3(n+1) + 1\right] = n\left[\frac{4(2n^2+n+2n+1)+9(n+1)+3}{3}\right]$$
$$= n\frac{[8n^2+12n+4+9n+9+3]}{3} = \frac{n}{3}(8n^2+21n+16)$$

5. $1^2+3^2+5^2+\ldots\ldots\ldots$ **Multan 2009**

Sol. $1^2+3^2+5^2+\ldots\ldots\ldots+n\ term$

$$T_k = [1+(k-1)2]^2 = (1+2k-2)^2 = (2k-1)^2$$
$$T_k = 4k^2-4k+1$$
$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(4k^2-4k+1)$$
$$= 4\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + n$$
$$= \frac{4n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + n$$
$$= n\left[\frac{2(n+1)(2n+1)}{3} - 2(n+1) + 1\right] = n\left[\frac{4n^2+6n+2}{3} - 2n - 1\right]$$
$$= n\left[\frac{4n^2+\cancel{6n}+2-\cancel{6n}-3}{3}\right] = \frac{n}{3}(4n^2-1)$$

6. $2^2+5^2+8^2+\ldots\ldots\ldots$ **Multan 2008**

Sol. $2^2+5^2+8^2+\ldots\ldots\ldots+n\ term$

$$T_k = [2+(k-1)3]^2 = (2+3k-3)^2 = (3k-1)^2$$
$$T_k = 9k^2-6k+1$$

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(9k^2 - 6k + 1) = 9\sum_{k=1}^{n} k^2 - 6\sum_{k=1}^{n} k + n$$

$$= 9\frac{n(n+1)(2n+1)}{6} - \frac{6n(n+1)}{2} + n$$

$$= n\left[\frac{3(n+1)(2n+1)}{2} - 3(n+1) + 1\right] = n\left[\frac{6n^2+9n+3}{2} - 3n - 2\right]$$

$$= n\left(\frac{6n^2+9n+3-6n-4}{2}\right) = \frac{n}{2}(6n^2+3n-1)$$

7. $2\times1^2 + 4\times2^2 + 6\times3^2 + \ldots\ldots\ldots$

Sol. $2\times1^2 + 4\times2^2 + 6\times3^2 + \ldots\ldots\ldots + n \text{ term}$

$$T_k = [2+(k-1)2].[1+(k-1)1]^2$$

$$= (2+2k-2)(1+k-1)^2 = 2k(k^2) = 2k^3$$

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} 2k^3 = 2\sum_{k=1}^{n} k^3 = 2\left(\frac{n(n+1)}{2}\right)^2$$

$$= \frac{2n^2(n+1)^2}{4} = \frac{n^2(n+1)^2}{2}$$

8. $3\times2^2 + 5\times3^2 + 7\times4^2 + \ldots\ldots\ldots$

Sol. $3\times2^2 + 5\times3^2 + 7\times4^2 + \ldots\ldots\ldots + n \text{ term}$

$$T_k = [3+(k-1)2]\times[2+(k-1)1]^2$$

$$= (3+2k-2)(2+k-1)^2 = (2k+1)(k+1)^2$$

$$= (2k+1)(k^2+2k+1) = 2k^3+4k^2+2k+k^2+2k+1 = 2k^3+5k^2+4k+1$$

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(2k^3+5k^2+4k+1)$$

$$= 2\sum_{k=1}^{n} k^3 + 5\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + n$$

$$= 2\left(\frac{n(n+1)}{2}\right)^2 + \frac{5n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} + n$$

$$= 2\left[\frac{n^2(n^2+2n+1)}{4}\right] + \frac{5n(2n^2+2n+n+1)}{6} + 2n(n+1) + n$$

$$=n\left[\frac{n(n^2+2n+1)}{2}+\frac{10n^2+10n+5n+5}{6}+2n+2+1\right]$$

$$=n\left[\frac{3n^3+6n^2+3n+10n^2+15n+5+12n+18}{6}\right]$$

$$=\frac{n}{6}(3n^3+16n^2+30n+23)$$

9. $\mathbf{2\times4\times7+3\times6\times10+4\times8\times13+\ldots\ldots\ldots}$

Sol. $2\times4\times7+3\times6\times10+4\times8\times13+\ldots\ldots\ldots$ *to n term*

$$T_k=[2+(k-1)1]\times[4+(k-1)2]\times[7+(k-1)3]$$

$$=(2+k-1)(4+2k-2)\times(7+3k-3)$$

$$=(k+1)(2k+2)(3k+4)=(k+1)(6k^2+14k+8)$$

$$=6k^2+14k^2+8k+6k^2+14k+8=6k^3+20k^3+22k+8$$

$$S_n=\sum_{k=1}^{n}T_k=\sum_{k=1}^{n}(6k^3+20k^3+22k+8)$$

$$=6\sum_{k=1}^{n}k^3+20\sum_{k=1}^{n}k^2+22\sum_{k=1}^{n}k+8n$$

$$=6\left[\frac{n(n+1)}{2}\right]^2+\frac{20n(n+1)(2n+1)}{6}+\frac{22n(n+1)}{2}+8n$$

$$S_n=\frac{6n^2(n+1)^2}{4}+\frac{20n(n+1)(2n+1)}{6}+\frac{22n(n+1)}{2}+8n$$

$$=n\left[\frac{3n(n^2+2n+1)}{2}+\frac{10(2n^2+3n+1)}{3}+11(n+1)+8\right]$$

$$=n\left[\frac{3n^3+6n^2+3n}{2}+\frac{20n^2+30n+10}{3}+11n+11+8\right]$$

$$=n\left[\frac{9n^3+18n^2+9n+40n^2+60n+20+66n+66+48}{6}\right]$$

$$=\frac{n}{6}(9n^3+58n^2+135n+134)$$

10. $\mathbf{1\times4\times6+4\times7\times10+7\times10\times14+\ldots\ldots\ldots}$

Sol. $1\times4\times6+4\times7\times10+7\times10\times14+\ldots\ldots\ldots+$ *n terms*

$$T_k=[1+(k-1)3]\times[4+(k-1)3]\times[6+(k-1)4]$$

$$=(1+3k-3)\times(4+3k-3)\times(4k+2)$$
$$=(3k-2)\times(3k+1)\times(4k+2)$$
$$=(3k-2)(12k^2+10k+2)=36k^3+30k^2+6k-24k^2-20k-4$$
$$=36k^3+6k^2-14k-4$$

$$S_n=\sum_{k=1}^{n}T_k=\sum_{k=1}^{n}(36k^3+6k^2-14k-4)$$
$$=36\sum_{k=1}^{n}k^3+6\sum_{k=1}^{n}k^2-14\sum_{k=1}^{n}k-4n$$
$$=36\left(\frac{n(n+1)}{2}\right)^2+\frac{\cancel{6}n(n+1)(2n+1)}{\cancel{6}}-\frac{14n(n+1)}{2}-4n$$
$$=\cancel{36}\frac{n^2(n^2+2n+1)}{\cancel{4}}+n(n+1)(2n+1)-7n(n+1)-4n$$
$$=n\left[9n(n^2+2n+1)+(2n^2+2n+n+1)-7(n+1)-4\right]$$
$$=n(9n^3+18n^2+9n+2n^2+3n+1-7n-7-4)$$
$$=n(9n^3+20n^2+5n-10)$$

11. $\mathbf{1+(1+2)+(1+2+3)+\ldots\ldots\ldots}$

Sol. $1+(1+2)+(1+2+3)+\ldots\ldots\ldots+$ *n term*

$$T_n=1+2+3+\ldots\ldots\ldots n=\frac{n(n+1)}{2}$$
$$T_k=\frac{k(k+1)}{2}$$

$$\boxed{\begin{aligned}S_n&=\frac{n}{2}\left(2a_1+(n-1)d\right)=\frac{n}{2}\left(2(1)+(n-1)1\right)\\&=\frac{n}{2}(2+n-1)=\frac{n(n+1)}{2}\end{aligned}}$$

$$S_n=\sum_{k=1}^{n}T_k=\sum_{k=1}^{n}\frac{k(k+1)}{2}=\frac{1}{2}(k^2+k)$$
$$=\frac{1}{2}\left[\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k\right]=\frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right]$$
$$=\frac{1}{2}\cdot\frac{n(n+1)}{2}\left(\frac{2n+1}{3}+1\right)=\frac{n(n+1)}{4}\left(\frac{2n+1+3}{3}\right)$$
$$=\frac{n(n+1)(2n+4)}{4\times3}=\frac{n(n+1)\cancel{2}(n+2)}{\cancel{2}\times2\times3}=\frac{n(n+1)(n+2)}{6}$$

12. $\mathbf{1^2+(1^2+2^2)+(1^2+2^2+3^2)+.........}$

Sol. $1^2+(1^2+2^2)+(1^2+2^2+3^2)+..........+ n\ term$

$$T_n = 1^2+2^2+3^2+..........n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$T_k = \frac{k(k+1)(2k+1)}{6} \qquad \boxed{\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+.....+n^2 = \frac{n(n+1)2n+1)}{6}}$$

$$T_k = \frac{1}{6}\left[k(2k^2+2k+k+1)\right] = \frac{1}{6}\left[2k^3+3k^2+k\right]$$

$$S_n = \sum_{k=1}^{n} T_k = \frac{1}{6}\left[2\sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k\right]$$

$$= \frac{1}{6}\left[2\left[\frac{(n(n+1))}{2}\right]^2 + \frac{\cancel{3}n(n+1)(2n+1)}{\cancel{6}_2} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{6}\left[\cancel{2}_1\frac{n(n+1)}{4_2} + \frac{2n+1}{2} + \frac{1}{2}\right]$$

$$= \frac{n(n+1)}{6\times 2}\left[n^2+n+2n+1+1\right]$$

$$= \frac{n(n+1)}{12}\left[n^2+3n+2\right]$$

13. $\mathbf{2+(2+5)+(2+5+8)+.................}$

Sol. $2+(2+5)+(2+5+8)+.................+ n\ term$

$T_n = 2+5+8.........+n\ term$

$a_1 = 2,\ d = 3,\ n = n$

$$T_n = \frac{n}{2}[2a_1+(n-1)d] = \frac{n}{2}[2(2)+(n-1)3] = \frac{n}{2}[4+3n-3]$$

$$T_n = \frac{n}{2}(3n+1) = \frac{3n^2+n}{2} \Rightarrow T_k = \frac{3k^2+k}{2}$$

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{3k^2+k}{2}$$

$$= \frac{1}{2}\left[3\ \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k\right]$$

$$=\frac{1}{2}\left[\frac{\cancel{3}n(n+1)(2n+1)}{\cancel{6}_2}+\frac{n(n+1)}{2}\right]=\frac{n(n+1)}{2\times2}[2n+1+1]$$

$$=\frac{n(n+1)}{4}[2n+2]=\frac{n(n+1)}{\cancel{4}_2}\cancel{2}(n+1)$$

$$=\frac{n(n+1)(n+1)}{2}=\frac{n(n+1)^2}{4}$$

14. Sum the series

i. $1^2-2^2+3^2-4^2+\dots\dots\dots+(2n-1)^2-(2n)^2$

Sol. $T_n=(2n-1)^2-(2n)^2=4n^2-4n+1-4n^2=-4n+1$

$T_k=-4k+1$

$$S_n=\sum_{k=1}^{n}T_k=\sum_{k=1}^{n}(-4k+1)=-\sum_{k=1}^{n}k+n$$

$$=-4\frac{n(n+1)}{2}+n=-2n(n+1)+n=-2n^2-2n+n$$

$$=-2n^2-n=-n(2n+1)$$

ii. $1^2-3^2+5^2-7^2+\dots\dots\dots+(4n-3)^2-(4n+1)^2$

Sol. $T_n=(4n-3)^2-(4n-1)^2=16n^2-24n+9-16n^2+8n-1$

$$=-16n+8=-8(2n-1)$$

$T_k=8(2k-1)$

$$S_n=\sum_{k=1}^{n}T_k=-8\left[2\sum_{k=1}^{n}k-n\right]$$

$$=-8\left[2\frac{n(n+1)}{2}-n\right]=-8[n(n+1)-n]=-8(n^2+n-n)=-2n^2$$

iii. $\frac{1^2}{1}+\frac{1^2+2^2}{2}+\frac{1^2+2^2+3^2}{3}+\dots\dots\dots+n\ term$

Sol. $\frac{1^2}{1}+\frac{1^2+2^2}{1+1}+\frac{1^2+2^2+3^2}{1+1+1}+\dots\dots\dots+n\ term$

$$T_n=\frac{1^2+2^2+3^2+\dots\dots\dots+n\,term}{1+1+1+\dots\dots\dots n\ term}$$

$$=\frac{1}{n}\times\frac{n(n+1)(2n+1)}{6}=\frac{(n+1)(2n+1)}{6}$$

$$T_k=\frac{(k+1)(2k+1)}{6}=\frac{2k^2+3k+1}{6}$$

$$S_n=\sum_{k=1}^{n}T_k=\frac{1}{6}\left[2\sum_{k=1}^{n}k^2+3\sum_{k=1}^{n}k+n\right]$$

$$=\frac{1}{6}\left[2\times\frac{n(n+1)(2n+1)}{6}+\frac{3n(n+1)}{2}+n\right]$$

$$=\frac{1}{6}\left[\frac{n(n+1)(2n+1)}{3}+\frac{3n(n+1)}{2}+n\right]=\frac{n}{6}\left[\frac{2n^2+3n+1}{3}+\frac{3n+3}{2}+1\right]$$

$$=\frac{n}{6}\left[\frac{4n^2+6n+2+9n+9+6}{6}\right]=\frac{n}{6}\left(\frac{4n^2+15n+17}{6}\right)$$

$$=\frac{n(4n^2+15n+17)}{36}$$

15. Find the sum to n terms of the series whose nth terms are given:

i. $3n^2+n+1$

Sol. Given $T_n=3n^2+n+1\Rightarrow T_k=3k^2+k+1$

$$S_n=\sum_{k=1}^{n}T_k=3\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k+n=3\frac{n(n+1)(2n+1)}{6}+n\left[\frac{n+1}{2}\right]+n$$

$$=n\left[\frac{(n+1)(2n+1)}{2}+\frac{n+1}{2}+1\right]=n\left[\frac{2n^2+3n+1+n+1+2}{2}\right]$$

$$=\frac{n(2n^2+4n+4)}{2}=n(n^2+2n+2)$$

ii. n^2+4n+1

Sol. Given $T_n=3n^2+n+1\Rightarrow Tk=3k^2+k+1$

$$S_n=\sum_{k=1}^{n}T_k=\sum_{k=1}^{n}k^2+4\sum_{k=1}^{n}k+n$$

$$=\frac{n(n+1)(2n+1)}{6}+\frac{\cancel{4}^{2}n(n+1)}{\cancel{2}}+n=n\left[\frac{n^2+2n+n+1}{6}+2n+2+1\right]$$

$$=n\left[\frac{2n^2+2n+n+1+12n+12+6}{6}\right]=\frac{n}{6}\left[2n^2+15n+19\right]$$

16. **Find the nth terms of the series, find the sum to 2n terms.**

i. $3n^2+2n+1$ **Multan 2008**

Sol. Given $T_n = 3n^2+2n+1 \Rightarrow T_k = 3k^2+2k+1$

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}\left(3k^2+2k+1\right)$$

$$= 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k + n$$

$$= \cancel{3}\frac{n(n+1)(2n+1)}{\cancel{6}_2} + \frac{\cancel{2}(n)(n+1)}{\cancel{2}} + n$$

$$= n\left[\frac{2n^2+n+2n+1}{2} + n+1+1\right]$$

$$= \frac{n}{2}\left[2n^2+3n+1+2n+2+2\right]$$

$$S_n = \frac{n}{2}\left[2n^2+5n+5\right]$$

$$S_{2n} = \frac{(2n)}{2}\left[2(2n^2)+5(2n)+5\right] = n\left[8n^2+10n+5\right]$$

ii. n^3+2n+3

Sol. Given $T_n = n^3+2k+3 \Rightarrow T_k = k^3+2k+3$

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(k^3+2k+3) = \sum_{k=1}^{n} k^3 + 2\sum_{k=1}^{n} k + 3n$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + \frac{2n(n+1)}{2} + 3n$$

$$= n\left[\frac{n(n^2+2n+1)}{4} + (n+1)+3\right] = \frac{n}{4}\left[n^3+2n^2+n+4n+4+12\right]$$

$$= \frac{n}{4}\left[n^3+2n^2+5n+16\right]$$

$$= \frac{2n}{4}\left[(2n)^3+2(2n)^2+5(2n)+16\right] \quad (\textit{Replace } n \textit{ by } 2n)$$

$$= \frac{n}{2}\left[8n^3+8n^2+10n+16\right] = n\left[4n^3+4n^2+5n+8\right]$$

TEST YOUR SKILLS Marks: 50

Q # 1. Select the Correct Option **(10)**

i. A.M between $3\sqrt{5}$ and $5\sqrt{5}$ is:

a) $4\sqrt{5}$ b) $5\sqrt{5}$

c) 10 d) $2\sqrt{5}$

ii. The series $1+\frac{x}{2}+\frac{x^2}{2}+$.............is convergent if:

a) $x \in R$ b) $x \in [-2,2]$

c) $x \in (-2,2)$ d) $x \in Z$

iii. The sum of an infinite geometric series exists if:

a) $|r|<1$ b) $|r|>1$

c) $r=1$ d) $r=-1$

iv. $\sum_{k=1}^{n} k^2 = ?$

a) $\frac{n(n+1)}{2}$ b) $\frac{n^2(n+1)^2}{4}$

c) $\frac{n(n+1)(2n+1)}{6}$ d) None of these

v. $\frac{1}{2},\frac{1}{7},\frac{1}{12},$............ is

a) $An\ A.P$ b) $G.P$

c) $H.P$ d) None of these

vi. If $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ is A.M between a and b then n is:

a) 0 b) 1

c) $\frac{1}{2}$ d) -1

vii. $\frac{1}{1-\sqrt{x}},\frac{1}{1-x},\frac{1}{1+\sqrt{x}}$ to n term is:

a) $A.P$ b) $G.P$

c) $H.P$ d) Geometric Series

viii. The sum of cube of first n natural numbers

a) $\frac{n(n+1)}{4}$ b) $\frac{n^2(n+1)^2}{4}$

c) $\dfrac{n(n+1)(2n+1)}{6}$ d) $\dfrac{n(n+1)}{2}$

ix. If $|r|>1$ then infinite geometric series is

a) Oscillatory b) Decreasing
c) Convergent d) Divergent

x. The 8^{th} term of the sequence 3, 6, 12, ... is

a) $\dfrac{1}{48}$ b) -48

c) $-\dfrac{1}{48}$ d) 48

Q # 2. Short Questions: **(10 X 2 = 20)**

i. Which term of the A.P $-2,4,10,......$ is 148?

ii. Find the 12^{th} term of the sequence $1+i, 2i, -2+2i, ...$

iii. Sum the series up to n term $3+33+333+.............$

iv. If 5 and 8 are two A.Ms between a & b. Find a and b.

v. If 5 is H.M between 2 and b, find b?

vi. If $a_{n-3}=2n-5$ find the nth term?

vii. Find the sum of the infinite geometric series $2+1+0.5+..........$

viii. Find the 9^{th} term of harmonic sequence $\dfrac{1}{3},\dfrac{1}{5},\dfrac{1}{7},...............$

ix. If the numbers $\dfrac{1}{K},\dfrac{1}{2K+1},\dfrac{1}{4K-1}$ are in H.P find K?

x. Sum the series $-3+(-1)+1+3+5...........+a_{16}$

Long Questions: **(2 X 10 = 20)**

Q # 3. (a) Find four A.Ms between $\sqrt{2}$ and $12/\sqrt{2}$

(b) A ball is dropped from height of 27m, it rebounds two third of the distance it falls if it continue to fall in the same way what distance will it travel before coming to rest?

Q # 4. (a) if the numbers $\dfrac{1}{2},\dfrac{4}{21},\dfrac{1}{36}$ are subtracted from three consecutive terms of a G.P, the resulting numbers are in H.P. Find numbers if their product is $\dfrac{1}{27}$

(b) If l,m,n are pth, qth, rth terms of an A.P, Show that $l(q-r)+m(r-p)+n(p-q)=0$

PERMUTATION, COMBINATION AND PROBABILITY

7

Exercise 7.1

Theorem: $0!=1$

Proof: we know that $n!=n(n-1)!$

$(put\ n=1)1!=1(1-1)! \Rightarrow 1=0! \Rightarrow 0!=1$

1. Evaluate each of the following.

i. $4!$

Sol. $4!=4.3.2.1=24$

ii. $6!$

Sol. $6!=6.5.4.3.2.1=720$

iii. $\dfrac{8!}{7!}$

Sol. $\dfrac{8!}{7!}=\dfrac{8.\cancel{7!}}{\cancel{7!}}=8$

iv. $\dfrac{10!}{7!}$

Sol. $\dfrac{10!}{7!}=\dfrac{10.9.8.\cancel{7!}}{\cancel{7!}}=720$

v. $\dfrac{11!}{4!7!}$

Sol. $\dfrac{11!}{4!7!}=\dfrac{11.10.9.8.\cancel{7!}}{4.3.2.1.\cancel{7!}}$

$=\dfrac{11.10.9.8}{4.3.2.1}=330$

vi. $\dfrac{6!}{3!3!}$

Sol. $\dfrac{6!}{3!3!}=\dfrac{6.5.4.\cancel{3!}}{3.2.1.\cancel{3!}}=\dfrac{6.5.4}{6}=20$

vii. $\dfrac{8!}{4!2!}$

Sol. $\dfrac{8!}{4!2!}=\dfrac{8.7.6.5.\cancel{4!}}{\cancel{4!}.2!}=\dfrac{8.7.6.5}{2}=840$

viii. $\dfrac{11!}{2!4!5!}$

Sol. $\dfrac{11!}{2!4!5!}=\dfrac{11.10.9.8.7.6.\cancel{5!}}{2.1.4.3.2.1.\cancel{5!}}=770$

ix. $\dfrac{9!}{2!(9-2)!}$

Sol. $\dfrac{9!}{2!(9-2)!}=\dfrac{9.8.\cancel{7!}}{2.\cancel{7!}}=36$

x. $\dfrac{15!}{15!(15-15)!}$

Sol. $\dfrac{\cancel{15!}}{\cancel{15!}(15-15)!}=\dfrac{1}{0!}=\dfrac{1}{1}=1$

xi. $\dfrac{3!}{0!}$

Sol. $\dfrac{3!}{0!}=\dfrac{3.2.1}{1}=6$

xii. **4!.0!.1!**

Sol. $4!.0!.1!=4.3.2.1.1=24$

2. **Write each of the following in factorial form:**

i. **6.5.4**

Sol. Multiplying and divided by $3!$

$$6.5.4=\frac{6.5.4.3!}{3!}=\frac{6!}{3!}$$

ii. **12.11.10**

Sol. $12.11.10$('×' & '÷' by 9!)

$$\frac{12.11.10.9!}{9!}=\frac{12!}{9!}$$

iii. **20.19.18.17**

Sol. $20.19.18.17$ ('×' & '÷' by 16!)

$$\frac{20.19.18.17.16!}{16!}=\frac{20!}{16!}$$

iv. $\dfrac{10.9}{2.1}$

Sol. $\dfrac{10.9}{2.1}$ ('×' & '÷' by 8!)

$$\frac{10.9.8!}{2.1.8!}=\frac{10!}{2!.8!}$$

v. $\dfrac{8.7.6}{3.2.1}$

Sol. $\dfrac{8.7.6}{3.2.1}$ ('×' & '÷' by 5!)

$$\frac{8.7.6.5!}{3.2.1.5!}=\frac{8!}{3!5!}$$

vi. $\dfrac{52.51.50.49}{4.3.2.1}$

Sol. $\dfrac{52.51.50.49}{4.3.2.1}$ ('×' & '÷' by 48!)

$$\frac{52.51.50.49.48!}{4!.48!}=\frac{52!}{4!.48!}$$

vii. $n(n-1)(n-2)$

Sol. $n(n-1)(n-2)$ ('×' & '÷' by $(n-3)!$)

$$\frac{n(n-1)(n-2)(n-3)!}{(n-3)!}=\frac{n!}{(n-3)!}$$

viii. $(n+2)(n+1)(n)$

Sol. $(n+2)(n+1)(n)$'×' & '÷' by $(n-1)!$

$$\frac{(n+2)(n+1)(n)(n-1)!}{(n-1)!}=\frac{(n+2)!}{(n-1)!}$$

ix. $\dfrac{(n+1)(n)(n-1)}{3.2.1}$ Sargodha 2006

Sol. $\dfrac{(n+1)(n)(n-1)(n-2)!}{3.2.1.(n-2)!}=\dfrac{(n+1)!}{3!(n-2)!}$

x. $n(n-1)(n-2)......(n-r+1)$

Sol. $n(n-1)(n-2)......(n-r+1)$

'×' & '÷' by $(n-r)!$ *Multan 2008*

$$\frac{n(n-1)(n-2).......(n-r+1)(n-r)!}{(n-r)!}$$

$$=\frac{n!}{(n-r)!}$$

Example 1:- **How many 4 digit nos can be formed by 1,2,3,4,5,6 when no digit is repeated**

Sol: n=6, r=4

$$\text{No of 4 digit numbers} = 6p_4 = \frac{6!}{(6-4)!} = \frac{6.5.4.3.2!}{2!} = 360$$

Example 2:- **How many signal with 4 different flags can be given when any no of flags can be used .** **Multan 2008, Faisalabad 2009**

Sol. **n=4, r=1,2,3,4**

$$\text{No of signal using 1 flag} = 4p_1 = 4$$

$$\text{No of signal using 2 flag} = 4p_2 = 12$$

$$\text{No of signal using 3 flag} = 4p_3 = 24$$

$$\text{No of signal using 4 flag} = 4p_4 = 24$$

$$\textbf{\textit{Total no of signal}} = \mathbf{4+12+24+24 = 64}$$

Exercise 7.2

$${}^nP_r = \frac{n!}{(n-r)!}$$ **(Formula for Permutation)** **Sargodha 2011**

1. **Evaluate the following.**

i. ${}^{20}P_3$

Sol. $${}^{20}P_3 = \frac{20!}{(20-3)!} = \frac{20.19.18.\cancel{17!}}{\cancel{17!}} = 6840$$

ii. ${}^{16}P_4$

Sol. $${}^{16}P_4 = \frac{16!}{(16-4)!} = \frac{16.15.14.13.\cancel{12!}}{\cancel{12!}} = 43680$$

iii. ${}^{12}P_5$

Sol. $${}^{12}P_5 = \frac{12!}{(12-5)!} = \frac{12.10.9.8.\cancel{7!}}{\cancel{7!}} = 95040$$

iv. ${}^{10}P_7$

Sol. $${}^{10}P_7 = \frac{10!}{(10-7)!} = \frac{10.9.8.7.6.5.4.\cancel{3!}}{\cancel{3!}} = 604800$$

v. ${}^9P_8$

Sol. $${}^9P_8 = \frac{9!}{(9-8)!} = \frac{9.8.7.6.5.4.3.2.\cancel{1!}}{\cancel{1!}} = 362880$$

2. Find the value of n

i. $^{n}P_2 = 30$ **Multan 2007, Sargodha 2008, Faisalabad 2008, Lahore 2009**

Sol. $\dfrac{n!}{(n-2)!} = 30$

$\dfrac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 6.5$

$n(n-1) = 6.5 \Rightarrow n = 6$

ii. $^{11}P_n = 11.10.9$ **Sgd 2009, Fsd 2007, 2008 Lahore 2009, Multan 2008, 2009**

Sol. $\dfrac{11!}{(11-n)!} = \dfrac{11.10.9.8!}{8!}$

$\dfrac{11!}{(11-n)!} = \dfrac{11!}{8!}$

$\Rightarrow 11 - n = 8 \Rightarrow 11 - 8 = n \Rightarrow n = 3$

iii. $^{n}P_4 : {}^{n-1}P_3 = 9:1$ **Faisalabad 2009**

Sol. $\dfrac{^{n}P_4}{^{n-1}P_3} = \dfrac{9}{1}$

$$\dfrac{\dfrac{n!}{(n-4)!}}{\dfrac{(n-1)!}{(n-1-3)!}} = 9$$

$\dfrac{n\cancel{(n-1)!}}{\cancel{(n-4)!}} \times \dfrac{\cancel{(n-4)!}}{\cancel{(n-1)!}} = 9 \Rightarrow n = 9$

3. Prove from the first principle that:

i. $^{n}P_r = n.{}^{n-1}P_{r-1}$

Sol. R.H.S $= n.{}^{n-1}P_{r-1}$

$= n.\dfrac{(n-1)!}{[n-1-(r-1)]!} = \dfrac{n.(n-1)!}{(n-1-r+1)!} = \dfrac{n!}{(n-r)!}\ {}^{n}P_r =$ L.H.S

ii. $^{n}P_r = {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$ **Lahore 2009**

Sol. R.H.S $= {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$

$$=\frac{(n-1)!}{(n-1-r)!}+\frac{r.(n-1)!}{[n-1-(r-1)]!}=\frac{(n-1)!}{(n-1-r)!}+\frac{r.(n-1)!}{(n-\cancel{1}-r+\cancel{1})!}$$

$$=\frac{(n-1)!}{(n-1-r)!}+\frac{r.(n-1)!}{(n-r)!}=\frac{(n-1)!}{(n-1-r)!}+\frac{r.(n-1)!}{(n-r)(n-r-1)!}$$

$$=\frac{(n-1)!}{(n-1-r)!}\left[1+\frac{r}{n-r}\right]=\frac{(n-1)!}{(n-1-r)!}\left[\frac{n-r+r}{n-r}\right]$$

$$=\frac{(n-1)!}{(n-1-r)!}\left(\frac{n}{n-r}\right)=\frac{n.(n-1)!}{(n-r)(n-1-r)!}=\frac{n!}{(n-r)!}={}^{n}P_{r}=\text{L.H.S}$$

4. How many signals can be given by 5 flags of different colours, using 3 flags at a time: **Rawalpindi 2009, Multan 2007,2008**

Sol. $n=5,\ r=3$

Number of signals $={}^{5}P_{3}$

$$=\frac{5!}{(5-3)!}=\frac{5.4.3.\cancel{2!}}{\cancel{2!}}=60$$

5. How many signals can be given by 6 flags of different colours when any number of flags can be used at a time?

Sol. $n=6,\ r=1,2,3,4,5,6$

$${}^{6}P_{1}=\frac{6!}{(6-1)!}=\frac{6.\cancel{5!}}{\cancel{5!}}=6$$

$${}^{6}P_{2}=\frac{6!}{(6-2)!}=\frac{6.5.\cancel{4!}}{\cancel{4!}}=30$$

$${}^{6}P_{3}=\frac{6!}{(6-3)!}=\frac{6.5.4.\cancel{3!}}{\cancel{3!}}=120$$

$${}^{6}P_{4}=\frac{6!}{(6-4)!}=\frac{6.5.4.3.\cancel{2!}}{\cancel{2!}}=360$$

$${}^{6}P_{5}=\frac{6!}{(6-5)!}=\frac{6.5.4.3.2.\cancel{1!}}{\cancel{1!}}=720$$

$${}^{6}P_{6}=\frac{6!}{(6-6)!}=\frac{6!}{0!}=\frac{6.5.4.3.2.1}{1}=720$$

Number of Signals.

$$=6+30+120+360+720+720=1956$$

6. How many words can be formed from the letters of the following words using all letters when no letter is to be repeated:

I. PLANE

Sol. $n=5,\ r=5$

Number of words $= {}^5P_5$

$$=\frac{5!}{(5-5)!}=\frac{5.4.3.2.1}{0!}=\frac{120}{1}=120$$

ii. OBJECT

Sol. $n=6,\ r=6$

Number of words $= {}^6P_6$

$$=\frac{6!}{(6-6)!}=\frac{6.5.4.3.2.1}{0!}=\frac{720}{1}=720$$

iii. FASTING

Sol. $n=7,\ r=7$

Number of words $= {}^7P_7$

$$=\frac{7!}{(7-7)!}=\frac{7.6.5.4.3.2.1}{0!}=\frac{5040}{1}=5040$$

7. How many digits numbers can be formed by using each one of the digits 2, 3, 5, 7, 9 only once?

Sol. $2,3,5,7,9$ 3 digits numbers = ?

3-digits numbers $= {}^5P_3$

$$=\frac{5.4.3.2!}{(5-3)!}=\frac{60.\cancel{2!}}{\cancel{2!}}=60$$

8. Find the numbers greater than 23000 that can be formed from the digits 1, 2, 3, 5, 6, without repeating any digit.

Sol. $1,2,3,4,5,6$ $n=5,\ r=5$

Total numbers $= {}^5P_5$

$$=\frac{5!}{(5-5)!}=\frac{5.4.3.2.1}{0!}=120$$

For less the 23000 If 1 is fixed at extreme left then permutation of 2, 3, 5, 6

$$={}^4P_4=\frac{4!}{(4-4)!}=\frac{4.3.2.1}{0!}=24$$

21 is fixed at extreme left then permutation of 3, 5, 6

$$= {}^3P_3 = \frac{3!}{(3-3)!} = \frac{3.2.1}{0!} = 6$$

Less than 23000 = 24 + 6 = 30

For greater than 23000

= Total – less than 23000

= 120 – 30 = 90

9. Find the number of 5-digit numbers that can be formed from the digits 1, 2, 4, 6, 8 (when no digit is repeated), but

(i) The digits 2 and 8 are next to each other;

(ii) The digits 2 and 8 are not next to each other.

Sol. $1,2,4,6,8$ permutation of 1, $\boxed{28}$,4, 6, is = ${}^4P_4$

$$= \frac{4!}{(4-4)!} = \frac{4.3.2.1!}{0!} = 24$$

Now $1,4,\boxed{82},6$

Permutation = ${}^4P_4 = 24$

(i) 2, 8 are next to each other $= 24 + 24 \Rightarrow 48$

Total permutation of $1,2,3,4,6,8 = {}^5P_5 = 120$

(ii) 2, 8 are not next to each other

= Total – 48 =120 – 48 = 72

10. How many 6 digit numbers can be formed, without repeating any digit from the digits 0, 1, 2, 3, 4, 5? In how many of them will 0 be at the tens place?

Multan 2008

Sol. $0,1,2,3,4,5$

Total 6 digits number $= {}^6P_6 = \frac{6!}{(6-6)!} = \frac{6.5.4.3.2.1}{0!} = 720$

When 0 at first place then $= {}^5P_5 = 120$ (because when 0 is at first place (not count))

Required 6 digits number = Total – 120 = 720 –120 = 600

When 0 at tens place ${}^5P_5$ (Because 0 is fixed at tens place) $= \frac{5.4.3.2.1}{0!} = 120$

11. How many 5 digit multiples of 5 can be formed from the digits 2, 3, 5, 7, 9, When no digit is repeated.

Sol. $2,3,7,9,\boxed{5}$ (Multiple of 5 , 5 at unit place fixed)

$$No.of\ 5\ digit\ multiple\ of\ 5 = {}^4P_4 = \frac{4.3.2.1}{0!} = 24$$

12. In how many ways can 8 books including 2 on English be arranged on a shelf in such a way that the English books are never together? Sargodha 2008

Sol. Suppose E_1, E_2 are two English books and $B_1, B_2, B_3, B_4, B_5, B_6$, remaining books.

Total $= {}^8P_8 = \dfrac{8!}{(8-8)!} = \dfrac{8.7.6.5.4.3.2.1}{0!} = 40320$

English books are Together

Case I $B_1, B_2, B_3, B_4, B_5, B_6, \boxed{E_1E_2}$

$= {}^7P_7 = \dfrac{7.6.5.4.3.2.1}{(7-7)!} = 5040$

Case II $B_1, B_2, B_3, B_4, B_5, B_6, \boxed{E_2E_1}$

$= {}^7P_7 = 5040$

English books together $= 5040 + 5040 = 10080$

English books not together = Total – Together = 40320-10080=30240

13. **Find the number of arrangements of 3 books on English and 5 books on urdu for placing them on a shelf such that the books on the same subjects are together.**

Sol. E_1, E_2, E_3 are English and U_1, U_2, U_3, U_4, U_5, are Urdu books. **Federal**

Case I $U_1, U_2, U_3, U_4, U_5 \times E_1, E_2, E_3 = {}^5P_5 \times {}^3P_3 = \dfrac{5!}{(5-5)!} \times \dfrac{3!}{(3-3)!}$

$= \dfrac{5.4.3.2.1}{(5-5)!} \times \dfrac{3.2.1}{(3-3)!} = \dfrac{120}{0!} \times \dfrac{6}{0!} = 120 \times 6 = 720$

Case II $E_1, E_2, E_3 \times U_1, U_2, U_3, U_4, U_5 = {}^3P_3 \times {}^5P_5$

$= \dfrac{3!}{(3-3)!} \times \dfrac{5}{(5-5)!} = 3.2.1 \times 5.4.3.2.1$

$= 6 \times 120 = 720$

Answer $= 720 + 720 = 1440$

14. **In how many ways can 5 boys and 4 girls be seated on a bench so that the girls and the boys occupy alternate seats?**

Sol: B represent Boys and G girls

Total no of ways $B_1G_1B_2G_2B_3G_3B_4G_4B_5G_5$

$= {}^5P_1 \times {}^4P_1 \times {}^4P_1 \times {}^3P_1 \times {}^3P_1 \times {}^2P_1 \times {}^1P_1 \times {}^1P_1$

$= 5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 = 2880$

Alternate sol.

$= {}^5P_5 \times {}^4P_4 = 120 \times 24$

$= 2880$

Circular permutation **Lahore 2009**

The permutation of things which can be represented by the points on a circle is called circular permutation.

Exercise 7.3

1. How many arrangements of the letters of the following words, taken all together, can be made:

i. PAKPATTAN Fsd 2008, 2009

Sol. Total letters $= n = 9$

P repeated time = 2

A repeated time = 3

K repeated time = 1

T repeated time = 2

N repeated time = 1

Total arrangements

$$=\frac{9!}{2!.3!.1!.2!.1!}$$

$$=\frac{9.8.7.6.5.4.\cancel{3!}}{2.1.\cancel{3!}.1.2.1.1}=15120$$

ii. PAKISTAN
(Sargodha 2009, Fsd 2008,09)

Sol. Total letters $= n = 8$

P repeated = 1

A repeated = 2

K repeated =1

I repeated = 1

S repeated = 1

T repeated = 1

N repeated = 1

Total arrangements

$$=\frac{8!}{1!.2!.1!.1!.1!.1!.1!}$$

$$=\frac{8.7.6.5.4.3.\cancel{2!}}{\cancel{2!}}=20160$$

iii. MATHAMETICS

Sol. Total letters $= 11$

M repeated = 2

A repeated = 2

T repeated =2

H repeated = 1

E repeated = 1

I repeated = 1

C repeated = 1

S repeated = 1

Total arrangements

$$=\frac{11!}{2!.2!.2!.1!.1!.1!.1!.1!}$$

$$=\frac{11.10.9.8.7.6.5.4.3.2.1}{2.1.2.12.1.1.1.1.1.1}$$

$$=\frac{11.10.9.8.7.6.5.4.3.2.1}{8}$$

$$=4989600$$

iv. ASSASSINATION
(Sargodha 2006, Multan 2007)

Sol. Total letters $= n = 13$

A repeated = 3

S repeated = 4

I repeated = 2

N repeated = 2

T repeated = 1

O repeated = 1

Total arrangements $=\dfrac{13!}{3!.4!.2!.2!.1!}$

$$=\frac{13.12.11.10.9.8.7.6.5.\cancel{4!}}{3.2.1.2.\cancel{4!}.2.1.2.1.1.1}=43243200$$

2. How many Permutation of the letters of the word PANAMA can be made, If P is to be the first letter in each arrangement?

Sol. **PANAMA**

If P is first letter then $\boxed{P}$ ANAMA, $n=5$

A repeated time =3

N repeated time =1

M repeated time =1

Total arrangements $=\dfrac{5!}{3!.1!.1!}=\dfrac{5.4.\cancel{3!}}{\cancel{3!}.1.1}=20$

3. How many arrangements of the letters of the word ATTACKED can be made, if each arrangement being with C and with K?

Sol. ATTACKED IF C is first and K is last letter than $\boxed{C}$ ATTAED $\boxed{K}$, $n=6$

A repeated time =2

T repeated time =2

E repeated time =1

D repeated time =1

Total arrangements $=\dfrac{6!}{2!.2!.1!.1!}=\dfrac{6.5.4.3.\cancel{2!}}{\cancel{2!}.2.1.1.1}=\dfrac{360}{2}=180$

4. How many numbers greater then 1000,000 can be formed from the digits 0, 2, 2, 2, 3, 4, 4, ?

Sol. **0, 2, 2, 2, 3, 4, 4,** $n=7$

0 repeated time =1

2 repeated time =3

3 repeated time =1

4 repeated time =2

Total $=\dfrac{7!}{1!.3!.1!.2!}=\dfrac{7.6.5.4.3!}{1.3!.1.2.1}=420$

For less than 1000000 fix 0 at extreme left $\boxed{0}, 2,2,2,3,4,4=\dfrac{6!}{3!.1!.2!}=\dfrac{6.5.4.\cancel{3!}}{\cancel{3!}.1.2.1}=60$

Number greater than 1000000 = 420-60=360

5. How many 6- digits numbers can be formed from the digits 2, 2, 3, 3, 4, 4,? How many of them will lie between 400,000 and 430,000?

Sol. **2, 2, 3, 3, 4, 4,** $n=6$ **Faisalabad 2008, Multan 2009**

2 repeated time =2

3 repeated time =2

4 repeated time =2

Total 6 digits numbers $= \frac{6!}{.2!.2!.2!} = \frac{6.5.4.3.2.1!}{2.1.2.1.2.1} = 90$

For numbers between 400,000 and 430,000 fixed 42 at first place then numbers =

$\boxed{42}$,2, 3, 3, 4, $n = 4$

2 repeated time =1

3 repeated time =2

4 repeated time =1

Total numbers $= \frac{4!}{1!.2!.1!} = \frac{4.3.\cancel{2!}}{\cancel{2!}} = 12$

6. 11 members of a club from 4 committees of 3, 4, 2, 2 members so that no member is a member of more then one committee. Find the number of committees?

Sol. $n = 11$

4 committees have 3, 4, 2, 2 members

No of committees $= \frac{11!}{3!.4!.2!.2!} = \frac{11.10.9.8.7.6.5.\cancel{4!}}{3.2.1.\cancel{4!}.2.1.2.1} = 69300$

7. The D.C.Os of 11 districts meet to discuss the law and order situation in their districts . In how many ways can they be seated at a round table, when two particular D.C.Os insist on sitting together?

Sol. Number of ways= ${}^9P_9 \times {}^2P_2$ (when 2 particular D.C.Os sit together)

$= \frac{9!}{(9-9)!} \times \frac{2!}{(2-2)!} = \frac{9!}{0!} \times \frac{2!}{0!} = 9! \times 2! = 725760$

Note for Round table formula $= (n-1)!$

8. The Governor of the Punjab calls a meeting of 12 officers. In how many ways can they be seated at a round table?

Sol. No of ways when one chair is fixed for Chairperson $= (12-1)! = 11! = 39916800$

9. Fatima invites 14 people to a dinner. There are 9 males and 5 females who are seated at two different tables so that guests of one sex sit at one round table and the guest of other sex at the second table. Find the number of ways in which all guests are seated.

Sol. Male = 9 & Female = 5

Number of ways $= {}^8P_8 \times {}^4P_4 = 8! \times 4! = 967680$

10. Find the number of ways in which 5 men and 5 women can be seated at a round in such a way no person of the same sex sit together.

Sol. Male = 5 & Women = 5

Both are sitting at one table so one chair is fixed then number of ways

$= {}^4P_4 \times {}^5P_5 = 4! \times 5! = 2880$

Note Circular permutation $= \frac{(n-1)!}{2}$

11. In how many ways can 4 keys be arranged on a circular key ring?

Sol. $=n=4$ **Faisalabad 2007, Sargodha 2009, Multan 2008)**

No of ways $=\dfrac{(n-1)!}{2}=\dfrac{(4-1)!}{2}=\dfrac{3!}{2}=\dfrac{3.2.1}{2}=3$

12. How many necklaces can be made from 6 beads of different colours?

Sol. $=n=6$ **Sargodha 2010, Multan 2009**

Number of necklace $=\dfrac{(n-1)!}{2}=\dfrac{(6-1)!}{2}=\dfrac{5!}{2}=\dfrac{120}{2}=60$

COMBINATION

THEOREM $^nC_r={}^nC_{n-r}$

Proof: R.H.S $={}^nC_{n-r}$

$$=\frac{n!}{(n-r)![n-(n-r)]!}=\frac{n!}{(n-r)!(\cancel{n}-\cancel{n}+r)}=\frac{n!}{(n-r)!r!}=\frac{n!}{r!(n-r)!}={}^nC_r = \text{L.H.S}$$

Example 2. **How many diagonal can be formed by 6 sided polygon.**

Faisalabad 2007, Multan 2007, 2008

Sol. No. of diagonal $={}^6C_2-6=\dfrac{6!}{2!(6-2)!}-6=\dfrac{6.5.4!}{2.1.4!}-6=15-6=9$

Example 3. $^{n-1}C_r+{}^{n-1}C_{r-1}={}^nC_r$ **Federal, Sargodha 2010, Faisalabad 2008**

Sol: L.H.S= $^{n-1}C_r+{}^{n-1}C_{r-1}$

$$=\frac{(n-1)!}{r!(n-1-r)!}+\frac{(n-1)!}{(r-1)!(n-\cancel{1}-r+\cancel{1})!}=\frac{(n-1)!}{r!(n-1-r)!}+\frac{(n-1)!}{(r-1)!(n-r)!}$$

$$=\frac{(n-1)!}{r(r-1)!(n-1-r)!}+\frac{(n-1)!}{(r-1)!(n-r)(n-r-1)!}$$

$$=\frac{(n-1)!}{(r-1)!(n-1-r)!}\left[\frac{1}{r}+\frac{1}{n-r}\right]=\frac{(n-1)!}{(r-1)!(n-r-1)!}\left[\frac{n-\cancel{r}+\cancel{r}}{r(n-r)}\right]$$

$$=\frac{(n-1)!}{(r-1)!(n-1-r)!}\left(\frac{n}{r(n-r)}\right)$$

$$=\frac{n(n-1)!}{r(r-1)!(n-r)(n-r-1)!}=\frac{n!}{r!(n-r)!}={}^nC_r = \text{R.H.S}$$

THEOREM: **Prove that** $^nC_r.r!={}^nP_r$ **Faisalabad 2008**

Proof: $L.H.S\ \ {}^nC_r.r!=\dfrac{n!}{\cancel{r!}(n-r)!}\cancel{r!}=\dfrac{n!}{(n-r)!}={}^nP_r=R.H.S$

Exercise 7.4

Formula $^{n}C_r = \dfrac{n!}{r!(n-r)!}$

1. **Evaluate the following:**

i. $^{12}C_3$

Sol. $^{12}C_3 = \dfrac{12!}{3!(12-3)!} = \dfrac{12!}{3!9!} = \dfrac{12.11.10.\cancel{9!}}{3.2.1.\cancel{9!}} = 220$

ii. $^{20}C_{17}$ **Multan 2009**

Sol. $^{20}C_{17} = \dfrac{20!}{17!(20-17)!} = \dfrac{20!}{17!3!} = \dfrac{20.19.18.\cancel{17!}}{\cancel{17!}3.2.1} = 1140$

iii. $^{n}C_4$

Sol. $^{n}C_4 = \dfrac{n!}{4!(n-4)!} = \dfrac{n(n-1)(n-2)(n-3)\cancel{(n-4)!}}{4!\cancel{(n-4)!}} = \dfrac{n(n-1)(n-2)(n-3)}{4!}$

2. **Find the values of n and r, when:**

i. $^{n}C_5 = {}^{n}C_4$ **Multan 2008**

Sol. $^{n}C_5 = {}^{n}C_4 \longrightarrow I$

We know that

$^{n}C_r = {}^{n}C_{n-r} \Rightarrow {}^{n}C_5 = {}^{n}C_{n-5}$

I become

$^{n}C_{n-5} = {}^{n}C_4 \Rightarrow n-5=4 \Rightarrow \boxed{n=9}$

ii. $^{n}C_{10} = \dfrac{12\times 11}{2!}$ **Multan 2007**

Sol. $^{n}C_{10} = \dfrac{12\times 11\times 10!}{2!10!}$ ('×'& ÷ by 10!)

$^{n}C_{10} = \dfrac{12!}{2!10!} = \dfrac{12!}{10!.(12-10)!} = {}^{12}C_{10} \Rightarrow {}^{n}C_{10} = {}^{12}C_{10} \Rightarrow \boxed{n=12}$

iii. $^{n}C_{12} = {}^{n}C_6$ **Multan 2009, Sargodha 2008, Faisalabad 2007**

Sol. $^{n}C_{12} = {}^{n}C_6 \longrightarrow 1$

use $^{n}C_r = {}^{n}C_{n-r} \Rightarrow {}^{n}C_{12} = {}^{n}C_{n-12}$ I become

$^{n}C_{n-12} = {}^{n}C_6 \Rightarrow n-12 = 6 \Rightarrow \quad n = 18$

3. Find the value of n and r, when

i. $^nC_r = 35 \; and \; ^nP_r = 210$ **Sargodha 2008, Multan 2009**

Sol. $^nC_r = 35 \; \& \; ^nP_r = 210$

Find n & r =?

We know that

$$^nC_r = \frac{^nP_r}{r!} \Rightarrow r! = \frac{^nP_r}{^nC_r} = \frac{210}{35} = 6 \Rightarrow r! = 3.2.1 = 3! \Rightarrow r = 3$$

Also $^nP_r = 210$

$$\Rightarrow \frac{n!}{(n-r)!} = 210 \Rightarrow \frac{n!}{(n-3)!} = 210 \qquad (put\, r = 3)$$

$$\frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}} = 210$$

$$n(n-1)(n-2) = 7.6.5 \Rightarrow \boxed{n = 7}$$

ii. $^{n-1}C_{r-1} : {^nC_r} : {^{n+1}C_{r+1}} = 3:6:11$ **Federal, Sargodha 2009, Faisalabad 2009, Lahore 2009, Multan 2008**

Sol. $$\Rightarrow \frac{^{n-1}C_{r-1}}{^nC_r} = \frac{3}{6} \; \& \; \frac{^nC_r}{^{n+1}C_{r+1}} = \frac{6}{11}$$

$$\frac{\dfrac{(n-1)!}{(r-1)!(n-\cancel{1}-r+\cancel{1})!}}{\dfrac{n!}{r!(n-r)!}} = \frac{1}{2} \; \& \; \frac{\dfrac{n!}{r!(n-r)!}}{\dfrac{(n+1)!}{(r+1)!(n+\cancel{1}-r-\cancel{1})!}} = \frac{6}{11}$$

$$\frac{(n-1)!}{(r-1)!\cancel{(n-r)}} \times \frac{r!\cancel{(n-r)!}}{n!} = \frac{1}{2} \quad \& \quad \frac{n!}{r!\cancel{(n-r)}} \times \frac{(r+1)!\cancel{(n-r)!}}{(n+1)!} = \frac{6}{11}$$

$$\frac{\cancel{(n-1)}! \times r\cancel{(r-1)!}}{\cancel{(r-1)!}\; n\cancel{(n-1)!}} = \frac{1}{2} \quad \& \quad \frac{\cancel{n!} \times (r+1)\cancel{r!}}{\cancel{r!}\;(n+1)\cancel{n!}} = \frac{6}{11}$$

$$\frac{r}{n} = \frac{1}{2} \Rightarrow 2r = n —— I \;\; \& \;\; \frac{r+1}{n+1} = \frac{6}{11}$$

$$\& \; 11r + 11 = 6n + 6 —— II$$

Put I in II

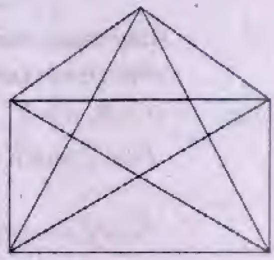
$$11r+11=6(2r)+6$$

$$\Rightarrow 11r+11=12r+6 \Rightarrow 11-6=12r-11r \Rightarrow 5=r \;\; put\ in\ I \Rightarrow n=2(5) \Rightarrow n=10$$

4. How many (a) diagonals and (b) triangles cab be formed by joining the vertices of the polygon having: Sargodha 2008, Lahore 2009

i. 5 sides

Sol. $n=5$



No of diagonal $= {}^5C_2-5$ $\left[\begin{array}{l} For\ Diagonal = {}^nC_2-n \\ For\ Triangle = {}^nC_3 \end{array}\right]$

$$=\frac{5!}{2!(5-2)!}-5=\frac{5.4.\cancel{3!}}{2.1.\cancel{3!}}-5=10-5=5$$

No of Triangles $= {}^5C_3=\dfrac{5!}{3!(5-3)!}=\dfrac{5.4.\cancel{3!}}{\cancel{3!}.2!}=\dfrac{5.4}{2.1}=10$

ii. 8 sides Sargodha 2008, 2009, 2011

Sol. $n=8$, No of diagonal $= {}^8C_2-8$

$$=\frac{8!}{2!(8-2)!}-8=\frac{8.7.\cancel{6!}}{2.1.\cancel{6!}}-8=28-8=20$$

No of Triangles $= {}^8C_3=\dfrac{8!}{3!(8-3)!}=\dfrac{8.7.6.\cancel{5!}}{3.2.1.\cancel{5!}}=42$

iii. 12 sides Rawalpindi 2009

Sol. $n=12$, No of diagonal $= {}^{12}C_2-12$

$$=\frac{12!}{2!(12-2)!}-12=\frac{12.11.\cancel{10!}}{2.1.\cancel{10!}}-12=54$$

No of Triangles $= {}^{12}C_3=\dfrac{12!}{3!(12-3)!}=\dfrac{12.11.10.\cancel{9!}}{3.2.1.\cancel{9!}}=220$

5. The members of a club are 12 boys and 8 girls. In how many ways can a committee of 3 boys and 2 girls be formed? Multan 2008

Sol. **Boys:** $n=12$, $r=3$

Girls: $n=8$, $r=2$

No of ways $= {}^{12}C_3 \times {}^8C_2$

$$=\frac{12!}{3!(12-3)!}\times\frac{8!}{2!(8-2)!}$$
$$=\frac{12!}{3!9!}\times\frac{8!}{2!\times 6!}$$
$$=\frac{12.11.10.\cancel{9!}}{3.2.1.\cancel{9!}}\times\frac{8.7.6!}{2.1.6!}$$
$$=220\times 28=6160$$

6. How many committees of 5 members can be chosen from a group of 8 persons when each committee must include 2 particular persons?

Sol. $n=8,\ r=5$

For 2 particular person $n=6,\ r=3$

No. of committees $={}^{6}C_{3}=\frac{6!}{3!(6-3)!}$

$$=\frac{6!}{3!.3!}=\frac{6.5.4.\cancel{3!}}{3.2.1.\cancel{3!}}=\frac{\cancel{6}.5.4}{\cancel{6}}=20$$

7. In how many ways can a hockey team of 11 players be selected out of 15 players? How many of them will include a particular player?

Sol. $n=15,\ r=11$

No of ways hockey team is selected $={}^{15}P_{11}=\frac{15!}{11!(15-11)!}$

$$=\frac{15!}{11!4!}=\frac{15.14.13.12.\cancel{11!}}{\cancel{11!}.4.3.2.1}=1365$$

If we included one particular player then $n=14,\ r=10$

No of ways $={}^{14}C_{10}=\frac{14!}{10!(14-10)!}=\frac{14.13.12.11.\cancel{10!}}{\cancel{10!}.4!}=1001$

8. Show that ${}^{16}C_{11}+{}^{16}C_{10}={}^{17}C_{11}$ **Sargodha 2011**

Sol. L.H.S $={}^{16}C_{11}+{}^{16}C_{10}$

$$=\frac{16!}{11!(16-11)!}+\frac{16!}{10!(16-10)!}$$
$$=\frac{16!}{11!5!}+\frac{16!}{10!6!}=\frac{16!}{11.10!.5!}+\frac{16!}{10!.6.5!}$$

$$=\frac{16!}{10!5!}\left(\frac{1}{11}+\frac{1}{6}\right)=\frac{16!}{10!5!}\left(\frac{6+11}{11.6}\right)=\frac{16!}{10!5!}\left(\frac{17}{11.6}\right)$$

$$=\frac{17.16!}{11.10!6.5!}=\frac{17!}{11!6!}=\frac{17!}{11!(17-11)!}={}^{17}C_{11}=\text{R.H.S}$$

9. There are 8 men and 10 women members of a club. How many committees of seven can be formed, having:

i. 4 Women

Sol. Men = 8 and women = 10

No of committees $= {}^{10}C_4\times{}^{8}C_3$

$$=210\times 55=11760$$

ii. At the most 4 Women

Sol. No of committees $= {}^{10}C_0\times{}^{8}C_7+{}^{10}C_1\times{}^{8}C_6+{}^{10}C_2\times{}^{8}C_5+{}^{10}C_3\times{}^{8}C_4+{}^{10}C_4\times{}^{8}C_3$

$$=1\times 8+10\times 28+45\times 56+120\times 70+20\times 56=22968$$

iii. At least 4 Women

Sol. $= {}^{10}C_4\times{}^{8}C_3+{}^{10}C_5\times{}^{8}C_2+{}^{10}C_6\times{}^{8}C_1+{}^{10}C_7\times{}^{8}C_0$

$$=210\times 56+252\times 28+210\times 8+120\times 1=20616$$

10. Prove that ${}^{n}C_r+{}^{n}C_{r-1}={}^{n+1}C_{r-1}$ **Sgd 2009, Faisalabad 2008, Multan 2007 Gujranwala 2009, Rawalpindi 2009,**

Sol. **L.H.S** $= {}^{n}C_r+{}^{n}C_{r-1}$

$$=\frac{n!}{r!(n-r)!}+\frac{n!}{(r-1)!(n-r+1)!}$$

$$=\frac{n!}{r(r-1)!(n-r)!}+\frac{n!}{(r-1)!(n-r+1)(n-r)!}$$

$$=\frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r}+\frac{1}{n-r+1}\right]$$

$$=\frac{n!}{(r-1)!(n-r)!}\left[\frac{n-\cancel{r}+1+\cancel{r}}{r(n-r+1)}\right]$$

$$=\frac{n!}{(r-1)!(n-r)!}\left(\frac{n+1}{r(n-r+1)}\right)$$

$$=\frac{(n+1).n!}{r(r-1)!(n-r+1)(n-r)!}$$

$$=\frac{(n+1)!}{r!(n-r+1)!}={}^{n+1}C_r=\text{R.H.S}$$

Example 1: **A die in rolled what is the probability that dot in the top is greater than 4.**

Sol. $S=\{1,2,3,4,5,6\}, n(s)=6$

$A=\{5,6\}, n(A)=2$

$P(A)=\dfrac{n(A)}{n(S)}=\dfrac{2}{6}=\dfrac{1}{3}$ **Multan 2007, Lahore2009, Fsd 2008**

Example 2: **What is probability that slip of numbers dividable by 4 picked from the numbers 1,2,3,.....10.** **Multan 2007, 2008**

Sol. $S=\{1,2,3,4,5,6,7,8,9,10\}, n(s)=10$

$A=\{4,8\}, n(A)=2$

$P(A)=\dfrac{n(A)}{n(S)}=\dfrac{2}{10}=\dfrac{1}{5}$

Exercise 7.5

Probability: **Sargodha 2008**

Probability is the numerical evaluation of a chance that a particular event would occur. OR Measurement of uncertainty.

Sample Space:

The set S consisting of all possible outcome of a given experiment is called a sample space.

Event:

An event is a subset of the sample space.

Formula $P(A)=\dfrac{n(A)}{n(S)}$

Total numbers $=n(S)$

For the following experiments, find the probability is each case:

1. **Experiment:**

Form a box containing orange flavoured sweets, Bilal takes out one sweet without looking.

Events Happening:

i. **The sweet is orange falvoured**

Sol. 'A' represent sweet is orange $P(A)=\dfrac{n(A)}{n(S)}=\dfrac{1}{1}=1$

ii. **The sweet is lemon falvoured**

Sol. 'B' represent sweet is lemon $P(B)=\dfrac{n(B)}{n(S)}=\dfrac{0}{1}=0$

2. **Experiment:**

Pakistan and India play a cricket match. The result is:

Events Happening:

i. **Pakistan wins** **Sargodha 2010, Faisalabad 2009**

Sol. $n(S)=3$

'A' represents "Pakistan wins" $n(A)=1$

$$P(A)=\frac{n(A)}{n(S)}=\frac{1}{3}$$

ii. **India does not lose**

Sol. 'B' represent 'India does not lose' $n(B)=2$

$$P(B)=\frac{n(B)}{n(S)}=\frac{2}{3}$$

3. **Experiment:**

There are 5 green and 3 red balls in a box, one ball is taken out:

Events Happening:

i. **The ball is green** **Sargodha 2009, Lahore 2009**

Sol. Green balls = 5, $n(S)=8$

'A' represents "green balls" $n(A)=5$

$$P(A)=\frac{n(A)}{n(S)}=\frac{5}{8}$$

ii. **The ball is Red**

Sol. Red balls = 3 ,$n(S)=8$

'B' represents "Red balls" $n(B)=3$

$$P(B)=\frac{n(B)}{n(S)}=\frac{3}{8}$$

4. **Experiment:**

A fair coin is tossed three times. It shows:

Events Happening:

i. **One tail** **Lahore 2009**

Sol. Sample space of coins 3 times

$=S=\{HHH,HHT,HTH,THH,TTH,THT,HTT,TTT\},n(S)=8$

'A' represents " one tail "

$A=\{THH,HTH,HHT\}.n(A)=3$

$$P(A)=\frac{n(A)}{n(S)}=\frac{3}{8}$$

ii. **Atleast one head**

Sol. 'B' represents "at leant one head"

$B=\{HHH, HHT, HTH, THH, TTH, THT, HTT\}$ $n(B)=8$

$$P(B)=\frac{n(B)}{n(S)}=\frac{7}{8}$$

5. **Experiment:**

A die is rolled. The top shows

Events Happening:

i. **3 or 4 dots**

Sol. $S=\{1,2,3,4,5,6\}$ $n(S)=6$

'A' represents " 3 or 4 dots"

$A=\{3,4\}$ $n(A)=2$

$$P(A)=\frac{n(A)}{n(S)}=\frac{2}{6}=\frac{1}{3}$$

ii. **Dots less then 5** **Rawalpindi 2009**

Sol. 'B' represents "dots less then 5"

$B=\{1,2,3,4\}$ $n(B)=4$

$$P(B)=\frac{n(B)}{n(S)}=\frac{4}{6}=\frac{2}{3}$$

6. **Experiment:**

From a box containing slips numbered 1,2,3,........,5 one slip is picked up.

Events Happening:

i. **The number on the slip is a prime number**

Sol. $S=\{1,2,3,4,5\}$ $n(S)=5$ **Lahore 2009, Sargodha 2009, Fsd 2008**

A represents "Prime numbers"

$A=\{2,3,5\}$ $n(A)=3$

$$P(A)=\frac{n(A)}{n(S)}=\frac{3}{5}$$

ii. **The number on the slip is a Multiple of 3.**

Sol. 'B' represents "multiple of 3"

$B=\{3\}$ $n(B)=1$

$$P(B)=\frac{n(B)}{n(S)}=\frac{1}{3}$$

7. **Experiment:**

Two dice, one red and the other blue, are rolled simultaneously. The numbers of dots on the tops are added. The total of the two scores is:

Events Happening:

(i) 5 **(ii) 7** **(iii) 11**

Sol. Sample space of two dice is

$$S=\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1)(2,2),(2,3),(2,4),(2,5),(2,6),$$
$$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$$
$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}\ n(S)=36$$

i. 5 **Faisalabad 2009**

Sol. A represents "sum of dots is 5"

$$A=\{(1,4),(2,3),(3,2),(4,1)\}\ n(A)=4\Rightarrow P(A)=\frac{n(A)}{n(S)}=\frac{4}{36}=\frac{1}{9}$$

ii. 7

Sol. B represents "sum of dots is 7"

$$B=\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}\ n(B)=6\Rightarrow P(B)=\frac{n(B)}{n(S)}=\frac{6}{36}=\frac{1}{6}$$

iii. 11

Sol. **C** represents "sum of dots is 11"

$$C=\{(5,6),(6,5)\}\ n(C)=2\Rightarrow P(C)=\frac{n(C)}{n(S)}=\frac{2}{36}=\frac{1}{18}$$

8. **Experiment:**

A bag contains 40 balls out of which 5 are green, 15 are black and the remaining are yellow. A ball is taken out of the bag. **Faisalabad 2008**

Events Happening:

i. The ball is black

Sol. $n(S)=40$

Green balls	**= 5**
Black balls	**= 15**
Yellow balls	**= 20**

A represents "black balls" $n(A)=15\Rightarrow P(A)=\dfrac{n(A)}{n(S)}=\dfrac{15}{40}=\dfrac{3}{8}$

i. **The ball is Green**

Sol. B represents the " Green Balls" $n(B)=5$

$$P(B)=\frac{n(B)}{n(S)}=\frac{5}{40}=\frac{1}{8}$$

iii. **The ball is not Green**

Sol. C represents the " Balls is not green" $n(C)=35$

$$P(C)=\frac{n(C)}{n(S)}=\frac{35}{40}=\frac{7}{8}$$

9. Experiment:

One chit out of 30 containing the names of 30 students of a class of 18 boys and 12 girls is taken out at random, for nomination as the monitor of the class.

Events Happening:

Boys = 18, Girls =12

Sol. $S=\{1,2,3,..........,30\}$, $n(S)=30$

i. **The monitor is a boy**

Sol. A represents "monitor is Boy" $n(A)=18$

$$P(A)=\frac{n(A)}{n(S)}=\frac{18}{30}=\frac{3}{5}$$

ii. **The monitor is a girl**

Sol. B represents "monitor is girl" $n(B)=12$

$$P(B)=\frac{n(B)}{n(S)}=\frac{12}{30}=\frac{2}{5}$$

10. Experiment:

A coin is tossed four times. The tops show

Events Happening:

(i) All heads **(ii) 2 heads and 2 tails**

Sol. Coins Tossed 4 times $n(S)=2^n=2^4=16$

$$S=\{HHHH,HHHT,HHTH,HTHH,THHH,TTHH,HTHT,THTH,HHTT$$
$$THHT,HTTH,TTTH,TTHT,THTT,HTTT,TTTT\}$$

i. **All heads**

Sol. A represents "all heads" $A=\{HHHH\}$

$$P(A)=\frac{n(A)}{n(S)}=\frac{1}{16}$$

ii. **2 heads and 2 tails**

Sol. B represents "2 heads 2 tails"

$$B=\{HHTT,TTHH,THHT,HTTH,HTHT,THTH\},n(B)=6$$

$$P(B)=\frac{n(B)}{n(S)}=\frac{6}{16}=\frac{3}{8}$$

Exercise 7.6

1. **A fair coin is tossed 30 times, The tops show:**
Events Happening:

Event	Tally Marks	Frequency
Head	𝍸 𝍸 ||||	14
Tail	𝍸 𝍸 𝍸 |	16

i. **How many times does 'head' appear?**
ii. **How many times does 'tail' appear?**
iii. **Estimate the probability of the appearance of head?**
iv. **Estimate the probability of the appearance of tail?**

Sol. (i) Head appear = 14
(ii) Tail appear = 16

(iii) $$P(Head) = \frac{n(Head)}{n(S)} = \frac{14}{30} = \frac{7}{15}$$

(iv) $$P(Tail) = \frac{n(Tail)}{n(S)} = \frac{16}{30} = \frac{8}{15}$$

2. **A die tossed 100 times. The result is tabulated below. Study the table and answer the questions given below the table:**

3.

Event	Tally Marks	Frequency
1	𝍸 𝍸 ||||	14
2	𝍸 𝍸 𝍸 ||	17
3	𝍸 𝍸 𝍸 𝍸	20
4	𝍸 𝍸 𝍸 |||	18
5	𝍸 𝍸 𝍸	15
6	𝍸 𝍸 𝍸 |	16

i. **How many times do 3 dots appear?**
ii. **How many times do 5 dots appear?**
iii. **How many times does an even number of dots appear?**
iv. **How many times a prime number of dots appear?**
v. **Find the probability of each one of the above cases?**

Sol. (i) 3 dots appear $= n(3) = 20$

(ii) 5 dots appear $= n(5) = 15$

(iii) Even dots $= n(Even) = 17 + 18 + 16 = 51$

(iv) Prime nos $= n(prime) = 17 + 20 + 15 = 52$

$$P(3) = \frac{n(3)}{n(S)} = \frac{20}{100} = \frac{1}{5}$$

$$P(5) = \frac{n(5)}{n(S)} = \frac{15}{100} = \frac{3}{20}$$

$$P(\Pr ime) = \frac{n(prime)}{n(S)} = \frac{52}{100} = \frac{13}{25} \text{ and } P(Even) = \frac{n(Even)}{n(S)} = \frac{51}{100}$$

4. **The eggs supplied by a poultry farm during a week broke during transit as follows:**

$1\%,\ 2\%,\ 1\frac{1}{2}\%,\ \frac{1}{2}\%,\ 1\%,\ 2\%,\ 1\%.$

Find the probability of the eggs that broke in a day. Calculate the number of eggs that will be broken in transiting the following number of eggs:

(i) 7,000 (ii) 8,400 (iii) 10,500

Sol. Eggs broken in week $= 1\% + 2\% + 1\frac{1}{2}\% + \frac{1}{2}\% + 1\% + 2\% + 1\% = \frac{18}{2}\% = 9\% = \frac{9}{100}$

Eggs broken in one day $= \frac{9}{100} \times \frac{1}{7} = \frac{9}{700}$

I. 7,000

Sol. Eggs are 7,000 then Broken eggs $= 7000 \times \frac{9}{700} = 90$

ii. 8,400

Sol. Eggs are 8400 then broken eggs $= 8400 \times \frac{9}{100} = 108$

iii. 10,500

Sol. Eggs are 10500 then broken eggs $= 10500 \times \frac{9}{700} = 135$

Exercise 7.7

Formula:

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$ Gujranwala 2009

1. **If sample space $S=\{1,2,3,..........9\}$, Event $A=\{2,4,6,8\}$ and Event $B=\{1,3,5\}$ Find $P(A\cup B)$.** Sargodha 2009, Faisalabad 2008

Sol. $S=\{1,2,3,..........9\}$, $n(S)=9$

$A=\{2,4,6,8\}$, $n(A)=4$

$B=\{1,3,5\}$, $n(B)=3$

$$P(A)=\frac{n(A)}{n(S)}=\frac{4}{9}, P(B)=\frac{n(B)}{n(S)}=\frac{3}{9}=\frac{1}{3}$$

Note

$$P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{0}{9}=0$$

$$P(A\cup B)=P(A)+P(B)$$

$$=\frac{4}{9}+\frac{1}{3}=\frac{4+3}{9}=\frac{7}{9}$$

2. **A box contains 10 red, 30 white and 20 black marbles. A marble is drawn at random. Find the probability that it is either red or white.**

Sol. Red = 10, White = 30, Black = 20 Sargodha 2011

$n(S)=60$

A represents 'Red' and B represents 'White'

$n(A)=10, n(B)=30$

$$P(A)=\frac{n(A)}{n(S)}=\frac{10}{60}=\frac{1}{6}$$

$$\boxed{A\cap B=\varphi}$$

$$P(B)=\frac{n(B)}{n(S)}=\frac{30}{60}=\frac{1}{2}$$

$$P(A\ or\ B)=P(A\cup B)=P(A)+P(B)$$

$$=\frac{1}{6}+\frac{1}{2}=\frac{1+3}{6}=\frac{4}{6}=\frac{2}{3}$$

3. **A natural number is chosen out of the first fifty natural numbers. What is the probability that the chosen numbers is a multiple of 3 or of 5?**

Sol. $S=\{1,2,3,4,.........,50\}$ Multan 2007

$n(S)=50$

A represents "Multiple of 3"

$A=\{3,6,9,12,15,18,21,24,27,30,33,36,39,42,45,48\}$

$$n(A)=16 \Rightarrow P(A)=\frac{16}{50}$$

B represents "multiple of 5"

$$B=\{5,10,15,20,25,30,35,40,45,50\}$$

$$n(B)=10 \Rightarrow P(B)=\frac{n(B)}{n(S)}=\frac{10}{50}$$

$$A\cap B=\{15,30,45\}, \quad n(A\cap B)=3 \Rightarrow P(A\cap B)=\frac{3}{50}$$

$$P(A \text{ or } B)=P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$=\frac{16}{50}+\frac{10}{50}-\frac{3}{50}=\frac{23}{50}$$

4. **A card is drawn from a deck of 52 playing cards. What is the probability that it is a diamond card or an ace? Sargodha 2008, Multan 2007,2009**

Sol. $n(S)=$ **Total Cards = 52**

Diamond Cards $=n(A)=13$

Ace Cards $=n(B)=4$

$n(A\cap B)=1$ (Because in diamond also one card is ace)

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$=\frac{n(A)}{n(S)}+\frac{n(B)}{n(S)}-\frac{n(A\cup B)}{n(S)}=\frac{13}{52}+\frac{4}{52}-\frac{1}{52}$$

$$=\frac{13+4-1}{52}=\frac{16}{52}=\frac{4}{13}$$

5. **A die is thrown twice. What is the probability that the sum of the number of dots shown is 3 or 11? Faisalabad 2009, Multan 2008**

Sol. $n(s)=36$

A represents "sum of dots is 3" $\Rightarrow A=\{(1,2),(2,1)\}, \quad n(A)=2 \Rightarrow P(A)=\frac{2}{36}$

B represents "sum of dots is 11"

$$B=\{(5,6),(6,5)\}, \; n(B)=2$$

$$P(B)=\frac{n(B)}{n(S)}=\frac{2}{36} \qquad \boxed{A\cap B=\varphi}$$

$$P(A \text{ or } B)=P(A\cup B)=P(A)+P(B)$$

$$=\frac{2}{36}+\frac{2}{36}=\frac{4}{36}=\frac{1}{9}$$

6. Two dice are thrown. What is the probability that the sum of the number of dots appearing on them is 4 or 6?

Sol. $n(S)=36$

A represents sum is 4.
B represents sum is 6.

$A=\{(1,3),(2,2),(3,1)\},\quad n(A)=3$

$B=\{(1,5),(2,4),(3,3),(4,2),(5,1)\},\quad n(B)=5$

$$P(A)=\frac{n(A)}{n(S)}=\frac{3}{36}$$

$$P(B)=\frac{n(B)}{n(S)}=\frac{5}{36} \qquad \boxed{A\cap B=\varphi}$$

$$P(A\ or\ B)=P(A\cup B)=P(A)+P(B)=\frac{3}{36}+\frac{5}{36}=\frac{8}{36}=\frac{2}{9}$$

7. Two dice are thrown simultaneously. If the event A is that the sum of the number of dots shown is an odd number and the event B is that the number of dots shown on at least one die is 3.
Find $P(A\cup B)$

Sol. $n(S)=36$

A represents "sum is odd"
B represents "one dice is 3"

$A=\{(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),(3,2),(3,4),(3,6),(4,1),(4,3)$
$(4,5),(5,2),(5,4),(5,6),(6,1),(6,3),(6,5)\}$

$$n(A)=18\Rightarrow P(A)=\frac{n(A)}{n(S)}=\frac{18}{36}$$

$B=\{(1,3),(2,3),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,3),(5,3),(6,3)\}$

$$n(B)=11\Rightarrow P(B)=\frac{n(B)}{n(S)}=\frac{11}{36}$$

$A\cap B=\{(2,3),(3,2),(3,4),(3,6),(4,3),(6,3)\},\ n(A\cap B)=6$

$$P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{6}{36}=\frac{1}{6}$$

$$P(A\ or\ B)=P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$=\frac{18}{36}+\frac{11}{36}-\frac{6}{36}=\frac{23}{36}$$

8. **There are 10 girls and 20 boys in a class. Half of the boys and half of the girls have blue eyes. Find the probability that one student chosen as monitor is either a girl or has blue eyes.**

Sol. Girls = 10, Boys =20, $n(S)=30$

A represents girls $\Rightarrow n(A)=10$

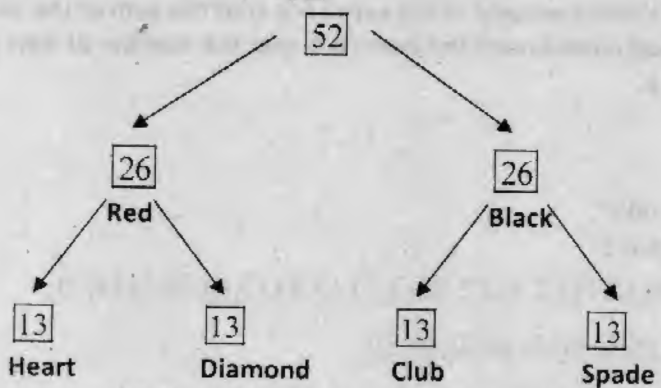
B represent students have blue eyes $\Rightarrow n(B)=15 \text{ and } n(A\cap B)=5$

$$P(A\cup B)=\frac{n(A)}{n(S)}+\frac{n(B)}{n(S)}-\frac{n(A\cap B)}{n(S)}$$

$$=\frac{10}{30}+\frac{15}{30}-\frac{5}{30}=\frac{20}{30}=\frac{2}{3}$$

Note: **"Playing Cards"**

Total



Each Type (Heart, Diamond, Club, Spade)
Consists of one – Ace
one – King
one – Queen
one – Jack

Exercise 7.8

Formula:

$$P(A \text{ and } B) = P(A \cap B) = P(A).P(B)$$

1. The probability that a person A will be alive 15 years hence is 5/7and the probability that another person B will be alive 15 years hence is 7/9. Find the Probability that both will be alive 15 years hence.

Sol. $P(A) = \frac{5}{7}, P(B) = \frac{7}{9}$

$$P(A \cap B) = P(A).P(B) = \frac{5}{7} \cdot \frac{7}{9} = \frac{5}{9}$$

2. A die is rolled twice: Even E_1 is the appearance of even number of dots and even E_2 is the appearance of more than 4 dots. Prove that:

Sol. $S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1)(2,2),(2,3),(2,4),(2,5),(2,6),$
$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$
$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$ $n(S) = 36$

$E_1 = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$, $n(E_1) = 9$

$E_2 = \{(5,5),(5,6),(6,5),(6,6)\}$, $n(E_2) = 4$

$E_1 \cap E_2 = \{(6,6)\}$

$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{9}{36}$ and $P(E_2) = \frac{n(E_2)}{n(S)} = \frac{4}{36}$

$$P(E_1).P(E_2) = \frac{9}{36} \cdot \frac{4}{36} = \frac{36}{36 \times 36} = \frac{1}{36} \longrightarrow I$$

$$P(E_1 \cap E_2) = \frac{n(E_1 \cap E_2)}{n(s)} = \frac{1}{36} \longrightarrow II$$

From I & II

$$P(E_1 \cap E_2) = P(E_1).P(E_2)$$

3. Determine the probability of getting 2 heads in two successive tosses of a balanced coin. **Gujranwala 2009, Rawalpindi 2009**

Sol. $S = \{HH, HT, TH, TT\}$, $n(S) = 4$

Let A represents 2 heads.

$$A = \{HH\}, \quad n(A) = 1 \Rightarrow P(A) = \frac{n(A)}{n(S)} = \frac{1}{4}$$

4. Two coins are tossed twice each. Find the probability that the head appears ●the first toss and the same faces appear in the two tosses.

Sol. $S = \{HH, HT, TH, TT\}$, $n(S) = 4$

Let A represents "head appears first"

$A = \{HH, HT\}$, $n(A) = 2$

B represents " Same faces"

$B = \{HH, TT\}$, $n(B) = 2$

$P(A \cap B) = P(A).P(B)$

$$= \frac{n(A)}{n(S)} \cdot \frac{n(B)}{n(S)} = \frac{2}{4} \cdot \frac{2}{4} = \frac{1}{4}$$

5. Two cards are drawn from a deck of 52 playing cards. If one card is drawn and replaced before drawing the second card, Find the probability that both the cards are aces.

Sol. $n(S) = 52$

Let A represents "aces"

B represents " aces"

$n(A) = 4$, $n(B) = 4$

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{1}{13}$$

$$P(A \cap B) = P(A).P(B) = \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$$

6. Two cards from a deck of 52 playing cards are drawn in such a way that the card is replaced after the firs draw. Find the probabilities in the following cases:

i. First Card is king and the second is a queen.

Sol. Let A represents "King"

$n(S) = 52$, $n(A) = 4$

B represents "Queen" , $n(B) = 4$

$P(A \cap B) = P(A).P(B)$

$$= \frac{4}{52} \cdot \frac{4}{52} = \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$$

ii. Both the cards are faced cards i.e. king, queen, jack.

Sol. A represents face Cards

B represents face Cards.

$n(A)=12, \quad n(B)=12$

$P(A\cap B)=P(A).P(B)$

$$=\frac{12}{52}.\frac{12}{52}=\frac{3}{13}.\frac{3}{13}=\frac{9}{169}$$

7. Two dice are thrown twice. What is probability that um of the dots shown in the first thrown is 7 and that of the second throw is 11? Federal

Sol. $n(s)=36$

Let A represents "sum is 7"

B represents " sum is 11"

$A=\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$, $n(A)=6$, $P(A)=\frac{n(A)}{n(S)}\Rightarrow P(A)=\frac{6}{36}$

$B=\{(5,6),(6,5)\}$, $n(B)=2$, $P(B)=\frac{n(B)}{n(S)}=\frac{2}{36}$

$P(A\cap B)=P(A).P(B)$

$$=\frac{6}{36}.\frac{2}{36}=\frac{1}{108}$$

8. Find the probability that the sum of dots appearing in two successive throws of two dice is every time 7?

Sol. $n(S)=36$

Let A represents "sum is 7"

B represents "sum is 7"

$A=\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$, $n(A)=6$, $P(A)=\frac{6}{36}=\frac{1}{6}$

$B=\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$, $n(B)=2$, $P(B)=\frac{6}{36}=\frac{1}{6}$

$P(A\cap B)=P(A).P(B)=\frac{1}{6}.\frac{1}{6}=\frac{1}{36}$

). A fair die is thrown twice. Find the probability that a prime number of dots appear in the first throw and the number of dots in the second throw is less then 5.

ol. $S=\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1)(2,2),(2,3),(2,4),(2,5),(2,6),$
$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$
$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$ $n(S)=36$

$A=\{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5)$
$(3,6),(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\}$

$$n(A)=18,\ P(A)=\frac{n(A)}{n(S)}=\frac{18}{36}=\frac{1}{2}$$

$$B=\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),$$
$$(4,1),(4,2),(4,3),(4,4),(5,1),(5,2),(5,3),(5,4),(6,1),(6,2),(6,3),(6,4)\}$$

$$n(B)=24,\ \ P(B)=\frac{n(B)}{n(S)}=\frac{24}{36}=\frac{2}{3}$$

$$P(A\cap B)=P(A).P(B)=\frac{1}{2}.\frac{2}{3}=\frac{1}{3}$$

10. **A bag contains 8 red, 5 white and 7 black balls. 3 balls are drawn from the bag. What is the probability that the first ball is red, the second ball is white and the third ball is black, when every time the ball is replaced?**

Hint: $\left(\frac{8}{20}\right)\left(\frac{5}{20}\right)\left(\frac{7}{20}\right)$ **is the probability.**

Sol. **Red = 8, White = 5, Black = 7**

n(Red) = 8, n (White) = 20, n(Black) = 7

P(Red, White, Black) = P(Red). P(White). P(Black)

$$=\left(\frac{8}{20}\right)\left(\frac{5}{20}\right)\left(\frac{7}{20}\right)=\frac{280}{8000}=\frac{7}{200}$$

TEST YOUR SKILLS

Marks: 50

Q # 1. Select the Correct Option **(10)**

i. For two events A and B if $p(A) = p(B) = \frac{1}{2}$ then $p(A \cap B) =$ is:

a) $\frac{1}{4}$ b) $\frac{1}{2}$

c) 1 d) 0

ii. If ${}^{n}C_{8} = {}^{n}C_{12}$ then $n =$

a) 4 b) 8

c) 12 d) 20

iii. If A and B are disjoint events then $p(A \cup B) = ?$

a) $p(A) + p(B)$ b) $p(A) - p(B)$

c) $p(A) + p(B) - p(A \cap B)$ d) None

iv. ${}^{n}C_{r} =$

a) ${}^{n}C_{r-n}$ b) ${}^{n}C_{n-r}$

c) ${}^{n}C_{n/r}$ d) ${}^{r}C_{n}$

v. For an event E, the range of its probability is:

a) $-1 \leq p(E) \leq 1$ b) $0 < p(E) < 1$

c) $0 \leq p(E) \leq 1$ d) $-1 < p(E) < 1$

vi. Value of ${}^{4}P_{2}$ is equal to:

a) 12 b) 4

c) 6 d) 8

vii. The set consisting of all possible out comes of a given experiment is called

a) Events b) Sample Space

c) Combination d) Probability

viii. What is the probability of an ace from 52 cards:

a) $1/52$ b) $3/13$

c) $1/13$ d) $1/26$

Pakistan and India play a cricket match then number of possible outcomes are

a) 1 b) 2

c) 3 d) 4

x. $P(E)$ equals

a) $1 + P(E)$ b) $P(E) - 1$

c) $1 - P(E)$ d) $2 - P(E)$

Q # 2. Short Questions: **(10 X 2 = 20)**

i. Define permutation and write formula of ${}^nP_r$:

ii. How many triangles can be formed by joining 8 sided polygon.

iii. A die is rolled what is the probability that dots on top are greater then 4.

iv. A box contains 10 red, 30 white, 20 black, marbles, A marble is drawn. Find the probability it is either red or white.

v. How many necklaces can be made from 6 beads of different colours.

vi. A card is drawn from deck of 52 cards, Find probability card is king.

vii. Find n when ${}^{11}P_n = 11.10.9$

viii. How many arrangements of the letters of PAKISTAN taken all together can be made:

ix. If ${}^nP_4 : {}^{n-1}P_3 = 9:1$ Find n?

x. Pakistan and India plays a Cricket match What is the probability that match is draw:

Long Questions: **(2 X 10 = 20)**

Q # 3. (a) Show that ${}^{16}C_{11} + {}^{16}C_{10} = {}^{17}C_{11}$

(b) Show that ${}^{n-1}C_r + {}^{n-1}C_{r-1} = {}^nC_r$

Q # 4. (a) Find n and r if ${}^{n-1}C_{r-1} : {}^nC_r : {}^{n+1}C_{r+1} = 3:6:11$

(b) How many signals can be made with 4 different flags when any number of them are to be used at a time?

Mathematical Induction and Binomial Theorem

8

Exercise 8.1

Example:6 $show\,that\; 4^n > 3^n + 4 \;\; for\; n \geq 2 \; (only\; for\; n = 2, 3)$ **Multan 2007,09**

Sol. $S(n): 4^n > 3^n + 4$

$For\; n = 2$

$S(2): 4^2 > 3^2 + 4 \Rightarrow 16 > 13\; True$

$For\; n = 3$

$S(3): 4^3 > 3^3 + 4 \Rightarrow 64 > 31\; True$

Use mathematical induction to prove the following formula for every positive integer *n*.

1. $\mathbf{1+5+9+\ldots\ldots\ldots+(4n-3)=n(2n-1)}$ **Multan 2008, Sargodha 2008**

Sol. $S(n): 1+5+9+\ldots\ldots\ldots+(4n-3) = n(2n-1)$

$C-1$: Put n = 1 then S(1): 4(1) – 3 = 4 – 3 = 1= 1(2(1) – 1) = 2 – 1 = 1

$C-1$ is satisfied

$C-2$: Let it be true for $n = k \in N$ then

$S(k): 1+5+9+\ldots\ldots\ldots+(4k-3) = k(2k-1) \longrightarrow (A)$

For $n = k+1$ statements is

$S(k+1): 1+5+9+\ldots\ldots\ldots+(4(k+1)-3) = (k+1)(2(k+1)-1)$

or $\quad 1+5+9+\ldots\ldots\ldots+(4k+4-3) = (k+1)(2k+2-1)$

or $\quad 1+5+9+\ldots\ldots\ldots+(4k+1) = (k+1)(2k+1) \longrightarrow (B)$

Adding both side $4k+1$ in (A) we get.

$$1+5+9+\ldots\ldots\ldots+(4k-3)+(4k+1) = k(2k-1)+4k+1$$

$$= 2k^2 - k + 4k + 1$$

$$= 2k^2 + 3k + 1$$

$$= 2k^2 + 2k + k + 1$$

$$= 2k(k+1) + 1(k+1)$$

$$=(k+1)(2k+1)$$

Which is B, so it is true for $n=k+1$, $C-2$ is satisfied.

Hence given statement is true for every +ve integer n.

2. $\mathbf{1+3+5+.........+(2n-1)=n^2}$ **Multan 2008**

Sol. $S(n):1+3+5+.........+(2n-1)=n^2$

$C-1$: Put n = 1 then S(1): $2(1)-1=2-1=1=(1)^2=1$

$C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$S(k):1+3+5+..........+(2k-1)=k^2 \longrightarrow (A)$

For $n=k+1$ statements is

$S(k+1):1+3+5+..........+(2(k+1)-1)=(k+1)^2$

$1+3+5+..........+(2k+2-1)=k^2+2k+1$

$1+3+5+..........+(2k+1)=k^2+2k+1 \longrightarrow (B)$

Adding both side $2k+1$ in (A) we get.

$1+3+5+..........+(2k-1)+(2k+1)=k^2+2k+1$

Which is (B), so it is true for every +ve integers n.

3. $\mathbf{1+4+7+.........+(3n-2)=\dfrac{n(3n-1)}{2}}$

Sol. $C-1$: Put n = 1 then S(1): $3(1)-2=3-2=1$

$=\dfrac{1(3(1)-1)}{2}=\dfrac{3-1}{2}=\dfrac{2}{2}=1 \quad \Rightarrow C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$S(k):1+4+7+.........+(3k-2)=\dfrac{k(3k-1)}{2} \longrightarrow (A)$

For $n=k+1$ statements is

$S(k+1):1+4+7+..........+(3(k+1)-2)=\dfrac{(k+1)(3(k+1)-1)}{2}$

$1+4+7+..........+(3k+3-2)=\dfrac{(k+1)(3k+3-1)}{2}$

$1+4+7+..........+(3k+1)=\dfrac{(k+1)(3k+2)}{2} \longrightarrow (B)$

Adding both sides $(3k+1)$ in (A) get.

$$1+4+7+..........+(3k-2)+(3k+1)=\frac{k(3k-1)}{2}+(3k+1)$$

$$=\frac{3k^2-k+6k+2}{2}=\frac{3k^2+5k+2}{2}$$

$$1+4+7+..........+(3k-2)+(3k+1)=\frac{3k^2+5k+2}{2}=\frac{3k^2+3k+2k+2}{2}$$

$$=\frac{3k(k+1)+2(k+1)}{2}=\frac{(k+1)(3k+2)}{2}$$

Which is (B) so it is true for $n=k+1$

$C-2$ is satisfied

Hence given statement is true for very +ve integer n.

4. $\mathbf{1+2+4+.........+2^{n-1}=2^n-1}$ **Sargodha 2008**

Sol. $S(n):1+2+4+ \quad - \quad +2^{n-1}=2^n-1$

$C-1:$ Put n = 1 then S(1): $=2^{1-1}=2^0=1=2^1-1=2-1=1$

$C-1$ is satisfied

$C-2:$ Let it be true for $n=k\in N$ then

$S(k):1+2+4+..........+2^{k-1}=2^{k+1}-1 \longrightarrow (A)$

For $n=k+1$ statements is

$S(k+1):1+2+4+..........+2^{k+1-1}=2^k-1$

$1+2+4+..........+2^k=2^{k+1}-1 \longrightarrow (B)$

Adding both side 2^k in (A) we get.

$$1+2+4+..........+2^{k-1}+2^k=2^k-1+2^k$$

$$=2^k+2^k-1$$

$$=2.2^k-1=2^{k+1}-1$$

Which is (B) so it is true for $n=k+1$, $C-2$ is satisfied

Hence given statement in true for every +ve integer n.

5. $\mathbf{1+\frac{1}{2}+\frac{1}{4}+.........+\frac{1}{2^{n-1}}=2\left[1-\frac{1}{2^n}\right]}$ **Multan 2009**

Sol. $S(n):\ 1+\frac{1}{2}+\frac{1}{4}+.........+\frac{1}{2^{n-1}}=2\left[1-\frac{1}{2^n}\right]$

$C-1:$ Put n = 1 then S(1): $\frac{1}{2^{1-1}}=\frac{1}{2^0}=\frac{1}{1}=1$

$$=2\left[1-\frac{1}{2^1}\right]=2\left[1-\frac{1}{2}\right]=1$$

$C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$$S(k):1+\frac{1}{2}+\frac{1}{4}+\ldots\ldots\ldots\ldots+\frac{1}{2^{k-1}}=2\left[1-\frac{1}{2^k}\right]\longrightarrow(A)$$

For $n=k+1$ statements is

$$S(k+1):1+\frac{1}{2}+\frac{1}{4}+\ldots\ldots\ldots\ldots+\frac{1}{2^{k+1-1}}=2\left[1-\frac{1}{2^{k+1}}\right]$$

$$1+\frac{1}{2}+\frac{1}{4}+\ldots\ldots\ldots+\frac{1}{2^k}=2\left[1-\frac{1}{2.2^k}\right]=2-2\times\frac{1}{2.2^k}=2-\frac{1}{2^k}\longrightarrow(B)$$

Adding both side $\frac{1}{2^k}$ in (A) we get.

$$1+\frac{1}{2}+\frac{1}{4}+\ldots\ldots\ldots+\frac{1}{2^{k-1}}+\frac{1}{2^k}=2\left(1-\frac{1}{2^k}\right)+\frac{1}{2^k}$$

$$=2-\frac{2}{2^k}+\frac{1}{2^k}=2-\frac{1}{2^k}$$

Which is (B) so it is true for $n=k+1$, $C-2$ is satisfied

Hence given statement is true for every +ve integer n.

6. $2+4+6+\ldots\ldots\ldots+2n=n(n+1)$ **Sargodha 2009**

Sol. $S(n)$: $2+4+6+\ldots\ldots\ldots\ldots+2n=n(n+1)$

$C-1$: Put n = 1 then S(1): $2(1)=2=1(1+1)=1(2)=2$

$C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$$S(k):2+4+6+\ldots\ldots\ldots+2k=k(k+1)\longrightarrow(A)$$

For $n=k+1$ statements is

$$S(k+1):2+4+6+\ldots\ldots\ldots+2(k+1)=(k+1)(k+1+1)$$

$$2+4+6+\ldots\ldots\ldots+2(k+1)=(k+1)(k+2)\longrightarrow(B)$$

Adding both side $2(k+1)$ in (A) we get.

$$2+4+6+\ldots\ldots\ldots+2k+2(k+1)=k(k+1)+2(k+1)$$

$=(k+1)(k+2)$ Which is true

Which is (B) so it is true for $n=k+1$, $C-2$ is satisfied

Hence given statement in true for every +ve integer n.

7. $\mathbf{2+6+18+\dots\dots\dots+2\times3^{n-1}=3^n-1}$

Sol. $S(n): 2+6+18+\dots\dots\dots+2\times3^{n-1}=3^n-1$

$C-1$: Put n = 1 then S(1): $=2\times3^{1-1}=2\times3^0=2\times1=2=3^1-1=3-1=2$

$C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$S(k): 2+6+18+\dots\dots\dots+2\times3^{k-1}=3^k-1 \longrightarrow (A)$

For $n=k+1$ statements is

$S(k+1): 2+6+18+\dots\dots\dots+2\times3^{k+1-1}=3^{k+1}-1$

$2+6+18+\dots\dots\dots+2\times3^k=3^{k+1}-1 \longrightarrow (B)$

Adding both side 2×3^k in (A) we get.

$$2+6+18+\dots\dots\dots+2\times3^{k-1}+2\times3^k=3^k-1+2\times3^k$$
$$=3^k+2.3^k-1$$
$$=3^k(1+2)-1=3^k.3-1$$

Which is (B) so it is true for $n=k+1$ $=3^{k+1}-1$

$C-2$ is satisfied.

Hence given statement in true for every +ve integer n.

8. $\mathbf{1\times3+2\times5+3\times7+\dots\dots\dots+n\times(2n+1)=\dfrac{n(n+1)(4n+5)}{6}}$

Sol. $S(n): 1\times3+2\times5+3\times7+\dots\dots\dots+n\times(2n+1)=\dfrac{n(n+1)(4n+5)}{6}$

$C-1$: Put n = 1 then S(1): $=1\times(2(1)+1)=2+1=3$

$$=\frac{1(1+1)(4(1)+5)}{6}=\frac{2(9)}{6}=\frac{18}{6}=3$$

$C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$$S(k): 1\times3+2\times5+3\times7+\dots\dots\dots+k\times(2k+1)=\frac{k(k+1)(4k+5)}{6} \longrightarrow (A)$$

For $n=k+1$ statements is

$$S(k+1): 1\times3+2\times5+3\times7+\dots\dots\dots+(k+1)(2(k+1)+1)=\frac{(k+1)(k+1+1)(4(k+1)+5)}{6}$$

$$1\times3+2\times5+3\times7+\dots\dots\dots+(k+1)\times(2k+2+1)=\frac{(k+1)(k+2)(4k+4+5)}{6}$$

$$1\times3+2\times5+3\times7+\dots\dots\dots+(k+1)\times(2k+3)=\frac{(k+1)(k+2)(4k+9)}{6}\longrightarrow(B)$$

Adding both side $(k+1)\times(2k+3)$ in (A) we get.

$$1\times3+2\times5+3\times7+\dots..+k\times(2k+1)+(k+1)\times(2k+3)=\frac{k(k+1)(4k+5)}{6}+(k+1)\times(2k+3)$$

$$=(k+1)\left[\frac{k(4k+5)}{6}+(2k+3)\right]$$

$$=(k+1)\left[\frac{k(4k+5)}{6}+(2k+3)\right]=(k+1)\left[\frac{4k^2+5k+12k+18}{6}\right]$$

$$=(k+1)\left[\frac{4k^2+17k+18)}{6}\right]=(k+1)\left[\frac{4k^2+8k+9k+18}{6}\right]$$

$$=\frac{(k+1)}{6}\left[\frac{4k(k+2)+9(k+2)}{6}\right]$$

$$=\frac{(k+1)}{6}\left[\frac{(k+2)(4k+9)}{6}\right]$$

Which is (B) so it is true for $n=k+1, C-2$ is satisfied.

Hence given statement in true for every +ve integer n.

9. $$1\times2+2\times3+3\times4+\dots\dots\dots+n\times(n+1)=\frac{n(n+1)(n+2)}{3}$$

Sol. $$S(n):1\times2+2\times3+3\times4+\dots\dots\dots+n\times(n+1)=\frac{n(n+1)(n+2)}{3}$$

$C-1$: Put n = 1 then S(1): $1\times(1+1)=2=\frac{1(1+1)(1+2)}{3}=\frac{2\times3}{3}=2$

$C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$$S(k):1\times2+2\times3+3\times4+\dots\dots\dots+k\times(k+1)=\frac{k(k+1)(k+2)}{3}\longrightarrow(A)$$

For $n=k+1$ statements is

$$S(k+1):1\times2+2\times3+3\times4+\dots\dots\dots+(k+1)\times(k+1+1)=\frac{(k+1)(k+1+1)(k+1+2)}{3}$$

$$1\times2+2\times3+3\times4+\dots\dots\dots+(k+1)\times(k+2)=\frac{(k+1)(k+2)(k+3)}{3}\longrightarrow(B)$$

Adding both side $(k+1)\times(k+2)$ in (A) we get.

$$1\times2+2\times3+3\times4+......+k\times(k+1)+(k+1)\times(k+2)=\frac{k(k+1)(k+2)}{3}+(k+1)\times(k+2)$$

$$=(k+1)\left[\frac{k(k+2)}{3}+k+2\right]=(k+1)\left[\frac{k^2+2k+3k+6}{3}\right]$$

$$=\frac{(k+1)}{3}[k(k+1)+3(k+2)]=\frac{(k+1)}{3}[(k+2)(k+3)]$$

$$=\frac{(k+1)(k+2)(k+3)}{3}$$

Which is (B) so it is true for $n=k+1, C-2$ is satisfied.

Hence given statement in true for every +ve integer n.

10. $$\mathbf{1\times2+3\times4+5\times6+.........+(2n-1)\times2n=\frac{n(n+1)(4n-1)}{3}}$$

Sol. $S(n): 1\times2+3\times4+5\times6+.........+(2n-1)\times2n=\dfrac{n(n+1)(4n-1)}{3}$

$C-1$: Put n = 1 then S(1): $(2(1)-1)\times2(1)=(2-1)\times2=2$

$$=\frac{1(1+1)(4(1)-1)}{3}=\frac{2(3)}{3}=2$$

$\Rightarrow C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$$S(n): 1\times2+3\times4+5\times6+.........+(2k-1)\times2k=\frac{k(k+1)(4k-1)}{3}$$

For $n=k+1$ statements is

$$S(k+1): 1\times2+3\times4+5\times6+......+(2(k+1)-1)\times2(k+1)=\frac{(k+1)(k+1+1)(4(k+1)-1)}{3}$$

$$1\times2+3\times4+5\times6+.........+(2k+2-1)\times2(k+1)=\frac{(k+1)(k+2)(4k+4-1)}{3}$$

$$1\times2+3\times4+5\times6+.......+(2k+1)\times2(k+1)=\frac{(k+1)(k+2)(4k+3)}{3}\longrightarrow(B)$$

Adding both side $(2k+1)\times2(k+1)$ in (A).

$$1\times2+3\times4+5\times6+.........+(2k-1)\times2k+(2k+1)\times2(k+1)$$

$$=\frac{k(k+1)(4k-1)}{3}+(2k+1)\times2(k+1)$$

$$=(k+1)\left[\frac{k(4k-1)}{3}+2(2k+1)\right]$$

$$=(k+1)\left[\frac{4k^2-k}{3}+4k+2\right]=\frac{(k+1)}{3}\left[4k^2-k+12k+6\right]$$

$$=\frac{(k+1)}{3}\left[4k^2+11k+6\right]=\frac{(k+1)}{3}\left[4k^2+8k+3k+6\right]$$

$$=\frac{(k+1)}{3}\left[4k(k+2)+3(k+2)\right]$$

$$=\frac{(k+1)}{3}\left[(k+2)(4k+3)\right]=\frac{(k+1)(k+2)(4k+3)}{3}$$

Which is (B) so it is true for $n=k+1, C-2$ is satisfied.

Hence given statement in true for every +ve integer n.

11. $$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\dots\dots\dots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$

Sol. $$S(n):\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\dots\dots\dots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$

$C-1:$ Put n = 1 then S(1): $\frac{1}{1(1+1)}=\frac{1}{2}=1-\frac{1}{1+1}=1-\frac{1}{2}=\frac{1}{2}$

$C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$$S(k):\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\dots\dots\dots+\frac{1}{k(k+1)}=1-\frac{1}{k+1}\longrightarrow (A)$$

For $n=k+1$ then statements is

$$S(k+1):\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\dots\dots\dots+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+1+1}\longrightarrow (B)$$

Adding both side $\frac{1}{(k+1)(k+2)}$ in (A).

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\dots\dots\dots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=1-\left[\frac{1}{k+1}-\frac{1}{(k+1)(k+2)}\right]=1-\left[\frac{k+2-1}{(k+1)(k+2)}\right]$$

$$=1-\left[\frac{k+2-1}{(k+1)(k+2)}\right]=1-\left[\frac{\cancel{(k+1)}}{\cancel{(k+1)}(k+2)}\right]=1-\frac{1}{k+2}$$

Which is (B) so it is true for $n=k+1, C-2$ is satisfied.

Hence given statement in true for every +ve integer n.

12. $$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+.........+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$ **Sargodha 2009**

Sol. $$S(n):\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+.........+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

$C-1$: Put n = 1 then S(1): $\frac{1}{(2(1)-1)(2(1)+1)}=\frac{1}{(2-1)(2+1)}=\frac{1}{3}$

$$=\frac{1}{2(1)+1}=\frac{1}{2+1}=\frac{1}{3}$$

$C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$$S(k):\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+.........+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}\longrightarrow(A)$$

For $n=k+1$ then statements is

$$S(k+1):\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+.........+\frac{1}{[2(k+1)-1][2(k+1)+1]}=\frac{k+1}{(2(k+1)+1)}$$

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+.........+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

Adding both side $\frac{1}{(2k+1)(2k+3)}$ in (A).

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+.........+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}=\frac{1}{(2k+1)}\left[k+\frac{1}{2k+3}\right]$$

$$=\frac{1}{(2k+1)}\left[\frac{2k^2+3k+1}{(2k+3)}\right]=\frac{1}{(2k+1)}\left[\frac{2k^2+2k+k+1}{(2k+3)}\right]$$

$$=\frac{1}{(2k+1)}\left[\frac{2k(k+1)+1(k+1)}{(2k+3)}\right]=\frac{1}{\cancel{(2k+1)}}\left[\frac{(k+1)\cancel{(2k+1)}}{2k+3}\right]=\frac{k+1}{2k+3}$$

Which is (B) so it is true for $n=k+1, C-2$ is satisfied.

Hence given statement in true for every +ve integer n.

13. $$\frac{1}{2\times5}+\frac{1}{5\times8}+\frac{1}{8\times11}+\dots\dots\dots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{2(3n+2)}$$

Sol. $$S(n):\frac{1}{2\times5}+\frac{1}{5\times8}+\frac{1}{8\times11}+\dots\dots\dots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{2(3n+2)}$$

$C-1$: Put n = 1 then S(1): $\frac{1}{(3(1)-1)(3(1)+2)}=\frac{1}{(2)(5)}=\frac{1}{10}$

$$=\frac{1}{2(3(1)+2)}=\frac{1}{2(5)}=\frac{1}{10}$$

$C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$$S(k):\frac{1}{2\times5}+\frac{1}{5\times8}+\frac{1}{8\times11}+\dots\dots\dots+\frac{1}{(3k-1)(3k+2)}=\frac{k}{2(3k+2)}$$

For $n=k+1$ then statements is

$$S(k+1):\frac{1}{2\times5}+\frac{1}{5\times8}+\frac{1}{8\times11}+\dots\dots\dots+\frac{1}{[3(k+1)-1][3(k+1)+2]}=\frac{k+1}{2(3(k+1)+2)}$$

$$\frac{1}{2\times5}+\frac{1}{5\times8}+\frac{1}{8\times11}+\dots\dots\dots+\frac{1}{(3k+3-1)(3k+3+2)}=\frac{k+1}{2(3k+3+2)}$$

$$\frac{1}{2\times5}+\frac{1}{5\times8}+\frac{1}{8\times11}+\dots\dots\dots+\frac{1}{(3k+2)(3k+5)}=\frac{k+1}{2(3k+5)}$$

Adding both side $\frac{1}{(3k+2)(3k+5)}$ in (A).

$$\frac{1}{2\times5}+\frac{1}{5\times8}+\frac{1}{8\times11}+\dots\dots\dots+\frac{1}{(3k-1)(3k+2)}+\frac{1}{(3k+2)(3k+5)}$$

$$=\frac{k}{2(3k+2)}+\left[\frac{1}{(3k+2)(3k+5)}\right]=\frac{1}{(3k+2)}\left[\frac{k}{2}+\frac{1}{3k+5}\right]$$

$$=\frac{1}{(3k+2)}\left[\frac{3k^2+5k+2}{2(3k+5)}\right]=\frac{1}{(3k+2)}\left[\frac{3k^2+3k+2k+2}{2(3k+5)}\right]$$

$$=\frac{1}{(3k+2)}\left[\frac{3k(k+1)+2(k+1)}{2(3k+5)}\right]=\frac{(k+1)\cancel{(3k+2)}}{2\cancel{(3k+2)}(3k+5)}=\frac{k+1}{2(3k+5)}$$

Which is (B) so it is true for $n=k+1, C-2$ is satisfied.

Hence given statement in true for every +ve integer n.

14. $$r+r^2+r^3+.........+r^n=\frac{r(1-r^n)}{1-r},(r\neq 1)$$

Sol. $$S(n): r+r^2+r^3+.........+r^n=\frac{r(1-r^n)}{1-r}$$

$C-1$: Put n = 1 then S(1): $r^1=r=\frac{r(1-r^1)}{1-r}=\frac{r(1-r)}{1-r}=r$

$C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$$S(k): r+r^2+r^3+.........+r^k=\frac{r(1-r^k)}{1-r}\longrightarrow(A)$$

For $n=k+1$ then statements is

$$S(k+1): r+r^2+r^3+.........+r^{k+1}=\frac{r(1-r^{k+1})}{1-r}$$

$$=\frac{r-r^{k+2}}{1-r}\longrightarrow(B)$$

Adding both side r^{k+1} in (A) get.

$$r+r^2+r^3+.........+r^k+r^{k+1}=\frac{r(1-r^k)}{1-r}+r^{k+1}$$

$$=\frac{r-r^{k+1}}{1-r}+r^{k+1}=\frac{r-\cancel{r^{k+1}}+\cancel{r^{k+1}}-r^{k+2}}{1-r}=\frac{r-r^{k+2}}{1-r}$$

Which is (B) so it is true for $n=k+1, C-2$ is satisfied.

Hence given statement in true for every +ve integer n.

15. $a+(a+d)+(a+2d)+........[a+(n-1)d]=\frac{n}{2}[2a+(n-1)d]$ **Faisalabad 2009**

Sol. $S(n)$: Let $a+(a+d)+(a+2d)+........[a+(n-1)d]=\frac{n}{2}[2a+(n-1)d]$

$C-1$: Put n = 1 then S(1): $a+(1-1)d=a+0.d=a$

$=\frac{1}{2}[2a+(1-1)d]=\frac{1}{2}[2a+0.d]=\frac{2a}{2}=a$

$C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$S(k):a+(a+2d)+........+[a+(k-1)d]=\frac{k}{2}[2a+(k-1)d]\longrightarrow(A)$

For $n=k+1$ then statements is

$S(k+1):a+(a+d)+(a+2d)+........+[a+(k+\cancel{1}-\cancel{1})d]=\frac{(k+1)}{2}[2a+(k+1-1)d]$

$S(k+1):a+(a+d)+(a+2d)+........+(a+kd)=\left(\frac{k+1}{2}\right)[2a+kd]$

Adding both side $a+kd$ in (A) get.

$a+(a+d)+(a+2d)+........+(a+(k-1)d)+(a+kd)$

$$=\frac{k}{2}[2a+(k-1)d]+a+kd$$

$$=\frac{k}{2}[2a+kd-d]+a+kd$$

$$=\frac{2ak+k^2d-kd}{2}+a+kd$$

$$=\frac{2ak+k^2d-kd+2a+2kd}{2}$$

$$=\frac{2a+2ak+kd+k^2d}{2}$$

$$=\frac{2a(1+k)+kd(1+k)}{2}$$

$$=\frac{(1+k)}{2}(2a+kd)$$

Which is (B) so it is true for $n=k+1, C-2$ is satisfied.
Hence given statement in true for every +ve integer n.

16. $1\lfloor\underline{1}+2\lfloor\underline{2}+3\lfloor\underline{3}\ldots\ldots\ldots+n\lfloor\underline{n}=\lfloor\underline{n+1}-1$

Sol. $S(n):\ 1\lfloor\underline{1}+2\lfloor\underline{2}+3\lfloor\underline{3}\ldots\ldots\ldots+n\lfloor\underline{n}=\lfloor\underline{n+1}-1$

$C-1:$ Put n = 1 then S(1): $1\lfloor\underline{1}=\lfloor\underline{n+1}-1$

$=\lfloor\underline{n+1}-1=\lfloor\underline{2}-1=2.1-1=2-1=1$

$C-1$ is satisfied

$C-2:$ Let it be true for $n=k\in N$ then

$S(k):\ 1\lfloor\underline{1}+2\lfloor\underline{2}+3\lfloor\underline{3}\ldots\ldots\ldots+k\lfloor\underline{k}=\lfloor\underline{k+1}-1 \longrightarrow (A)$

For $n=k+1$ then statements is

$S(k+1):\ 1\lfloor\underline{1}+2\lfloor\underline{2}+3\lfloor\underline{3}\ldots\ldots\ldots+(k+1)\lfloor\underline{k+1}=\lfloor\underline{k+1+1}-1$

$1\lfloor\underline{1}+2\lfloor\underline{2}+3\lfloor\underline{3}\ldots\ldots\ldots+(k+1)\lfloor\underline{k+1}=\lfloor\underline{k+2}-1 \longrightarrow (B)$

Adding both side $(k+1)\lfloor\underline{k+1}$ in (A) get.

$$1\lfloor\underline{1}+2\lfloor\underline{2}+3\lfloor\underline{3}\ldots\ldots\ldots+k\lfloor\underline{k}+(k+1)\lfloor\underline{k+1}=\lfloor\underline{k+1}-1+(k+1)\lfloor\underline{k+1}$$

$$=\lfloor\underline{k+1}+(k+1)\lfloor\underline{k+1}-1$$

$$=\lfloor\underline{k+1}(1+k+1)-1$$

$$=\lfloor\underline{k+1}(k+2)-1$$

$$=(k+2)\lfloor\underline{k+1}-1$$

$$=\lfloor\underline{k+2}-1$$

Which is (B) so it is true for $n=k+1, C-2$ is satisfied.

Hence given statement in true for every +ve integer n.

17. $a_n=a_1+(n-1)d$ When $a_1, a_1+d, a_1+2d, \ldots\ldots$ form an A.P.

Sol. $a_n=a_1+(n-1)d$

$C-1:$ Put n = 1 then S(1): $a_1=a_1+(1-1)d=a_1$

$C-1$ is satisfied

$C-2:$ Let it be true for $n=k\in N$ then

$S(k): a_k = a_1 + (k-1)d \longrightarrow (A)$

For $n = k+1$

$$S(k+1): a_{k+1} = a_1 + (k+1-1)d = a_1 + kd$$

L.H.S $= a_{k+1} = a_k + d$

$= a_1 + (k-1)d + d(use\ (A))$

$= a_1 + kd - d + d$

$= a_1 + kd =$ R.H.S

Which is (B) so it is true for $n = k+1, C-2$ is satisfied.

Hence given statement in true for every +ve integer n.

18. $\boldsymbol{a_n = a_1 r^{n-1}}$ When $a_1, a_1 r, a_1 r^2, ...$ form an G.P.

Sol. $S(n):\ Let\ a_n = a_1 r^{n-1}$

$C-1$: Put n = 1 then S(1): $a_1 = a_1 r^{1-1} = a_1 r^0 = a_1(1) = a_1$

$\because C-1$ is satisfied

$C-2$: Let it be true for $n = k \in N$ then

$S(k): a_k = a_1 r^{k-1} \longrightarrow (A)$

For $n = k+1$

$$S(k+1): a_{k+1} = a_1 r^{k+1-1} = a_1 r^k \longrightarrow (B)$$

L.H.S $= a_{k+1} = a_k r$

$= a_k r^{k-1}.r$

$= a_k r^{k-1+1} = a_1 r^k =$ R.H.S

Which is (B) so it is true for $n = k+1, C-2$ is satisfied.

Hence given statement in true for every +ve integer n.

19. $\boldsymbol{1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \dfrac{n(4n^2-1)}{3}}$ **Faisalabad 2007**

Sol. $S(n): 1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \dfrac{n(4n^2-1)}{3}$

$C-1$: Put n = 1 then S(1): $(2(1)-1)^2 = (1)^2 = 1 = \dfrac{1(4(1)^2-1)}{3} = \dfrac{4-1}{3} = \dfrac{3}{3} = 1$

$C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$$S(k): 1^2+3^2+5^2+\dots\dots\dots\dots+(2k-1)^2=\frac{k(4k^2-1)}{3} \longrightarrow (A)$$

For $n=k+1$ the statement is

$$S(k+1): 1^2+3^2+5^2+\dots\dots\dots\dots+(2(k+1)-1)^2=\frac{(k+1)(4(k+1)^2-1)}{3}$$

$$1^2+3^2+5^2+\dots\dots\dots\dots+(2k+2-1)^2=\frac{(k+1)(4(k^2+2k+1)-1)}{3}$$

$$1^2+3^2+5^2+\dots\dots\dots\dots+(2k+1)^2=\frac{(k+1)(4k^2+8k+4-1)}{3}$$

$$=\frac{(k+1)(4k^2+8k+3)}{3}=\frac{4k^3+8k^2+3k+4k^2+8k+3}{3}$$

$$1^2+3^2+5^2+\dots\dots\dots\dots+(2k+1)^2=\frac{4k^3+12k^2+11k+3}{3} \longrightarrow (B)$$

Adding both sides $(2k+1)^2$ in (A) we get.

$$1^2+3^2+5^2+\dots\dots\dots\dots+(2k-1)^2+(2k+1)^2=\frac{k(4k^2-1)}{3}+(2k+1)^2$$

$$=\frac{4k^3-k}{3}+4k^2+4k+1=\frac{4k^3-k+12k^2+12k+3}{3}$$

$$=\frac{4k^3+12k^2+11k+3}{3}$$

Which is (B) so it is true for $n=k+1, C-2$ is satisfied.

Hence given statement in true for every +ve integer n.

20. $\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\dots\dots\dots\dots+\binom{n+2}{3}=\binom{n+3}{4}$ **Faisalabad 2008, Sargodha 2009**

Sol. *Let* $S(n): \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\dots\dots\dots\dots+\binom{n+2}{3}=\binom{n+3}{4}$

$C-1$: Put n = 1 then S(1): $\binom{1+2}{3}=\binom{3}{3}=1=\binom{1+3}{3}=\binom{4}{4}=1$

$C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$$Let\ S(k): \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\dots\dots\dots+\binom{k+2}{3}=\binom{k+3}{4}\longrightarrow(A)$$

For $n=k+1$ the statement is

$$S(k+1): \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\dots\dots\dots+\binom{k+1+2}{3}=\binom{k+1+3}{4}$$

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\dots\dots\dots+\binom{k+3}{3}=\binom{k+4}{4}\longrightarrow(B)$$

Adding both sides $\binom{k+3}{3}$ in (A) we get.

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\dots\dots\dots+\binom{k+2}{3}+\binom{k+3}{3}=\binom{k+3}{4}+\binom{k+3}{3}$$

$$=\binom{k+3+1}{4}=\binom{k+4}{4}$$

Which is (B) so it is true for $n=k+1$

$C-2$ satisfied Hence proved.

21. Prove by mathematical induction that for all positive integral values of n:

i. n^2+n is divided by 2.

Sol. $C-1$: put $n=1$ then $1^2+1=1+1=2$ is divided by 2.

$C-1$ is satisfied

$C-2$ Let it be true for $n=k\in N$ then

So k^2+k is divided by 2 $\Rightarrow k^2+k=2Q$ ———— (A)

For $n=k+1$ the statement is

$$(k+1)^2+k+1=k^2+2k+1+k+1$$
$$=k^2+2k+k+2$$
$$=(k^2+k)+2(k+1)$$

(k^2+k) & $2(k+1)$ are separately divisible by 2.

So $=2Q$, $C-2$ is satisfied.

Hence proved for all +ve integer values n.

ii. **$5^n - 2^n$ is divided by 3.**

Sol. $C-1$: put $n=1$ then $5^1 - 2^1 = 5-2=3$ divisible by 3.

$C-1$ is satisfied

$C-2$: Let it be true for $n=k$ mean $5^k - 2^k$ is divisible by 3

$$\Rightarrow 5^k - 2^k = 3Q \text{———} (A) \; for \; n = k+1$$

$$5^{k+1} - 2^{k+1} = 5^k.5 - 2^k.2 = 5^k(3+2) - 2^k.2$$

$= 3.5^k + 2(5^k - 2^k) = 3Q,\quad C-2$ is satisfied.

Hence it is true for all +ve integral values n.

iii. **$5^n - 1$ is divided by 4.**

Sol. $C-1$ put $n=1$ then $5^1 - 1 = 5-1 = 4$ divisible by 4.

$C-1$: is satisfied

$C-2$: Let it be true for $n = k \in N$ then

$5^k - 1$ is divisible by $4 \Rightarrow 5^k - 1 = 4Q$ ———(A)

For $n = k+1$ then

$5^{k+1} - 1 = 5^k.5 - 1 = 5^k(4+1) - 1$

$= 4.5^k + 5^k - 1 = 4.5^k + (5^k - 1)$———*which is* (A)

Both terms are separately divisible by 4

$$= 4Q$$

$C-2$ is satisfied.

Hence given statement is true for all +ve integral values n.

iv. **$8 \times 10^n - 2$ is divisible by 6.** **Multan 2008**

Sol. $C-1$: put $n=1$ then $8\times 10^1 - 2 = 80-2 = 78$ divisible by 6

$C-1$ is satisfied

$C-2$: Let it be true for $n=k$ mean

$8\times 10^k - 2$ is divisible by $6 \Rightarrow 8\times 10^k - 2 = 6Q$ ———(A)

For $n = k+1$

$8\times 10^{k+1} - 2 = 8\times 10^k.10 - 2$

Adding and subtracting 20

$$= 8\times 10^k \times 10 - 2 + 20 - 20$$

$$=8\times10^k\times10+18-20$$
$$=8\times10^k\times10-20+18$$
$$=10(8\times10^k-2)+6\times3$$

Both terms of R.H.S are separately divisible by 6 so $=6Q$

$C-2$ is satisfied.

Hence given statement is true for every +ve integral n.

v. n^3-n **is divisible by 6.** **Sargodha 2010**

Sol. $C-1:$ put $n=1$ then $(1)^3-1=1-1=0$ is divisible by 6

$C-2:$ Let it be true for $n=k$ mean k^3-k is divisible by 6.

For $n=k+1$ *then* $(k+1)^3-(k+1)=k^3+3k^2+3k+1-k-1$

$$=(k^3-k)+3k^2+3k$$
$$=(k^3-k)+3k(k+1) \qquad k(k+1) \text{ is even so put } k(k+1)=2m$$
$$=(k^3-k)+3(2m)$$
$$=(k^3-k)+6m$$

Both terms of R.H.S are divisible by 6 separately $(k+1)^3-(k-1)=6Q$

$C-2$ is satisfied.

Hence given statement is true for every +ve integral n.

22. $\dfrac{1}{3}+\dfrac{1}{3^2}+\ldots\ldots\ldots\ldots+\dfrac{1}{3^n}=\dfrac{1}{2}\left[1-\dfrac{1}{3^n}\right]$

Sol. $Let\ S(n):\dfrac{1}{3}+\dfrac{1}{3^2}+\ldots\ldots\ldots\ldots+\dfrac{1}{3^n}=\dfrac{1}{2}\left[1-\dfrac{1}{3^n}\right]$

$C-1:$ Put n = 1 then S(1): $\dfrac{1}{3^1}=\dfrac{1}{3}=\dfrac{1}{2}\left[1-\dfrac{1}{3^1}\right]=\dfrac{1}{2}\left[\dfrac{3-1}{3}\right]=\dfrac{1}{2}\left[\dfrac{2}{3}\right]=\dfrac{1}{3}$

R.H.S $C-1$ is satisfied

$C-2:$ Let it be true for $n=k\in N$ then

$$then\ S(k):\frac{1}{3}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots\ldots+\frac{1}{3^k}=\frac{1}{2}\left[1-\frac{1}{3^k}\right]\longrightarrow(A)$$

For $n=k+1$ then

$$Let\ S(k+1): \frac{1}{3}+\frac{1}{3^2}+..............+\frac{1}{3^{k+1}}=\frac{1}{2}\left[1-\frac{1}{3^{k+1}}\right]$$

$$=\frac{1}{2}\left[1-\frac{1}{3^k.3}\right]=\frac{1}{2}-\frac{1}{6.3^k}\longrightarrow(B)$$

Adding both sides $\frac{1}{3^{k+1}}$ in (A) we get.

$$\frac{1}{3}+\frac{1}{3^2}+..............+\frac{1}{3^k}=\frac{1}{3^{k+1}}=\frac{1}{2}\left[1-\frac{1}{3^k}\right]+\frac{1}{3^{k+1}}$$

$$=\frac{1}{2}-\frac{1}{2.3^k}+\frac{1}{3.3^k}=\frac{1}{2}-\left(\frac{1}{2.3^k}-\frac{1}{3.3^k}\right)$$

$$=\frac{1}{2}-\frac{+3-2}{6.3^k}=\frac{1}{2}-\frac{1}{6.3^k}$$

Which is (B) so it is true for $n=k+1$ $C-2$ satisfied.

Hence given statement in true for every +ve integral n.

23. $$\mathbf{1^2-2^2+3^2-4^2+............+(-1)^{n-1}.n^2=\frac{(-1)^{n-1}.n(n+1)}{2}}$$

Sol. $$Let\ S(n): 1^2-2^2+3^2-4^2+............+(-1)^{n-1}.n^2=\frac{(-1)^{n-1}.n(n+1)}{2}$$

$C-1$: Put n = 1 then S(1): $(-1)^{1-1}.1^2=(-1)^0.1=1.1=1$

$$=\frac{(-1)^{1-1}.1(1+1)}{2}=\frac{(-1)^0.1.2}{2}=1.1=1$$

$C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$$S(k): 1^2-2^2+3^2-4^2+............+(-1)^{k-1}.k^2=\frac{(-1)^{k-1}.k(k+1)}{2}\longrightarrow(A)$$

$$S(k+1): 1^2-2^2+3^2-4^2+....+(-1)^k.(k+1)^2=\frac{(-1)^{k+1-1}(k+1)(k+1+1)}{2}$$

$$1^2-2^2+3^2-4^2+............+(-1)^k.(k+1)^2=\frac{(-1)^k(k+1)(k+2)}{2}\longrightarrow(B)$$

Adding both sides $(-1)^k(k+1)^2$ in (A) we get.

$$1^2-2^2+3^2-4^2+\ldots\ldots\ldots\ldots+(-1)^{k-1}k^2+(-1)^k(k+1)^2=\frac{(-1)^{k-1}k(k+1)}{2}+(-1)^k(k+1)^2$$

$$=\frac{(-1)^{k-1}k(k+1)}{2}+(-1).(-1)^{k-1}(k+1)^2$$

$$=(-1)^{k-1}.(k+1)\left[\frac{k}{2}+(-1)(k+1)\right]$$

$$=(-1)^{k-1}.(k+1)\left[\frac{k-2k-2}{2}\right]$$

$$=(-1)^{k-1}.(k+1)\left[\frac{-k-2}{2}\right]$$

$$=(-1)^{k-1}(-1)(k+1)\left[\frac{k+2}{2}\right]$$

$$=(-1)^{k-1+1}\frac{(k+1)(k+2)}{2}=\frac{(-1)^k(k+1)(k+2)}{1}$$

Which is (B) so it is true for $n=k+1$ $C-2$ satisfied.
Hence given statement in true for every +ve integral n.

24. $\mathbf{1^3+3^3+5^3+\ldots\ldots\ldots\ldots+(2n-1)^3=n^2\left[2n^2-1\right]}$

Sol. $S(n):\ 1^3+3^3+5^3+\ldots\ldots\ldots\ldots+(2n-1)^3=n^2\left[2n^2-1\right]$

$C-1$: Put n = 1 then S(1): $(2(1)-1)^3=(1)^3=1$

$=1^2\left[2(1)^2-1\right]=1(2-1)=1(1)=1$

$C-1$ is satisfied

$C-2$: Let it be true for $n=k\in N$ then

$S(k):\ 1^3+3^3+5^3+\ldots\ldots\ldots\ldots+(2k-1)^3=k^2\left[2k^2-1\right]\longrightarrow(A)$

For $n=k+1$ the statement is

$S(k+1):\ 1^3+3^3+5^3+\ldots\ldots\ldots\ldots+(2(k+1)-1)^3=(k+1)^2\left[2(k+1)^2-1\right]$

$1^3+3^3+5^3+\ldots\ldots\ldots\ldots+(2k+2-1)^3=(k+1)^2\left[2(k^2+2k+1)-1\right]$

$1^3+3^3+5^3+\ldots\ldots\ldots\ldots+(2k+1)^3=(k+1)^2\left[2k^2+4k+2-1\right]$

$$1^3+3^3+5^3+\dots\dots\dots\dots+(2k+1)^3=(k+1)^2\left[2k^2+4k+1\right]$$
$$=(k^2+2k+1)=(2k^2+4k+1)$$
$$=2k^4+4k^3+k^2+4k^3+8k^2+2k+2k^2+4k+1$$
$$=2k^4+8k^3+11k^2+6k+1 \longrightarrow (B)$$

Adding both sides $(2k+1)^3$ in (A) we get.

$$1^3+3^3+5^3+\dots\dots\dots\dots+(2k+1)^3+(2k+1)^3=k^2\left[2k^2-1\right]+(2k+1)^3$$
$$=2k^4-k^2+8k^3+12k^2+6k+1$$
$$=2k^4+8k^3+11k^2+6k+1$$

Which is (B) so it is true for $n=k+1$, $C-2$ satisfied.

Hence given statement in true for every +ve integral n.

25. $x+1$ **is a factor of** $x^{2n}-1; (x \neq -1)$

Sol. $x+1$ is a factor $x^{2n}-1$

$C-1$: Put n = 1 then

$x^{2(1)}-1=x^2-1=(x-1)(x+1)$ clearly $x+1$ is factor of $x^{2n}-1 \longrightarrow (A)$

let it be true for $n=k \in N$ *then* $x+1$ *is factor of* $x^{2k}-1$

For $n=k+1$ then $x^{2(k+1)}-1=x^{2k+2}-1=x^{2k}.x^2-1$

Adding and subtract x^2

$$=x^2.x^{2k}-x^2+x^2-1$$
$$=x^2(x^{2k}-1)+1(x^2-1)$$
$$=x^2(x^{2k}-1)+(x-1)(x+1)$$

$x-1$ is factor of both term so $(x+1)$ is factor of R.H.S $C-2$ satisfied hence proved

26. $x-y$ **is a factor of** $x^n-y^n;(x \neq y)$ **Sargodha 2011**

Sol. $x-y$ is a factor of x^n-y^n

$C-1$: Put n = 1 then $x^1-y^1=x-y$ *clearly* $x-y$ is its factor $C-1$ is satisfied.

Let it be true for $n=k \in N$

$x-y$ is factor of $x^k-y^k \longrightarrow (A)$

For $n=k+1$, $k^{k+1}-y^{k+1}=x^k.x-y^k.y$

Adding and subtracting xy^k

$=x^k.x-xy^k+xy^k-y^k.y=x(x^k-y^k)+y^k(x-y)$

So $(x-y)$ is factor of R.H.S $C-2$ is satisfied

Hence proved.

27. $x+y$ **is a factor of** $x^{2n-1}+y^{2n-1};(x\neq y)$ **Faisalabad 2008**

Sol. $x+y$ is a factor of $x^{2n-1}+y^{2n-1}$

$C-1$: Put n = 1 then $x^{2(1)-1}+y^{2-1}=x+y$

So $x+y$ is its factor, $C-1$ is satisfied.

$C-2$: Let it be true $n=k$ its mean

$x+y$ is factor of $x^{2k-1}+y^{2k-1}\longrightarrow(A)$

For $n=k+1$

$x^{2(k+1)-1}+y^{2(k+1)-1}=x^{2k+2-1}+y^{2k+2-1}$

$=x^{2k-1+2}+y^{2k-1+2}$

$=x^{2k-1}.x^2+y^{2k-1}.y^2$

Adding and subtracting x^2y^{2k-1} we get

$=x^{2k-1}.x^2+x^2y^{2k-1}-x^2y^{2k-1}+y^{2k-1}.y^2$

$=x^2(x^{2k-1}+y^{2k-1})-y^{2k-1}(x^2-y^2)$

$=x^2(x^{2k-1}+y^{2k-1})-y^{2k-1}(x-y)(x+y)$ by using A.

Clearly $x+y$ is factor of R.H.S $C-2$ is satisfied

Hence proved.

28. **Using mathematical induction to show that:**

$\mathbf{1+2+2^2+\dots\dots\dots+2^n=2^{n+1}-1}$ **for all non-negative integers n.**

Sol. $1+2^1+2^2+2^3\dots\dots\dots+2^n=2^{n+1}-1$

$C-1$: for $n=0$, S(1): $2^0=1=2^{0+1}-1=2-1=1$

$C-1$ is satisfied.

$C-2$: Let it be true for $n=k$ then

$1+2^1+2^2+2^3\dots\dots\dots+2^k=2^{k+1}-1\longrightarrow(A)$

For $n=k+1$ then

$1+2^1+2^2+2^3\dots\dots\dots+2^{k+1}=2^{k+1+1}-1$

$1+2^1+2^2+2^3\dots\dots\dots+2^{k+1}=2^{k+2}-1\longrightarrow(B)$

Adding both sides 2^{k+1} in A we get.

$$1+2^1+2^2+2^3..........+2^k+2^{k+1}=2^{k+1}-1+2^{k+1}$$
$$=2^{k+1}+2^{k+1}-1$$
$$=2.2^{k+1}-1$$
$$=2^{k+2}-1$$

Which both (B) so it is true for $n=k+1, C-2$ is satisfied.

Hence given statement is true for all non negative integral.

29. If A and B are square matrices and AB = BA, then show by mathematical induction that $AB^n = B^n A$ for any positive integer n.

Sol. Given $AB=BA$ to prove

$AB^n = B^n A$

$C-1$: put $n=1$, then $AB^1 = B^1 A \Rightarrow AB = BA \longrightarrow I$

$C-1$ is given satisfied which is given

$C-2$: Let it be true for $n=k$ then

$AB^k = B^k A \longrightarrow (A)$

For $n=k+1$, $AB^{k+1} = B^{k+1} A$

L.H.S $= AB^{k+1} = AB^k.B$

$= B^k A.B \longrightarrow use\ (A)$

$= B^k B.A \longrightarrow use\ I$

$= B^{k+1} A =$ R.H.S

It is true for $n=k+1$ $C-2$ is satisfied.

Hence it hold for any +ve integral n.

30. Prove by the Principle of mathematical induction that n^2-1 is divisible by 8 when n is an odd positive integer.

Sol. n^2-1 is divisible by 8 (n is odd +ve.)

$C-1$: put $n=1$, then $1^2-1=1-1=0$ is divisible by 8 $C-1$ is satisfied.

$C-2$: Let it be true for $n=k$ then k^2-1 is divisible 8.

$\Rightarrow k^2-1=8Q \longrightarrow (A)$ For $n=k+2$ then

$x+1$ is even so put $n=k+2$ which is odd.

$(k+2)^2-1=k^2+4k+4-1$

$=k^2-1+4(k+1)$ $k+1$ is even so Take $k+1=2m$

$=k^2-1+4(2m)$

$(k+1)^2-1=(k^2-1)+8m$ by using A clearly R.H.S is divisible by 8

$=8Q, C-2$ is satisfied.

Hence given statement is true for all odd +ve integral.

31. Use the Principle of mathematical induction to prove that $\ell n\ x^n = n\ \ell n\ x$ for any integer $n \geq 0$ if x is positive number.

Sol. In $x^n = n\ \ell n\ x$, $n \geq 0$

$C-1$: put $n=1$, then $\ell n x^1 = (1)\ell n x \Rightarrow \ell n x = \ell n x$

$C-1$ is satisfied.

$C-2$: Let it be true for $n=k$ then

$\ell n\ x^k = k\ \ell n\ x \longrightarrow (A)$

For $n=k+1$

$\ell n x^{k+1} = (k+1)\ \ell n\ x \longrightarrow (B)$

Adding $\ell n x$ on both sides of A.

$\ell n x^k + \ell n\ x = k \ell n\ x + \ell n\ x$

$\ell n\ (x^k . x) = (k+1)\ \ell n\ x$

$\ell n x^{k+1} = (k+1)\ \ell n\ x$

Which is (B) $C-2$ is satisfied.

Hence proved.

32. Use the Principle of extended mathematical induction to prove that $n! > 2^n - 1$ for integral values of $n \geq 4$ **Multan 2009**

Sol. $n! > 2^n - 1$ $For\ n \geq 4$

$C-1$: put $n=4$, then $4! > 2^4 - 1 \Rightarrow 4.3.2.1 > 16-1 \Rightarrow 24 > 15$ which is true.

$C-1$ is satisfied.

$C-2$: Let it be true for $n=k \geq 4$ then

$k! > 2^k - 1 \longrightarrow (A)$

For $n=k+1$ we have

$(k+1)! > 2^{k+1} - 1 \longrightarrow (B)$

'×' (A) both sides by $(k+1)$

$(k+1)k! > (k+1)\left[2^k - 1\right]$

$(k+1)k! > 2(2^k - 1)$ replace $k+1$ by 2 because $k+1 > 2$

$(k+1)! > 2.2^k - 2$

$(k+1)! > 2^{k+1} - 1 - 1$

$(k+1)! > 2^{k+1} - 1 \quad (-1\, Ignore)\, which is (B)$

$C-2$ is satisfied. Hence proved.

33. $n^2 > n+3$ **for integral values of** $n \geq 3$ **Gujranwala 2009**

Sol. $n^2 > n+3 \qquad n \geq 3$

$C-1$: put $n=3$, then $3^2 > 3+3 \Rightarrow 9 > 6$ true

$C-1$ is satisfied.

$C-2$: Let it be true for $n = k \geq 3$ then

$k^2 > k+3 \longrightarrow (A) \qquad k \geq 3$

for $n = k+1$

$(k+1)^2 > k+1+3 \Rightarrow (k+1)^2 > k+4 \longrightarrow (B)$

Adding both sides of (A) $2k+1$

$k^2 + 2k + 1 > 2k + 1 + k + 3$

$or\ (k+1)^2 > k+4+2k$

$(k+1)^2 > k+4$ Ignore 2k because $2k > 0$ which is (B) so

It is true for $n = k+1$, $C-2$ is satisfied.

Hence proved.

34. $4^n > 3^n + 2^{n-1} \qquad n \geq 2$

Sol. $C-1$: put $n=2$, then $4^2 > 3^2 + 2^{2-1} \Rightarrow 16 > 9+2 \Rightarrow 16 > 11$ true

$C-1$ is satisfied.

$C-2$: Let it be true for $n = k \geq 2$ then

$4^k > 3^k + 2^{k-1},\ k \geq 2 \longrightarrow (A)$

for $n = k+1$ then $4^{k+1} > 3^{n+1} + 2^{k+1-1}$

$\Rightarrow 4^{k+1} > 3^{k+1} + 2^k \longrightarrow (B)$

'×' both sides of (A) by 4.

$4.4^k > 4(3^k + 2^{k-1})$

$4^{k+1} > 4.3^k + 4.2^{k-1}$

$4^{k+1} > (3+1)3^k + 2^2.2^{k-1}$

$4^{k+1} > 3.3^k + 1.3^k + 2^{k-1+2}$

$4^{k+1} > 3^{k+1} + 3^k + 2^{k+1}$

$4^{k+1} > 3^{k+1} + 2^k$ (Because $2^{k+1} > 2^k$ replace 2^{k+1} by 2^k and ignore 3^k)

Which is (B) so it is for $n = k+1$ $C-2$ is satisfied.

Hence proved.

35. **$3^n < n!$ for integral values of $n < 6$**

Sol. $3^n < n!$ $\qquad n < 6$

$C-1$: put $n = 7$ then $3^7 < 7! \Rightarrow 2187 < 5040$ true $C-1$ is satisfied.

$C-2$: Let it be true for $n = k$

$3^k < 3!$ $\qquad k < 6 \longrightarrow (A)$

for $n = k+1$ then $3^{k+1} < (k+1)! \longrightarrow (B)$

'x' both sides of (A) by 3.

$3.3^k < 3.k!$ because $k+1 > 3$ replace 3 by $k+1$

$3^{k+1} < (k+1).k!$

$3^{k+1} < (k+1)!$ which is (B) It is true for $n = k+1$

$C-2$ is satisfied.

Hence proved.

36. **$n! > n^2$ for integral values of $n \geq 4$.**

Sol. $n! > n^2$ $\qquad n \geq 4.$

$C-1$: put $n = 4$ then $4! > 4^2 \Rightarrow 24 > 16$ $\qquad$ **Hint:** $4! = 4.3.2.1 = 24$

$C-1$ is satisfied.

$C-2$: Let it be true for $n = k \geq 4$ then

$k! > k^2$ $\qquad k \geq 4 \longrightarrow (A)$

for $n = k+1$ then

$(k+1)! > (k+1)^2 \longrightarrow (B)$

'x' both sides of $(k+1)$ we get.

$(k+1)k! > (k+1).k^2$ $\qquad k^2 > k+1$ so replace k^2 by $k+1$

$(k+1)! > (k+1)(k+1)$

$(k+1)! > (k+1)^2$ which is (B) $C-2$ is satisfied.

Hence proved

37. **$3+5+7+\ldots\ldots\ldots+(2n+5) = (n+2)(n+4)$ for integral values of $n \geq -1$.**

Sol. $3+5+7+\ldots\ldots\ldots+(2n+5) = (n+2)(n+4), n \geq -1$

$C-1$: put $n = 1$ then S(1): $2(-1)+5 = -2+5 = 3 = (-1+2)(-1+4) = (1)(3) = 3$

$C-1$ is satisfied.

$C-2$: Let it be true for $n=k$ then

$3+5+7+.......+(2k+5)=(k+2)(k+4)\longrightarrow(A)$

for $n=k+1$

$3+5+7+.......+(2(k+1)+5)=(k+1+2)(k+1+4)$

$3+5+7+.......+(2k+2+5)=(k+3)(k+5)$

$3+5+7+.......+(2k+7)=(k+3)(k+5)\longrightarrow(B)$

Adding both sides of (A) $2k+7$ we get.

$3+5+7+.......+(2k+5)+(2k+7)=(k+2)(k+4)+2k+7$

$=k^2+4k+2k+8+2k+7$

$=k^2+8k+15$

$=k^2+3k+5k+15$

$=k(k+3)+5(k+3)$

$=(k+3)(k+5)$

Which is (B) so it if true for $n=k+1$, $C-2$ is satisfied.

Hence proved

38. $\mathbf{1+nx\leq(1+x)^n}$ **for** $\mathbf{n\geq 2}$ **and** $\mathbf{x>-1}$

Sol. $(1+x)^n\geq 1+nx$

$C-1$: put $n=2$ then $(1+x)^2\geq 1+2x$

$\Rightarrow 1+2x+x^2\geq 1+2x$ true

$C-1$ is satisfied.

$C-2$: Let it be true for $n=k\geq 2$ then

$(1+x)^k\geq 1+kx\longrightarrow(A)$

for $n=k+1, (1+x)^{k+1}\geq 1+(1+k)x\longrightarrow(B)$

'×' both sides of (A) with $(1+x)$

$(1+x)(1+x)^k\geq(1+kx)(1+x)=1+kx+x+kx^2$

$(1+x)^{k+1}\geq 1+(1+k)x$ ignore $kx^2>0$

Which is (B) so it is true for $n=k+1$ $C-2$ is satisfied.

Hence proved

Binomial Theorem

Statement: If a & x are real numbers and n is natural then prove that **Sargodha 2011**

$$(a+x)^n = \binom{n}{0} a^n x^o + \binom{n}{1} a^{n-1} x^1 + \binom{n}{2} a^{n-2} x^2 + \binom{n}{3} a^{n-3} x^3$$

$$+ \dots\dots + \binom{n}{r-1} a^{n-(r-1)} x^{r-1} + \binom{n}{r} a^{n-r} x^r + \dots\dots + \binom{n}{n} a^0 x^n$$

Sol. Put $n=1$ then

$C-1$: S(1): $(a+x)^1 = a+x = \binom{1}{0} a^1 x^0 + \binom{1}{1} a^0 x^1 = a+x$

$C-1$ is satisfied.

$C-2$: let it to be true for $n=k \in N$ then

$$(a+k)^k = \binom{k}{0} a^k x^0 + \binom{k}{1} a^{k-1} x^1 \binom{k}{2} a^{k-2} x^2 + \binom{k}{3} a^{k-3} x^3$$

$$+ \dots\dots + \binom{k}{r-1} a^{k-(r-1)} x^{r-1} + \binom{k}{r} a^{k-r} x^r + \dots\dots + \binom{k}{k} a^0 x^k \longrightarrow (A)$$

For $n=k+1$

$$(a+k)^{k+1} = \binom{k+1}{0} a^{k+1} x^0 + \binom{k+1}{1} a^{k+1-1} x^1 \binom{k+1}{2} a^{k+1-2} x^2 + \binom{k+1}{3} a^{k+1-3} x^3$$

$$+ \dots\dots + \binom{k+1}{r-1} a^{k+1-r+1} x^{r-1} + \binom{k+1}{r} a^{k+1-r} x^r + \dots\dots + \binom{k+1}{k+1} a^0 x^{k+1}$$

$$(a+k)^{k+1} = \binom{k+1}{0} a^{k+1} x^0 + \binom{k+1}{1} a^k x^1 + \binom{k+1}{2} a^{k-1} x^2 + \binom{k+1}{3} a^{k-2} x^3$$

$$+ \dots\dots + \binom{k+1}{r-1} a^{k-r+2} x^{r-1} + \binom{k+1}{r} a^{k-r+1} x^r + \dots\dots + \binom{k+1}{k+1} a^0 x^{k+1} \longrightarrow (B)$$

Multiplying (A) by (a + x) both sides

$$(a+x)^k (a+x) = (a+x)\left[\binom{k}{0} a^k x^0 + \binom{k}{1} a^{k-1} x^1 + \binom{k}{2} a^{k-2} x^2 + \dots + \binom{k}{r-1} a^{k-r+1} x^{r-1} + \binom{k}{r} a^{k-r} x^r + \dots + \binom{k}{k} a^0 x^k\right]$$

$$(a+k)^{k+1} = \binom{k}{0} a^{k+1} x^0 + \binom{k}{1} a^k x^1 + \binom{k}{2} a^{k-1} x^2 + \dots\dots + \binom{k}{r-1}$$

$$a^{k-r+2}x^{r-1}+\binom{k}{r}a^{k-r+1}x^{r}+\dots\dots\dots+\binom{k}{k}a^{1}x^{k}+\binom{k}{0}a^{k}x^{1}+\binom{k}{1}a^{k-1}x^{2}+\binom{k}{2}a^{k-2}x^{3}$$

$$+\dots\dots\dots+\binom{k}{r-1}a^{k-r+1}x^{r}+\binom{k}{r}a^{k-r}x^{r+1}+\dots\dots\dots+\binom{k}{k}a^{0}x^{k+1}$$

$$=\binom{k}{0}a^{k+1}x^{0}+\left[\binom{k}{0}+\binom{k}{1}\right]a^{k}.x^{1}+\left[\binom{k}{1}+\binom{k}{2}\right]a^{k-1}x^{2}$$

$$+\dots\dots\dots+\left[\binom{k}{r-1}+\binom{k}{r}\right]a^{k-r+1}x^{r}+\dots\dots\dots+\binom{k}{k}a^{1}x^{k+1}$$

$$\boxed{\text{Note}\quad \binom{k}{0}=\binom{k+1}{0},\binom{k}{k}=\binom{k+1}{k+1},\binom{k}{0}+\binom{k}{1}=\binom{k+1}{1}}$$

$$(a+x)^{k+1}=\binom{k+1}{0}a^{k+1}x^{0}+\binom{k+1}{1}a^{k}x^{1}+\binom{k+1}{2}a^{k-1}x^{2}+\dots\dots\dots+\binom{k+1}{r}a^{k-r+1}x^{r}$$

$$+\dots\dots\dots+\binom{k+1}{k+1}a^{1}x^{k+1}$$

Which is (B) so it is true $n=k+1$ $C-2$ is satisfied.

Hence given statement is true for all natural number n.

Exercise 8.2

Example 3: **Find the term involving x^5 also find fifth term in the expansion of**

$$\left(\frac{3x}{2}-\frac{1}{3x}\right)^{11}$$

Sol. **Its general term is**

$$T_{r+1}=\binom{11}{r}\left(\frac{3x}{2}\right)^{11-r}\left(-\frac{1}{3x}\right)^{r}=\binom{11}{r}\left(\frac{3}{2}\right)^{11-r}(x)^{11-r}\left(-\frac{1}{3}\right)^{r}\frac{1}{x^r}$$

$$=\binom{11}{r}x^{11-r-r}\left(\frac{3}{2}\right)^{11-r}\left(-\frac{1}{3}\right)^{r}=\binom{11}{r}x^{11-2r}\left(\frac{3}{2}\right)^{11-r}\left(-\frac{1}{3}\right)^{r}$$

i. **Compare Exponent of x with exponent of x^5**

$11-2r=5\Rightarrow 2r=11-5=6\Rightarrow r=3$ **Sargodha 2009, 2010 Multan 2007**

put $r=3$

$$T_{3+1}=\binom{11}{3}x^{11-2(3)}\left(\frac{3}{2}\right)^{11-3}\left(-\frac{1}{3}\right)^{3}=\frac{11!}{3!.8!}x^{11-6}\left(\frac{3}{2}\right)^{8}(-1)^3\left(\frac{1}{3}\right)^{3}$$

$$T_4=\frac{11.10.9.8!}{3.2.1.8!}x^5.\frac{3^5}{2^8}(-1)=-\frac{165\times 243}{256}x^5=-\frac{40095}{256}x^5$$

ii. *For fifth term put* $r=4$ **Faisalabad 2007, 2008**

$$T_5=\binom{11}{4}x^{11-2(4)}\left(\frac{3}{2}\right)^{11-4}\left(-\frac{1}{3}\right)^{4}=\frac{11.10.9.8.\cancel{7!}}{4.3.2.1.\cancel{7!}}x^3\left(\frac{3}{2}\right)^{7}\left(\frac{1}{3^4}\right)=\frac{4455}{64}x^3$$

1. **Using Binomial theorem, expand the following:**

i. $(a+2b)^5$ **Multan 2008**

Sol. $(a+2b)^5=\binom{5}{0}a^5(2b)^0+\binom{5}{1}a^4(2b)^1+\binom{5}{2}a^3(2b)^2+\binom{5}{3}a^2(2b)^3+\binom{5}{4}a^1(2b)^4+\binom{5}{5}a^0(2b)^5$

$$=(1)a^5(1)+5a^4(2b)+10a^3(4b^2)+10a^2(8b^3)+5(a)(16b^4)+(1)(1)(32b^5)$$

$$=a^5+10a^4b+40a^3b^2+80a^2b^3+80ab^4+32b^5$$

ii. $\left(\dfrac{x}{2}-\dfrac{2}{x^2}\right)^6$

Sol. $\left(\frac{x}{2}-\frac{2}{x^2}\right)^6=\binom{6}{0}\left(\frac{x}{2}\right)^6\left(-\frac{2}{x^2}\right)^0+\binom{6}{1}\left(\frac{x}{2}\right)^5\left(\frac{-2}{x^2}\right)^1+\binom{6}{2}\left(\frac{x}{2}\right)^4\left(\frac{-2}{x^2}\right)^2+\binom{6}{3}\left(\frac{x}{3}\right)^3$

$$\left(-\frac{2}{x^2}\right)^3+\binom{6}{4}\left(\frac{x}{2}\right)^2\left(\frac{-2}{x^2}\right)^4+\binom{6}{5}\left(\frac{x}{2}\right)^1\left(-\frac{2}{x^2}\right)^5+\binom{6}{6}\left(\frac{x}{2}\right)^0\left(-\frac{2}{x^2}\right)^6$$

$$=(1)\left(\frac{x^6}{64}\right)(1)+(6)\left(\frac{x^5}{32}\right)\left(\frac{-2}{x^2}\right)+(15)\left(\frac{x^4}{16}\right)\left(\frac{4}{x^4}\right)+20\left(\frac{x^3}{8}\right)\left(-\frac{8}{x^6}\right)+15\left(\frac{x^2}{4}\right)\left(\frac{16}{x^8}\right)$$

$$6\left(\frac{x}{2}\right)\left(\frac{-32}{x^{10}}\right)+(1)(1)\left(\frac{64}{x^{12}}\right)$$

$$\frac{x^6}{64}-\frac{3x^3}{8}+\frac{15}{4}-\frac{20}{x^3}+\frac{60}{x^6}-\frac{96}{x^9}+\frac{64}{x^{12}}$$

iii. $\left(3a-\dfrac{x}{3a}\right)^4$ **Sargodha 2008**

Sol. $$\left(3a-\frac{x}{3a}\right)^4=\binom{4}{0}(3a)^4\left(-\frac{x}{3a}\right)^0+\binom{4}{1}(3a)^3\left(-\frac{x}{3a}\right)^1+\binom{4}{2}(3a)^2\left(-\frac{x}{3a}\right)^2$$

$$+\binom{4}{3}(3a)^1\left(-\frac{x}{3a}\right)^3+\binom{4}{4}(3a)^0\left(-\frac{x}{3a}\right)^4$$

$$=(1)(81a^4)(1)+4(27a^3)\left(-\frac{x}{3a}\right)+6(9a^2)\left(\frac{x^2}{9a^2}\right)+4(3a)\left(-\frac{x^3}{27a^3}\right)+(1)(1)\left(\frac{x^4}{81a^4}\right)$$

$$=81a^4-36a^2x+6a^2x^2-\frac{4x^3}{9a^2}+\frac{x^4}{81a^4}$$

iv. $\left(2a-\dfrac{x^2}{a}\right)^7$

Sol. $$\left(2a-\frac{x^2}{a}\right)^7=\left(2a+\left(\frac{-x^2}{a}\right)\right)^7=\binom{7}{0}(2a)^7\left(\frac{-x^2}{a}\right)^0+\binom{7}{1}(2a)^6\left(\frac{-x^2}{a}\right)^1+\binom{7}{2}(2a)^5\left(\frac{-x^2}{a}\right)^2$$

$$+\binom{7}{3}(2a)^4\left(\frac{-x^2}{a}\right)^3+\binom{7}{4}(2a)^3\left(\frac{-x^2}{a}\right)^4+\binom{7}{5}(2a)^2\left(\frac{-x^2}{a}\right)^5+\binom{7}{6}(2a)^1\left(\frac{-x^2}{a}\right)^6+\binom{7}{7}(2a)^0\left(\frac{-x^2}{a}\right)^7$$

$$=(1)(128a^7)(0)+7(64a^6)\left(\frac{-x^2}{a}\right)+21(32a^5)\left(\frac{x^4}{a^2}\right)+35(16a^4)\left(\frac{-x^6}{a^3}\right)+35(8a^3)\left(\frac{x^8}{a^4}\right)$$

$$+21(4a^2)\left(\frac{-x^{10}}{a^5}\right)+7(2a)\left(\frac{x^{12}}{a^6}\right)+(1)(1)\left(-\frac{x^{14}}{a^7}\right)$$

$$=128a^7-448a^5x^2+672a^3x^4-560ax^6+280\frac{x^8}{a}-84\frac{x^{10}}{a^3}+\frac{14x^{12}}{a^5}-\frac{x^{14}}{a^7}$$

v. $\left(\dfrac{x}{2y}-\dfrac{2y}{x}\right)^8$

Sol. $\left(\dfrac{x}{2y}-\dfrac{2y}{x}\right)^8=\left(\dfrac{x}{2y}+\left(\dfrac{-2y}{x}\right)\right)^8=\binom{8}{0}\left(\dfrac{x}{2y}\right)^8\left(\dfrac{-2y}{x}\right)^0$

$+\binom{8}{1}\left(\dfrac{x}{2y}\right)^7\left(\dfrac{-2y}{x}\right)^1+\binom{8}{2}\left(\dfrac{x}{2y}\right)^6\left(\dfrac{-2y}{x}\right)^2+\binom{8}{3}\left(\dfrac{x}{2y}\right)^5\left(\dfrac{-2y}{x}\right)^3$

$+\binom{8}{4}\left(\dfrac{x}{2y}\right)^4\left(\dfrac{-2y}{x}\right)^4+\binom{8}{5}\left(\dfrac{x}{2y}\right)^3\left(\dfrac{-2y}{x}\right)^5+\binom{8}{6}\left(\dfrac{x}{2y}\right)^2\left(\dfrac{-2y}{x}\right)^6$

$+\binom{8}{7}\left(\dfrac{x}{2y}\right)^1\left(-\dfrac{2y}{x}\right)^7+\binom{8}{8}\left(\dfrac{x}{2y}\right)^0\left(-\dfrac{2y}{x}\right)^8$

$=(1)\left(\dfrac{x^8}{256y^8}\right)(1)+8\left(\dfrac{x^7}{128y^7}\right)\left(\dfrac{-2y}{x}\right)+28\left(\dfrac{x^6}{64y^6}\right)\left(\dfrac{4y^2}{x^2}\right)+56\left(\dfrac{x^5}{3xy^5}\right)\left(\dfrac{-8y^3}{x^3}\right)$

$+70\left(\dfrac{x^4}{16y^4}\right)\left(\dfrac{16y^4}{x^4}\right)+56\left(\dfrac{x^3}{8y^3}\right)\left(\dfrac{-32y^5}{x^5}\right)$

$+28\left(\dfrac{x^2}{4y^2}\right)\left(\dfrac{64y^6}{x^6}\right)+8\left(\dfrac{x}{2y}\right)\left(-\dfrac{128y^7}{x^7}\right)+(1)(1)\left(\dfrac{256y^8}{x^8}\right)$

$=\dfrac{x^8}{256y^8}-\dfrac{x^6}{8y^6}+\dfrac{7x^4}{4y^4}-14\dfrac{x^2}{y^2}+70-224\dfrac{y^2}{x^2}+448\dfrac{y^4}{x^4}-512\dfrac{y^6}{x^6}+\dfrac{256y^8}{x^8}$

vi. $\left(\sqrt{\dfrac{a}{x}}-\sqrt{\dfrac{x}{a}}\right)^6$

Sol. $\left(\sqrt{\dfrac{a}{x}}-\sqrt{\dfrac{x}{a}}\right)^6=\binom{6}{0}\left(\sqrt{\dfrac{a}{x}}\right)^6\left(-\sqrt{\dfrac{x}{a}}\right)^0+\binom{6}{1}\left(\sqrt{\dfrac{a}{x}}\right)^5\left(-\sqrt{\dfrac{x}{a}}\right)^1$

$+\binom{6}{2}\left(\sqrt{\dfrac{a}{x}}\right)^4\left(-\sqrt{\dfrac{x}{a}}\right)^2+\binom{6}{3}\left(\sqrt{\dfrac{a}{x}}\right)^3\left(-\sqrt{\dfrac{x}{a}}\right)^3+\binom{6}{4}\left(\sqrt{\dfrac{a}{x}}\right)^2\left(-\sqrt{\dfrac{x}{a}}\right)^4$

$+\binom{6}{5}\left(\sqrt{\dfrac{a}{x}}\right)^1\left(-\sqrt{\dfrac{x}{a}}\right)^5+\binom{6}{6}\left(\sqrt{\dfrac{a}{x}}\right)^0\left(-\sqrt{\dfrac{x}{a}}\right)^6$

$$= (1)\left(\frac{a}{x}\right)^3 (1) + 6\left(\frac{a}{x}\right)^{5/2}\left(\frac{-x}{a}\right)^{1/2} + 15\left(\frac{a}{x}\right)^2\left(\frac{x}{a}\right)^1 + 20\left(\frac{a}{x}\right)^{3/2}\left(\frac{-x}{a}\right)^{3/2} + 15\left(\frac{a}{x}\right)^1\left(\frac{x}{a}\right)^2$$

$$+6\left(\frac{a}{x}\right)^{1/2}\left(\frac{-x}{a}\right)^{5/2} + (1)(1)\left(\frac{x}{a}\right)^3$$

$$= \frac{a^3}{x^3} - 6\frac{a^{5/2}}{x^{5/2}} \times \frac{x^{1/2}}{a^{1/2}} + 15\frac{a^2}{x^2} \times \frac{x}{a} - 20\frac{a^{3/2}}{x^{3/2}} \times \frac{x^{3/2}}{a^{3/2}} + 15\left(\frac{a}{x} \times \frac{x^2}{a^2}\right) - 6\frac{a^{1/2}}{x^{1/2}} \times \frac{x^{5/2}}{a^{5/2}} + 6\frac{x^3}{a^3}$$

$$= \frac{a^3}{x^3} - 6\left(\frac{a^2}{x^2}\right) + 15\frac{a}{x} - 20 + 15\frac{x}{a} - 6\frac{x^2}{a^2} + \frac{x^3}{a^3}$$

2. Using Binomial theorem, expand the following:

i. $(0.93)^3$ **Faisalabad 2008**

Sol. $(0.93)^3 = (1-0.03)^3$

$$= \binom{3}{0}(1)^3(-0.003)^0 + \binom{3}{1}(1)^2(-0.03)^1 + \binom{3}{2}(1)^1(-0.03)^2 + \binom{3}{3}(1)^0(-0.003)^3$$

$$= (1)(1)(1) + 3(1)(-0.03) + (3)(1)(0.0009) + 1(1)(-0.000027)$$

$$= 1 - 0.09 + 0.00027 - 0.000027$$

$$= 0.0910243$$

ii. $(2.02)^4$ **Faisalabad 2009, Multan 2008, 2009**

Sol. $(2.02)^4 = (2+0.02)^4 = \binom{4}{0}(2)^4(.02)^0 + \binom{4}{1}(2)^3(.02)^1 + \binom{4}{2}(2)^2(.02)^2$

$$+\binom{4}{3}(2)^1(.02)^3 + \binom{4}{4}(2)^0(.02)^4$$

$$= (1)(16)(1) + 4(8)(0.02) + 6(4)(0.0004) + 4(2)(0.00008) + (1)(1)(0.0000006)$$

$$= 16 + 0.6400 + 0.0096 + 0.000064 + 0.00000016$$

$$= 16.6496$$

iii. $(9.98)^4$ **Sargodha 2009**

Sol. $(9.98)^4 = (10-0.02)^4 = \binom{4}{0}(10)^4(-0.02)^0 + \binom{4}{1}(10)^3(-0.02)^1 + \binom{4}{2}(10)^2(-0.02)^2$

$$+\binom{4}{3}(10)^1(-0.02)^3 + \binom{4}{4}(10)^0(-0.02)^4$$

$$= (1)(10000)(1) + 4(1000)(-0.02) + 6(100)(0.0004) + 4(10)(-0.00008)$$

$$+(1)(1)(0.0000016)$$

$$= 10000 - 80 + 0.24 - 0.00032 + 0.00000016 = 9920.2397$$

iv. $(2.1)^5$

Sol. $(2.1)^5 = (2+0.1)^5 = \binom{5}{0}(2)^5(0.1)^0 + \binom{5}{1}(2)^4(0.1)^1 + \binom{5}{2}(2)^3(0.1)^2$

$+\binom{5}{3}(2)^2(0.1)^3 + \binom{5}{4}(2)^1(0.1)^4 + \binom{5}{5}(2)^0(0.1)^5$

$= (1)(32)(1) + 5(16)(0.1) + 10(8)(0.01) + 10(4)(0.001) + 5(2)(0.0001) + (1)(1)(0.00001)$

$= 32 + 8 + 0.8 + 0.4 + 0.001 + 0.00001 = 40.84101$

3. Expand and simplify the following:

i. $(a+\sqrt{2}x)^4 + (a-\sqrt{2}x)^4$

Sol. $= \binom{4}{0}a^4(\sqrt{2}x)^0 + \binom{4}{1}a^3(\sqrt{2}x)^1 + \binom{4}{2}a^2(\sqrt{2}x)^2 + \binom{4}{3}a^1(\sqrt{2}x)^3 + \binom{4}{4}a^0(\sqrt{2}x)^4$

$= \binom{4}{0}a^4(-\sqrt{2}x)^0 + \binom{4}{1}a^3(-\sqrt{2}x)^1 + \binom{4}{2}a^2(-\sqrt{2}x)^2 + \binom{4}{3}a^1(-\sqrt{2}x)^3 + \binom{4}{4}a^0(-\sqrt{2}x)^4$

$= (1)(a^4)(1) + 4a^3\sqrt{2}x + 6a^2(2x^2) + 4a(2\sqrt{2}x^3) + (1)(1)4x^4 + (1)a^4(1) + 4a^3$
$(-\sqrt{2}x) + 6a^2(+2x^2) + 4a(-2\sqrt{2}x^3) + (1)(1)(4x^4)$

$= a^4 + 4a^3\cancel{\sqrt{2}x} + 12a^2x^2 + 8a\cancel{\sqrt{2}x^3} + 4x^4 + a^4 - 4a^3\cancel{\sqrt{2}x} + 12a^2x^2 - \cancel{8a\sqrt{2}x^3} + 4x^4$

$= 2a^4 + 24a^2x^2 + 8x^4$

ii. $(2+\sqrt{3})^5 + (2-\sqrt{3})^5$

Sol. $= \binom{5}{0}(2)^5(\sqrt{3})^0 + \binom{5}{1}(2)^4(\sqrt{3})^1 + \binom{5}{2}(2)^3(\sqrt{3})^2 + \binom{5}{3}(2)^2(\sqrt{3})^3$

$+\binom{5}{4}(2)^1(\sqrt{3})^4 + \binom{5}{5}(2)^0(\sqrt{3})^5$

$+\binom{5}{0}(2)^5(-\sqrt{3})^0 + \binom{5}{1}(2)^4(-\sqrt{3})^1 + \binom{5}{2}(2)^3(-\sqrt{3})^2 + \binom{5}{3}(2)^2(-\sqrt{3})^3$

$+\binom{5}{4}(2)^1(-\sqrt{3})^4 + \binom{5}{5}(2)^0(-\sqrt{3})^5$

$= (1)(32)(1) + 5(16)(\sqrt{3}) + 10(8)(3) + 10(4)(3\sqrt{3}) + 5(2)(9) + (1)(1)(9\sqrt{3})$
$+(1)(32)(1) + 5(16)(-\sqrt{3}) + 10(8)(3) + 10(4)(-3\sqrt{3}) + 5(2)(9) + (1)(1)(-9\sqrt{3})$

$= 32 + \cancel{80\sqrt{3}} + 240 + \cancel{120\sqrt{3}} + 90 + \cancel{9\sqrt{3}} + 32 - \cancel{80\sqrt{3}} + 240 - \cancel{120\sqrt{3}} + 90 - \cancel{9\sqrt{3}}$

$= 64 + 480 + 180 = 724$

iii. $(2+i)^5-(2-i)^5$ $\boxed{i^3=i^2.i=(-1)i=-i \quad Similarly\ i^4=1\ \&\ i^5=i}$

Sol. $=\binom{5}{0}2^5i^0+\binom{5}{1}2^4i^1+\binom{5}{2}2^3i^2+\binom{5}{3}2^2i^3+\binom{5}{4}2^1i^4+\binom{5}{5}2^0i^5$

$-\left(\binom{5}{0}2^5(-i)^0+\binom{5}{1}2^4(-i)^1+\binom{5}{2}2^3(-i)^2+\binom{5}{3}2^2(-i)^3+\binom{5}{4}2^1(-i)^4+\binom{5}{5}2^0(-i)^5\right)$

$=32+5(16i)+10(8)(-1)+10(4)(-i)+5(2)(1)+(1)(1)(i)-32-5(-16i)-$
$10(4)(+i)-5(2)(+1)-(1)(1)(-i)$

$=\cancel{32}+80i-\cancel{80}-40i+\cancel{10}+i-\cancel{32}+80i+\cancel{80}-40i-\cancel{10}+i=82i$

iv. $(x+\sqrt{x^2-1})^3+(x-\sqrt{x^2-1})^3$

Sol. $=\binom{3}{0}x^3(\sqrt{x^2-1})^0+\binom{3}{1}x^2(\sqrt{x^2-1})^1+\binom{3}{2}x^1(\sqrt{x^2-1})^2+\binom{3}{3}x^0(\sqrt{x^2-1})^3$

$+\binom{3}{0}x^3(-\sqrt{x^2-1})^0+\binom{3}{1}x^2(-\sqrt{x^2-1})^1+\binom{3}{2}x^1(-\sqrt{x^2-1})^2+\binom{3}{3}x^0(-\sqrt{x^2-1})^3$

$=(1)x^3(1)+\cancel{3x^2(\sqrt{x^2-1})}+3x(x^2-1)+(1)\cancel{(1)(x^2-1)^{3/2}}$

$+(1)x^3(1)-\cancel{3x^2(\sqrt{x^2-1})}+3x(x^2-1)-(1)\cancel{(1)(x^2-1)^{3/2}}$

$=2x^3+6x\sqrt{x^2-1}$

4. **Expand the following in ascending power of x:**

i. $(2+x-x^2)^4$

Sol. $(2+x-x^2)^4=\left[(2+x)-x^2\right]^4$

$=\binom{4}{0}(2+x)^1(-x^2)^0+\binom{4}{1}(2+x)^3(-x^2)^1+\binom{4}{2}(2+x)^2(-x^2)^2$

$+\binom{4}{3}(2+x)^1(-x^2)^3+\binom{4}{4}(2+x)^0(-x^2)^4$

$=1\times(2+x)^4(1)+4(2+x)^3(-x^2)+(6)(2+x)^2(x^4)+4(2+x)(-x^6)+(1)(1)(x^8)$

$=\binom{4}{0}2^4x^0+\binom{4}{1}2^3x^1+\binom{4}{2}2^2x^2+\binom{4}{3}2^1x^3+\binom{4}{4}2^0x^4-4x^2(8+12x+6x^2+x^3)+$
$6(4+4x+x^2)(x^4)-8x^6-4x^7+x^8$

$=16+32x+24x^2+8x^3+x^4-32x^2-48x^3-24x^4-4x^5+24x^4+24x^5+6x^6-8x^6-4x^7+x^8$

$=16+32x-8x^2-40x^3+x^4+20x^5-2x^6-4x^7+x^8$

ii. $\mathbf{(1-x+x^2)^4}$

Sol. $(1-x+x^2)^4=\left[(1-x)+x^2\right]^4$

$$=\binom{4}{0}(1-x)^4(x^2)^0+\binom{4}{1}(1-x)^3(x^2)^1+\binom{4}{2}(1-x)^2(x^2)^2$$

$$+\binom{4}{3}(1-x)^1(x^2)^3+\binom{4}{4}(1-x)^0(x^2)^4$$

$$=1(1-x)^4(1)+4(1-x)^3(x^2)+6(1-x)^2x^4+4(1-x)(x^6)+(1)(1)(x^8)$$

$$=\binom{4}{0}1^4(-x)^0+\binom{4}{1}1^3(-x)^1+\binom{4}{2}1^2(-x)^2+\binom{4}{3}1^1(-x)^3+\binom{4}{4}1^0(-x)^4+4x^2(1-3x+3x^2-x^3)$$

$$+6x^4(1-2x+x^2)+4x^6(1-x)+x^8$$

$$=1-4x+6x^2-4x^3+x^4+4x^2-12x^3+12x^4+6x^4-12x^5+6x^6+4x^6-4x^7+x^8$$

$$=1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$$

iii. $\mathbf{(1-x-x^2)^4}$

Sol. $(1-x-x^2)^4=\left[(1-x)-x^2\right]^4$

$$=\binom{4}{0}(1-x)^4(-x^2)^0+\binom{4}{1}(1-x)^3(-x^2)^1+\binom{4}{2}(1-x)^2(-x^2)^2$$

$$+\binom{4}{3}(1-x)^1(-x^2)^3+\binom{4}{4}(1-x)^0(-x^2)^4$$

$$=1(1-x)^4(1)+4(1-x)^3(-x^2)+6(1-x)^2(x^4)+4(1-x)(-x^6)+(1)(1)(x^8)$$

$$1-4x+6x^2-4x^3+x^4-4x^2(1-3x+3x^2-x^3)+6x^4(1-2x+x^2)-4x^6(1-x)+x^8$$

$$=1-4x+6x^2-4x^3+x^4-4x^2+12x^3-12x^4+4x^5+6x^4-12x^5+6x^6-4x^6+4x^7+x^8$$

$$=1-4x+\quad{}^{2}+8x^3-5x^4-8x^5+2x^6+4x^7+x^8$$

5. **Expand the following in descending powers of x:**

i. $\mathbf{(x^2+x-1)^3}$

Sol. $(x^2+x-1)^3=\left[x^2+(x-1)\right]^3$

$$=\binom{3}{0}(x^2)^3(x-1)^0+\binom{3}{1}(x^2)^2(x-1)^1+\binom{3}{2}(x^2)^1(x-1)^2+\binom{3}{3}(x^2)^0(x-1)^3$$

$$=(1)(x^6)(1)+3(x^4)(x-1)+3(x^2)(x^2-2x+1)+(1)(1)(x^3-3x^2+3x-1)$$

$$=x^6+3x^5-\cancel{3x^4}+\cancel{3x^4}-6x^3+\cancel{3x^2}+x^3-\cancel{3x^2}+3x-1=x^6+3x^5-5x^3+3x-1$$

ii. $\left(x-1-\frac{1}{x}\right)^3$

Sol. $\left(x-1-\frac{1}{x}\right)^3=\left[(x-1)-\frac{1}{x}\right]^3$

$$=\binom{3}{0}(x-1)^3\left(\frac{-1}{x}\right)^0+\binom{3}{1}(x-1)^2\left(\frac{-1}{x}\right)^1+\binom{3}{2}(x-1)^1\left(\frac{-1}{x}\right)^2+\binom{3}{0}(x-1)^0\left(\frac{-1}{x}\right)^3$$

$$=(1)(x^3-3x^2+3x-1)(1)+3(x^2-2x+1)\left(\frac{-1}{x}\right)+3(x-1)\left(\frac{1}{x^2}\right)+(1)(1)\left(\frac{-1}{x^3}\right)$$

$$=x^3-3x^2+\cancel{3x}-1-\cancel{3x}+6-\frac{\cancel{3}}{\cancel{x}}+\frac{\cancel{3}}{\cancel{x}}-\frac{3}{x^2}-\frac{1}{x^3}$$

$$=x^3-3x^2+5-\frac{3}{x^2}-\frac{1}{x^3}$$

6. **Find the term involving:**

i. x^4 **in the expansion of** $(3-2x)^7$ **Faisalabad 2007**

Sol. $(3-2x)^7$

Its general term is

$$T_{r+1}=\binom{7}{r}(3)^{7-r}(-2x)^r=\binom{7}{r}(3)^{7-r}(-2)^r x^r \longrightarrow I$$

Compare exponent of x with 4 so $r=4$ put $r=4$ in I

$$T_{4+1}=\binom{7}{4}(3)^{7-4}(-2)^4x^4=\binom{7}{4}3^3(-2)^4x^4$$

$$T_5=35(27)(16)x^4=15120x^4$$

ii. x^{-2} **in the expansion of** $\left(x-\frac{2}{x^2}\right)^{13}$

Sol. Its general term is

$$T_{r+1}=\binom{13}{r}(x)^{13-r}\left(\frac{-2}{x^2}\right)^r$$

$$=\binom{13}{r}(x)^{13-r}(-2)^r\left(\frac{1}{x^{2r}}\right)$$

$$=\binom{13}{r}x^{13-r-2r}(-2)^r=\binom{13}{r}x^{13-3r}(-2)^r \longrightarrow I$$

Compare exponent of x with power -2

$$13-3r=-2 \Rightarrow 13+2=3r \Rightarrow 15=3r \Rightarrow r=\frac{15}{3}=5 \Rightarrow r=5$$

Put $r=5$ in I

$$T_{5+1}=\binom{13}{5}x^{13-3(5)}(-2)^5$$

$$=\frac{13!}{5!.8!}x^{13-15}(-2)^5=\frac{13.12.11.10.9.\cancel{8!}}{5.4.3.2.1.\cancel{8!}}(-32)x^{-2}$$

$$T_6=-41184x^{-2}$$

iii. a^4 in the expansion of $\left(\frac{2}{x}-a\right)^9$ **Sargodha 2008**

Sol. $\left(\frac{2}{x}-a\right)^9$ $\quad a^4=?$

Its general term is

$$T_{r+1}=\binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(-a)^r=\binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(-1)^r a^r$$

Compare exponent of a with power 4 $\Rightarrow r=4$

$$T_{4+1}=\binom{9}{4}\left(\frac{2}{x}\right)^{9-4}(-1)^4 a^4=\frac{9.8.7.6.\cancel{5!}}{\cancel{5!}.4!}\left(\frac{2}{x}\right)^5(1)a^4$$

$$=\frac{9.8.7.6.}{4.3.2.1}\frac{3.2}{x^5}a^4=\frac{4032}{x^5}a^4$$

iv. y^3 in the expansion of $(x-\sqrt{y})^{11}$

Sol. $(x-\sqrt{y})^{11}$

Its general term is

$$T_{r+1}=\binom{11}{r}(x)^{11-r}(-\sqrt{y})^r$$

$$=\binom{11}{r}(x)^{11-r}(-1)^r(y)^{r/2}\longrightarrow I$$

Put Power of y equal to 3 so

$$\frac{r}{2}=3 \Rightarrow r=6 \text{ so Put } r=6$$

$$T_{6+1}=\binom{11}{6}(x)^{11-6}(-1)^{6}y^{6/2}$$

$$T_7=\frac{11!}{6!.5!}x^5(1)y^3$$

$$=\frac{11.10.9.8.7.\cancel{6!}}{5.4.3.2.1.\cancel{6!}}x^5y^3=462x^5y^3$$

7. Find the Coefficient of: **Faisalabad 2007, 08, Multan 2009, Sargodha 2009, 10**

i. x^5 in the expansion of $\left(x^2-\frac{3}{2x}\right)^{10}$

Sol. Its general term is

$$T_{r+1}=\binom{10}{r}(x^2)^{10-r}\left(\frac{-3}{2x}\right)^r$$

$$=\binom{10}{r}x^{20-2r}\left(\frac{-3}{2}\right)^r\frac{1}{x^r}$$

$$=\binom{10}{r}x^{20-2r-r}\left(\frac{-3}{2}\right)^r=\binom{10}{r}x^{20-3r}\left(\frac{-3}{2}\right)^r\longrightarrow I$$

Compare exponent of x with $x^5\Rightarrow 20-3r=5\Rightarrow -3r=5-15\Rightarrow r=5$

$$T_{5+1}=\binom{10}{5}x^{20-3(5)}\left(\frac{-3}{2}\right)^5=\frac{10!}{5!.5!}x^{20-15}\left(\frac{-243}{32}\right)$$

$$T_6=\frac{10.9.8.7.6.\cancel{5!}}{5.4.3.2.1.\cancel{5!}}\left(\frac{-243}{32}\right)x^5$$

Coefficient of x^5 is $\dfrac{-15309}{8}$

ii. x^n in the expansion of $\left(x^2-\frac{1}{x}\right)^{2n}$

Sol. Its general term is

$$T_{r+1}=\binom{2n}{r}(x^2)^{2n-r}\left(-\frac{1}{x}\right)^r$$

$$=\binom{2n}{r}x^{4n-2r}(-1)^r\left(\frac{1}{x^r}\right)$$

$$=\binom{2n}{r}x^{4n-2r-r}(-1)^r=\binom{2n}{r}x^{4n-3r}(-1)^r\longrightarrow I$$

Put $4n-3r=n\Rightarrow -3r=n-4n$

$\Rightarrow -3r = -3n \Rightarrow r = n$

Put $r = n$ in I

$$T_{n+1} = \binom{2n}{n} x^{4n-3n}(-1)^n = \left(\frac{(2n)!}{n!(2n-n)!}\right)(-1)^n.x^n = \frac{(2n)!}{n!n!}(-1)^n x^n = \frac{(2n)!}{(n!)^2}(-1)^n x^n$$

Coefficient of x^n is $\frac{2n!}{(n!)^2}(-1)^n$

8. Find 6^{th} term in the expansion of $\left(x^2 - \frac{3}{2x}\right)^{10}$ **Sargodha 2006, Multan 2008**

Sol. Its general term is

$$T_{r+1} = \binom{10}{r}(x^2)^{10-r}\left(\frac{-3}{2x}\right)^r$$

Put $r = 5$

$$T_{5+1} = \binom{10}{5}(x^2)^{10-5}\left(\frac{-3}{2x}\right)^5$$

$$= \frac{10!}{5!.5!}(x^2)^5\left(\frac{-3}{2}\right)^5 \frac{1}{x^5}$$

$$= \frac{10.9.8.7.6.\cancel{5!}}{5.4.3.2.1.\cancel{5!}} \quad x^{10}\left(\frac{-243}{32}\right)\frac{1}{x^5}$$

$$= -252 \times \frac{243}{32}x^5$$

$$T_6 = -\frac{15309}{8}x^5$$

9. Find the term independent of x in the following expansions.

i. $\left(x - \frac{2}{x}\right)^{10}$ **Multan 2007, Sargodha 2011**

Sol. Its general term is

$$T_{r+1} = \binom{10}{r}(x)^{10-r}\left(\frac{-2}{x}\right)^r$$

$$= \binom{10}{r}(x)^{10-r}\left(\frac{1}{x^r}\right)(-2)^r$$

$$= \binom{10}{r}x^{10-r-r}(-2)^r = \binom{10}{r}x^{10-2r}(-2)^r \longrightarrow I$$

Put power of $x = 0$ so $10 - 2r = 0$

$\Rightarrow 2r = 10 \Rightarrow r = 5$

Put $r = 5$ in I

$$T_{5+1}=\binom{10}{5}x^{10-2(5)}(-2)^5=\frac{10!}{5!.5!}x^{10-10}(-32)$$

$$T_6=-252(32)x^0=-8064(1)=-8064$$

ii. $\left(\sqrt{x}+\frac{1}{2x^2}\right)^{10}$

Sol. Its general term is

$$T_{r+1}=\binom{10}{r}(\sqrt{r})^{10-r}\left(\frac{1}{2x^2}\right)^r=\binom{10}{r}x^{\frac{10-r}{2}}\left(\frac{1}{x^{2r}}\right)\left(\frac{1}{2}\right)^r$$

$$=\binom{10}{r}x^{\frac{10-r}{2}-2r}\left(\frac{1}{2}\right)^r=\binom{10}{r}x^{\frac{10-r-4r}{2}}\left(\frac{1}{2}\right)^r$$

$$=\binom{10}{r}x^{\frac{10-5r}{2}}\left(\frac{1}{2}\right)^r\longrightarrow I$$

Put $\frac{10-5r}{2}=0\Rightarrow 10-5r=0\Rightarrow 5r=10\Rightarrow r=2$

Put $r=2$ in I

$$T_{2+1}=\binom{10}{2}x^{\frac{10-5(2)}{2}}\left(\frac{1}{2}\right)^2$$

$$=\frac{10!}{2!.8!}x^{\frac{10-10}{2}}\frac{1}{4}=\frac{10.9.\cancel{8!}}{2.1.\cancel{8!}}x^0\left(\frac{1}{4}\right)$$

$$T_3=\frac{45}{4}(1)=\frac{45}{4}$$

iii. $(1+x^2)^3\left(x+\frac{1}{x^2}\right)^4$

Sol. $(1+x^2)^3\left(x+\frac{1}{x^2}\right)^4=(1+x^2)^3\left(\frac{1+x^2}{x^2}\right)^4$

$$=(1+x^2)^3\frac{(1+x^2)^4}{x^8}=\frac{(1+x^2)^{3+4}}{x^8}=\frac{(1+x^2)^7}{x^{\frac{8}{7}\times 7}}$$

$$=\left(\frac{1+x^2}{x^{\frac{8}{7}}}\right)^7=\left(\frac{1}{x^{8/7}}+\frac{x^2}{x^{8/7}}\right)^7=\left(\frac{1}{x^{8/7}}+x^{2-\frac{8}{7}}\right)^7=\left(\frac{1}{x^{8/7}}+x^{6/7}\right)^7=\left(x^{6/7}+\frac{1}{x^{8/7}}\right)^7$$

Its general term is

$$T_{r+1}=\binom{7}{r}(x^{6/7})^{7-r}\left(\frac{1}{x^{8/7}}\right)^r$$

$$T_{r+1}=\binom{7}{r}x^{\frac{42-6r}{7}-\frac{8r}{7}}=\binom{7}{r}x^{\frac{42-6r-8r}{7}}=\binom{7}{r}x^{\frac{42-14}{7}}$$

Put $\frac{42-14r}{7}=0 \Rightarrow 42-14r=0$

$\Rightarrow 14r=42 \Rightarrow r=3$

Put $r=3$in I

$$T_{3+1}=\binom{7}{3}x^{\frac{42-14(3)}{7}}=\frac{7!}{3!.4!}=x^{\frac{42-42}{7}}$$

$$=\frac{7.6.5.\cancel{4!}}{3.2.1.\cancel{4!}}x^0=35(1)=35$$

So term independent of x is 35.

10. Determine the middle term in the following expansions:

i. $\left(\frac{1}{x}-\frac{x^2}{2}\right)^{12}$ **Faisalabad 2008, Sargodha 2008, 2009**

Sol. Its general term is $\boxed{\text{If power is 12 then it has 13 term so middle term is 7}}$

$$T_{r+1}=\binom{12}{r}\left(\frac{1}{x}\right)^{12-r}\left(\frac{-x^2}{2}\right)^r$$

Put $r=6$

$$T_{6+1}=\binom{12}{6}\left(\frac{1}{x}\right)^{12-6}\left(\frac{-x^2}{2}\right)^6$$

$$=\frac{12!}{6!.6!}\left(\frac{1}{x}\right)^6\left(\frac{x^{12}}{2^6}\right)$$

$$T_7=\frac{12.11.10.9.8.7.\cancel{6!}}{6.5.4.3.2.1.\cancel{6!}}\frac{1}{x^6}\left(\frac{x^{12}}{64}\right)$$

$$=924\frac{x^{12-6}}{64}=\frac{231x^6}{16}$$

So middle term $=T_7=\frac{231x^6}{16}$

ii. $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$ **Federal**

Sol. Its general term is

$$T_{r+1}=\binom{11}{r}\left(\frac{3x}{2}\right)^{11-r}\left(\frac{-1}{3x}\right)^r$$

It has twelve (12) term in expansion so middle terms are 6^{th} & 7^{th} for 6^{th} term

Put $r=5$

$$T_{5+1}=\binom{11}{5}\left(\frac{3x}{2}\right)^{11-5}\left(\frac{-1}{3x}\right)^{5}$$

$$=\frac{11!}{5!.6!}\left(\frac{3x}{2}\right)^{6}\left(\frac{-1}{243x^{5}}\right)$$

$$=\frac{11.10.9.8.7.\cancel{6!}}{5.4.3.2.1.\cancel{6!}}\left(\frac{729x^{6}}{64}\right)\left(\frac{-1}{243x^{5}}\right)$$

$$=-462\times\frac{3x}{64}=\frac{693x}{32}$$

For 7^{th} term put $r=6$

$$T_{6+1}=\binom{11}{6}\left(\frac{3x}{2}\right)^{11-6}\left(\frac{-1}{3x}\right)^{6}$$

$$=462\left(\frac{3x}{2}\right)^{5}\left(\frac{1}{729x^{6}}\right)=462\left(\frac{243x^{5}}{32}\right)\left(\frac{1}{729x^{6}}\right)=\frac{231}{48x}=\frac{77}{16x}$$

iii. $\left(2x-\frac{1}{2x}\right)^{2m+1}$

Sol. Its general term is

$$T_{r+1}=\binom{2m+1}{r}(2x)^{2m+1-r}\left(-\frac{1}{2x}\right)^{r}$$

Its power 2m+1 is odd so middle term are (m+1) & (m+2)

$$T_{r+1}=\binom{2m+1}{r}2x^{2m+1-r}(-1)^{r}\left(\frac{1}{2x}\right)^{r}$$

$$T_{r+1}=\binom{2m+1}{r}(2x)^{2m+1-2r}(-1)^{r}\text{ ——— }I$$

for $(m+1)$ th term Put $r=m$ then

$$T_{m+1}=\binom{2m+1}{m}(2x)^{2m+1-2(m)}(-1)^{m}$$

$$=\frac{(2m+1)!}{m!(2m+1-m)!}(2x)^{\cancel{2m}+1-\cancel{2m}}(-1)^{m}$$

$$=\frac{(2m+1)!}{(m+1)!(m)!}2x(-1)^{m}$$

Note: If power is odd then

middle term $=\frac{n+1}{2}$ & $\frac{n+3}{2}$

put $n=2m+1$

middle terms are $=\frac{2m+1+1}{2}$ & $\frac{2m+1+3}{2}$

$$=\frac{2m+2}{2}\ \&\ \frac{2m+4}{2}=\frac{\cancel{2}(m+1)}{\cancel{2}}\ \&\ \frac{\cancel{2}(m+2)}{\cancel{2}}$$

For $(m+2)$ th term

Put $r=m+1$

$$T_{m+1+1}=\binom{2m+1}{m+1}(2x)^{2m+1-2(m+1)}(-1)^{m+1}$$

$$=\frac{(2m+1)!}{(m+1)!(2m+1-m-1)}(2x)^{2m+1-2m-2}(-1)^{m+1}$$

$$T_{m+2}=\frac{(2m+1)!}{(m+1)!(m)!}(2x)^{-1}(-1)^{m+1}$$

$$=\frac{(2m+1)!}{(m+1)!(m)!}\frac{1}{2x}(-1)^{m+1}$$

11. **Find $(2n+1)$th term from the end in the expansion of $\left(x-\frac{1}{2x}\right)^{3n}$:**

Sol. For $(2n+1)$ th term from beginning, the question become $\left(-\frac{1}{2x}+x\right)^{3n}$

Its general term is

$$T_{r+1}=\binom{3n}{r}\left(\frac{-1}{2x}\right)^{3n-r}(x)^r$$

Put $r=2n$

$$T_{2n+1}=\binom{3n}{2n}\left(\frac{-1}{2x}\right)^{3n-2n}(x)^{2n}$$

$$=\frac{3n!}{2n!(3n-2n)!}\left(\frac{-1}{2x}\right)^{n}(x)^{2n}$$

$$=\frac{(3n)!}{2n!n!}(-1)^n.\left(\frac{1}{2}\right)^n.\left(\frac{1}{x^n}\right)x^{2n}$$

$$=\frac{(3n)!}{2n!n!}(-1)^n.\frac{1}{2^n}x^{2n-n}$$

$$=\frac{(3n)!}{(2n!).n!}\frac{(-1)^n x^n}{2^n}$$

12. **Show that the middle term of $(1+x)^{2n}$ is $\frac{1.3.5........(2n-1)}{n!}2^n x^n$:**

Sol. Its general term is:

$$T_{r+1}=\binom{2n}{r}(1)^{2n-r}x^r$$

Power 2n is even so middle term
$=\frac{n+2}{2}=\frac{2n+2}{2}=\frac{\cancel{2}(n+1)}{\cancel{2}}=n+1$

Put $r=n$

$$T_{n+1}=\binom{2n}{n}(1)^{2n-n}x^n=\frac{(2n)!}{n!(2n-n)!}1^n.x^n$$

$$=\frac{(2n)!}{n!.n!}x^n=\frac{2n.(2n-1)......5.4.3.2.1}{n!.n!}x^n=\frac{1.2.3.4.5............(2n-1).2n}{n!.n!}x^n$$

$$=\frac{[1.3.5.........(2n-1)][(2.4.6.........2n)]}{n!.n!}x^n=\frac{(1.3.5........(2n-1))(1.2.3...n)2^n x^n}{n!.n!}$$

$$=\frac{[1.3.5.........(2n-1)]2^n.n!x^n}{n!.n!}=\frac{1.3.5........(2n-1)}{n!}2^n x^n$$

Hence proved.

13. *Show that* : $\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+......+\binom{n}{n-1}=2^{n-1}$

Sol. We know that

$$(1+x)^n=\binom{n}{0}(1)^n x^0+\binom{n}{1}(1)^{n-1}x^1+\binom{n}{2}(1)^{n-2}x^2+\binom{n}{3}(1)^{n-3}x^3+...+\binom{n}{n-1}(1)^{n-n+1}x^{n-1}+\binom{n}{n}(1)^0 x^n$$

$$(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+.......+\binom{n}{n-1}x^{n-1}+\binom{n}{n}x^n \text{——} I$$

Put $x=-1$

$$(1-1)^n=\binom{n}{0}+\binom{n}{1}(-1)+\binom{n}{2}(-1)^2+\binom{n}{3}(-1)^3+.....+\binom{n}{n-1}(-1)^{n-1}+\binom{n}{n}(-1)^n$$

$$0=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+......-\binom{n}{n-1}+\binom{n}{n}$$

$$\Rightarrow\binom{n}{0}+\binom{n}{2}+.......+\binom{n}{n}=\binom{n}{1}+\binom{n}{3}+......+\binom{n}{n-1}\text{——} II$$

Now put $x=1$ *in I*

$$(1+1)^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{3}+.....+\binom{n}{n-1}+\binom{n}{n}$$

$$2^n=\left[\binom{n}{0}+\binom{n}{2}+.....+\binom{n}{n}\right]+\left[\binom{n}{1}+\binom{n}{3}+.....+\binom{n}{n-1}\right]$$

$$2^n=\left[\binom{n}{1}+\binom{n}{3}+........+\binom{n}{n-1}\right]+\left[\binom{n}{1}+\binom{n}{3}+.....+\binom{n}{n-1}\right]$$

$$2^n = 2\left[\binom{n}{1}+\binom{n}{3}+\dots\dots+\binom{n}{n-1}\right] \Rightarrow \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\dots\dots+\binom{n}{n-1} = \frac{2^n}{2}$$

$$\Rightarrow \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\dots\dots+\binom{n}{n-1} = 2^{n-1}$$

14. **Show that:** $\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+\dots\dots\dots+\frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$

Sol. $\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+\dots\dots\dots+\frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$

$$\text{L.H.S} = \binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+\dots\dots\dots+\frac{1}{n+1}\binom{n}{n}$$

$$= 1+\frac{1}{2}(n)+\frac{1}{3}\frac{n(n-1)(n-\cancel{2})!}{2!(n-\cancel{2})!}+\frac{n(n-1)(n-2)\cancel{(n-3)!}}{4.3!\cancel{(n-3)!}}+\dots\dots\dots+\frac{1}{n+1}(1)$$

$$= 1+\frac{n}{2!}+\frac{n(n-1)}{3!}+\frac{n(n-1)(n-2)}{4!}+\dots\dots\dots+\frac{1}{n+1}$$

'$\times$' & '$\div$' by $(n+1)$

$$= \frac{1}{(n+1)}\left[(n+1)+\frac{(n+1)n}{2!}+\frac{(n+1)(n)(n-1)}{3!}+\frac{(n+1)n(n-1)(n-2)}{4!}+\frac{n+1}{n+1}\right]$$

$$= \frac{1}{(n+1)}\left[{}^{n+1}C_1+{}^{n+1}C_2+{}^{n+1}C_3+\dots\dots\dots\dots+{}^{n+1}C_{n+1}\right] \longrightarrow I$$

Consider $(1+x)^{n+1} = {}^{n+1}C_0(1)^{n+1}(x)^0+{}^{n+1}C_1(1)^{n+1-1}(x)^1+{}^{n+1}C_2(1)^{n+1-2}x^2$
$+{}^{n+1}C_3(1)^{n+1-3}x^3+\dots+{}^{n+1}C_{n+1}x^{n+1}$

Put $x=1$

$$(1+1)^{n+1} = {}^{n+1}C_0+{}^{n+1}C_1+{}^{n+1}C_2+{}^{n+1}C_3+\dots\dots\dots+{}^{n+1}C_{n+1}$$

$$2^{n+1} = 1+{}^{n+1}C_1+{}^{n+1}C_2+{}^{n+1}C_3+\dots\dots\dots+{}^{n+1}C_{n+1}$$

$$2^{n+1}-1 = {}^{n+1}C_1+{}^{n+1}C_2+{}^{n+1}C_3+\dots\dots\dots\dots{}^{n+1}C_{n+1} \longrightarrow II$$

Use II in I then I become

$$\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\dots+\frac{1}{n+1}\binom{n}{n} = \frac{1}{(n+1)}\left[2^{n+1}-1\right]$$

Hence proved.

Exercise 8.3

Example 7:- **If m and n are nearly equal Find value of** $\dfrac{m}{m+2n}$ *Faisalabad* 2008 Sargodha 2009

Sol. Put m=n+h (where h is very small that its higher power can be neglected)

$$\frac{m}{m+2n}=\frac{n+h}{n+h+2n}=\frac{n+h}{3n+h}=\frac{(n+h)}{3n\left(1+\frac{h}{3n}\right)}$$

$$=(n+h).\frac{1}{3n}\left(1+\frac{h}{3n}\right)^{-1}=\left(\frac{n}{3n}+\frac{h}{3n}\right)\left(1+(-1)\frac{h}{3n}+neglect\right)$$

$$=\left(\frac{1}{3}+\frac{h}{3n}\right)\left(1-\frac{h}{3n}\right)=\frac{1}{3}-\frac{h}{9n}+\frac{h}{3n}-\frac{h^2}{9n^2}(neglect)$$

$$=\frac{1}{3}+\frac{h}{3n}-\frac{h}{9n}=\frac{1}{3}+\frac{3h-h}{9n}=\frac{1}{3}+\frac{h}{9n}$$

1. **Expand the following upto 4 terms, taking the values of x such that the expansion in each case is valid.**

i. $(1-x)^{1/2}$ **Sargodha 2006, Faisalabad 2007, 2009**

Sol. $$(1-x)^{1/2}=1+\frac{1}{2}(-x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-x)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\times\left(\frac{1}{2}-2\right)}{3!}(-x)^3+\ldots\ldots\ldots$$

$$=1-\frac{x}{2}+\frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)x^2+\frac{1}{3.2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x^3)+\ldots\ldots\ldots$$

$$=1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (valid\ if\ |-x|=|x|<1)$$

ii. $(1+2x)^{-1}$ **Faisalabad 2008, Multan 2008**

Sol. $$(1+2x)^{-1}=1+(-1)(2x)+\frac{(-1)(-1-1)}{2!}(2x)^2+\frac{(-1)(-1-1)(-1-2)}{3!}(2x)^3+\ldots\ldots\ldots$$

$$=1-2x+\frac{(-1)(-2)}{2!}(4x^2)+\frac{(-1)(-2)(-3)}{3.2.1}(8x^3)+\ldots\ldots\ldots+$$

$$=1-2x+\frac{\cancel{2}}{\cancel{2}}(4x^2)-\frac{\cancel{6}}{\cancel{6}}(8x^3)+\ldots\ldots\ldots$$

$$=1-2x+4x^2-8x^3+\ldots\ldots\ldots \qquad (valid\ if\ |2x|<1\Rightarrow|x|<\frac{1}{2})$$

iii. $(1+x)^{-1/3}$ **Faisalabad 2007, Rawalpindi 2009**

Sol. $$(1+x)^{-1/3}=1+\left(-\frac{1}{3}\right)x+\frac{-\frac{1}{3}\left(-\frac{1}{3}-1\right)}{2!}x^2+\frac{-\frac{1}{3}\left(-\frac{1}{3}-1\right)\left(\frac{-1}{3}-2\right)}{3!}x^3+\text{.........}$$

$$=1-\frac{x}{3}+\frac{1}{2}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)x^2+\frac{1}{3.2}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)x^3+\text{.........}$$

$$=1-\frac{x}{3}+\frac{2}{9}x^2-\frac{14}{81}x^3+\text{...................}\quad (valid\ if\ |x|<1)$$

iv. $(4-3x)^{1/2}$ **Multan 2007**

Sol. $$(4-3x)^{1/2}=4^{1/2}\left(1-\frac{3x}{4}\right)^{1/2}=2^{2\times\frac{1}{2}}\left(1-\frac{3x}{4}\right)^{1/2}$$

$$=2\left[1+\frac{1}{2}\left(\frac{-3x}{4}x\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{-3x}{4}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{-3x}{4}\right)^3+\text{.........}\right]$$

$$=2\left[1-\frac{3}{8}x+\frac{1}{2.1}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{9}{16}x^2\right)+\frac{1}{3.2.1}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{27}{64}x^3\right)\right]$$

$$=2\left[1-\frac{3}{8}x+\frac{9}{128}x^2-\frac{27x^3}{1024}+\text{...........}\right]$$

$$=2-\frac{3}{4}x+\frac{9x^2}{64}-\frac{27x^3}{572}+\text{........}\quad\left(valid\ if\ \left|\frac{3x}{4}\right|<1\Rightarrow|x|<\frac{4}{3}\right)$$ *Faisalabad* 2008 *Sargodha* 2009

v. $(8-2x)^{-1}$ **Faisalabad 2009**

Sol. $$(8-2x)^{-1}=8^{-1}\left(1-\frac{2x}{8}\right)^{-1}=\frac{1}{8}\left(1-\frac{x}{4}\right)^{-1}$$

$$=\frac{1}{8}\left[1+(-1)\left(\frac{-x}{4}\right)+\frac{(-1)(1-1)}{2!}\left(\frac{-x}{4}\right)^2+\frac{(-1)(-1-1)(-1-2)}{3!}\left(\frac{-x}{4}\right)^3+\text{.......}\right]$$

$$=\frac{1}{8}\left[1+\frac{x}{4}+\frac{(-1)(-2)}{2}\left(\frac{x^2}{16}\right)+\frac{(-1)(-2)(-3)}{3.2.1}\left(\frac{-x^3}{64}\right)+\text{.......}\right]$$

$$=\frac{1}{8}\left[1+\frac{x}{4}+\frac{x^2}{16}+\frac{x^3}{64}+.......\right] \qquad \left(valid\ if\left|\frac{2x}{8}\right|<1\Rightarrow|x|<\frac{8}{2}\Rightarrow|x|<4\right)$$

vi. $(2-3x)^{-2}$

Sargodha 2009

Sol. $(2-3x)^{-2}=2^{-2}\left(1-\frac{3x}{2}\right)^{-2}$

$$=\frac{1}{2^2}\left[1+(-2)\left(\frac{-3x}{2}\right)+\frac{(-2)(-2-1)}{2!}\left(\frac{-3x}{2}\right)^2+\frac{(-2)(-2-1)(-2-2)}{3!}\left(\frac{-3x}{2}\right)^2+.......\right]$$

$$=\frac{1}{4}\left[1+3x+\frac{(-2)(-3)}{2}\left(\frac{9x^2}{4}\right)+\frac{(-2)(-3)(-4)}{3.2.1}\left(\frac{-27x^3}{8}\right)+.......\right]$$

$$=\frac{1}{4}\left[1+3x+\frac{27x^2}{4}+\frac{27x^3}{2}+.......\right] \qquad \left(valid\ if\left|\frac{3x}{2}\right|<1\Rightarrow|x|<\frac{2}{3}\right)$$

vii. $\frac{(1-x)^{-1}}{(1+x)^2}$

Sol. $\frac{(1-x)^{-1}}{(1+x)^2}=(1-x)^{-1}(1+x)^{-2}$

$$=\left[1+(-1)(-x)+\frac{(-1)(-1-1)}{2!}(-x)^2+\frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3+.......\right]$$

$$\times\left[1+(-2)x+\frac{(-2)(-2-1)}{2!}x^2+\frac{(-2)(-2-1)(-2-2)}{3!}x^3+.......\right]$$

$$=\left[1+x+\frac{(-1)(-2)}{2}x^2+\frac{(-1)(-2)(-3)}{3.2.1}(-x)^3+.......\right]\times$$

$$\left[1-2x+\frac{(-2)(-3)}{2.1}x^2+\frac{(-2)(-3)(-4)}{3.2.1}(x)^3+.......\right]$$

$$=\left[1+x+x^2+x^3+..........\right]\left[1-2x+3x^2-4x^3+...........\right]$$

$$=\left[1-2x+3x^2-4x^3+x-2x^2+3x^3-4x^4+x^2-2x^3+3x^4-4x^5+x^3-2x^4+3x^5-4x^6\right]$$

$$=1-x+2x^2-2x^3+............... \qquad (valid\ of\ |x|<1)$$

viii. $\dfrac{\sqrt{1+2x}}{1-x}$ **Sargodha 2011**

Sol. $\dfrac{\sqrt{1+2x}}{1-x}=(1+2x)^{1/2}(1-x)^{-1}$

$$=\left[1+\frac{1}{2}(2x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(2x)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(2x)^3+\ldots\ldots\right]\times$$

$$\left[1+(-1)(-x)+\frac{(-1)(-1-1)}{2}(-x)^2+\frac{(-1)(-1-2)(-1-2)}{3!}(-x)^3+\ldots\ldots\right]$$

$$=\left[1+x+\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)(4x^2)+\frac{1}{3.2.1}\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)(8x^3)+\ldots\ldots\right]\times$$

$$\left[1+x+\frac{1}{2}(-1)(-2)x^2+\frac{1}{3.2.1}(-1)(-2)(-3)(-x)^3+\ldots\ldots\right]$$

$$=\left(1+x-\frac{x^2}{2}+\frac{x^3}{2}\ldots\ldots\ldots\right)\times(1+x+x^2+x^3+\ldots\ldots\ldots)$$

$$=\left(1+x+x^2+x^3+x+x^2+x^3+x^4-\frac{x^2}{2}-\cancel{\frac{x^3}{2}}-\frac{x^4}{8}+\cancel{\frac{x^3}{2}}+\frac{x^4}{2}+\frac{x^5}{2}+\ldots\right)$$

$$=1+2x+\frac{3x^2}{2}+\frac{2x^3}{1}+\frac{11x^4}{2}+\ldots\ldots\ldots$$

ix. $\dfrac{(4+2x)^{1/2}}{2-x}$

Sol. $\dfrac{(4+2x)^{1/2}}{2-x}=4^{1/2}\left(1+\dfrac{2x}{4}\right)^{1/2}(2-x)^{-1}$

$$=2\left(1+\frac{x}{2}\right)^{1/2}(-2)^{-1}\left(1-\frac{x}{2}\right)^{-1}$$

$$=\cancel{2}\left(1+\frac{x}{2}\right)^{1/2}\frac{1}{\cancel{2}}\left(1-\frac{x}{2}\right)^{-1}$$

$$=\left(1+\frac{x}{2}\right)^{1/2}.\left(1-\frac{x}{2}\right)^{-1}$$

$$=\left(1+\frac{1}{2}\left(\frac{x}{2}\right)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{x}{2}\right)^3+\dots\dots\right)$$

$$\times\left[1+(-1)\left(\frac{-x}{2}\right)+\frac{(-1)(-1-1)}{2!}\left(\frac{-x}{2}\right)^2+\frac{(-1)(-1-1)(-1-2)}{3!}\left(\frac{-x}{2}\right)^3+\dots\dots\dots\right]$$

$$=\left(1+\frac{x}{4}+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{x^2}{4}\right)+\left(\frac{1}{6}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{x^3}{8}\right)\right)$$

$$\times\left(1+\frac{x}{2}+\left(\frac{1}{2}\right)(-1)(-2)\left(\frac{x^2}{4}\right)+\left(\frac{1}{6}\right)(-1)(-2)(-3)\left(-\frac{x^3}{8}\right)\right)$$

$$=\left(1+\frac{x}{4}-\frac{x^2}{32}+\frac{x^3}{128}+\dots\dots\dots\right)\times\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\dots\dots\dots\right)$$

$$=1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\frac{x}{4}+\frac{x^2}{8}+\frac{x^3}{16}-\frac{x^2}{32}-\frac{x^3}{64}+\frac{x^3}{128}+\dots\dots\dots(\textit{Ignore } x^4 \textit{ and higher})$$

$$=1+\frac{x}{2}+\frac{x}{4}+\frac{x^2}{4}+\frac{x^2}{8}-\frac{x^2}{32}+\frac{x^3}{8}+\frac{x^3}{16}-\frac{x^3}{64}+\frac{x^3}{128}+\dots\dots\dots$$

$$=1+\frac{2x+x}{4}+\frac{8x^2+4x^2-x^2}{32}+\frac{16x^3+8x^3-2x^2+x^3}{128}+\dots\dots\dots$$

$$=1+\frac{3x}{4}+\frac{11x^2}{32}+\frac{23x^3}{128}\qquad\left(\textit{Valid if }\left|\frac{x}{2}\right|<1\Rightarrow|x|<2\right)$$

x. $\quad(1+x-2x^2)^{1/2}$

Sol. $\quad(1+x-2x^2)^{1/2}=\left[1+(x-2x^2)\right]^{1/2}$

$$=1+\frac{1}{2}(x-2x^2)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(x-2x^2)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(x-2x^2)^3+\dots\dots\dots$$

$$=1+\frac{1}{2}x-x^2+\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)(x^2-4x^3+4x^4)+\frac{1}{3.2.1}\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)$$

$$(x^3-6x^4+12x^5-8x^6)+\dots\dots\dots\dots$$

$$=1+\frac{x}{2}-x^2-\frac{1}{8}(x^2-4x^3+4x^4)+\frac{1}{16}(x^3-6x^4+12x^5-8x^6)+\dots\dots\dots\dots$$

$$=1+\frac{x}{2}-x^2-\frac{x^2}{8}+\frac{x^3}{2}-\frac{x^4}{2}+\frac{x^3}{16}-\frac{3x^4}{8}+\frac{3x^5}{4}-\frac{x^6}{2}+\dots\dots$$

$$=1+\frac{x}{2}-x^2-\frac{x^2}{8}+\frac{x^3}{2}+\frac{x^3}{16}-\frac{x^4}{2}-\frac{3x^4}{8}+\dots\dots=1+\frac{x}{2}-\frac{9x^2}{8}+\frac{9x^3}{16}-\frac{7x^4}{4}+\dots\dots$$

$$=1+\frac{x}{2}-\frac{9x^2}{8}+\frac{9x^3}{16}+\dots\dots\quad\left(Valid\ if\ -\frac{1}{2}<x<1\ Because\ satify\left|x-2x^2\right|<1\right)$$

xi. $(1-2x+3x^2)^{1/2}$

Sol. $=\left[(1+(-2x+3x^2)\right]^{1/2}$

$$=1+\left(\frac{1}{2}\right)(-2x+3x^2)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-2x+3x^2)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-2x+3x^2)^3+\dots\dots$$

$$=1-x+\frac{3x^2}{2}+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)(4x^2+9x^4-12x^3)+\left(\frac{1}{6}\right)\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)(-8x^3+36x^4-54x^5+27x^6)$$

$$=1-x+\frac{3x^2}{2}-\frac{x^2}{2}-\frac{9x^4}{8}+\frac{3}{2}x^3-\frac{x^3}{2}+\frac{36x^4}{16}-\frac{54x^5}{16}+\frac{27x^6}{16}+\dots\dots$$

$$=1-x+\frac{3x^2-x^2}{2}+\frac{3x^3-x^3}{2}+\frac{36x^4-18x^4}{16}-\frac{54x^5}{16}+\frac{27x^6}{16}+\dots\dots$$

$$=1-x+x^2+x^3+\frac{9}{8}x^4-\frac{27}{8}x^5+\frac{27}{16}x^6+\dots\left(Valid\ if\ \frac{-2}{3}<x<1\ Because\left|-2x+3x^2\right|<1\right)$$

2. Using Binomial theorem find the value of the following to three places of decimals

i. $\sqrt{99}$ **Rawalpindi 2009, Multan 2009**

Sol. $\sqrt{99}=(99)^{1/2}=(100-1)^{1/2}=100^{1/2}\left(1-\frac{1}{100}\right)^{1/2}$

$$=10(1-0.01)^{1/2}=10\left[1+\frac{1}{2}(-0.01)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-0.01)^2+\dots\dots\dots\right]$$

$$=10\left[1-0.005+\frac{(0.5)(-0.5)}{2}(-0.0001)+\dots\dots\dots\right]$$

$$=10\left[1-0.005-0.000125+\dots\dots\dots\right]=10(0.9499)=9.499\ approx$$

ii. $(0.98)^{1/2}$ **Sargodha 2008, Faisalabad 2009, Federal**

Sol. $(0.98)^{1/2}=(1-0.02)^{1/2}=1+\frac{1}{2}(-0.02)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-0.02)^2+.........$

$=1-0.01+\frac{(0.5)(-0.5)}{2}(0.0004)+.............$

$=1-0.01-0.00005=0.989=0.990$ (approx)

iii. $(1.03)^{1/3}$ **Multan 2008**

Sol. $(1.03)^{1/3}=(1+0.03)^{1/3}=1+\left(\frac{1}{3}\right)(0.03)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(0.03)^2+.........$

$=1+0.01+\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)(0.0009)+.............$

$=1+0.01-0.0001+............=1.010$ (approx)

iv. $\sqrt[3]{65}$ **Sargodha 2008**

Sol. $\sqrt[3]{65}=(65)^{1/3}=(64+1)^{1/3}=64^{1/3}\left(1+\frac{1}{64}\right)^{1/3}$

$=4^{3\times\frac{1}{3}}(1+0.015625)^{1/3}$

$=4\left[1+\frac{1}{3}(0.015625)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(0.001562)^2+................\right]$

$=4\left[1+0.0052083+\frac{1}{2}\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)(0.00024414)\right]$

$=4[1+0.0052083-0.00027126]$

$=4(1.005181173)=4.021$ (Approx)

v. $\sqrt[4]{17}$ **Faisalabad 2009, Sargodha 2011**

Sol. $\sqrt[4]{17}=(17)^{1/4}=(16+1)^{1/4}=16^{1/4}\left(1+\frac{1}{16}\right)^{1/4}=2^{4\times\frac{1}{4}}(1+0.0625)^{1/4}$

$$=2\left(1+\frac{1}{4}(0.0625)\right)+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}(0.0625)^2+.............$$

$$=2\left(1+0.15625+\frac{1}{2}\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)(0.00390625)\right)$$

$$=2(1+0.015625-0.00036621+...........)$$

$$=2(1.0525)=2.31 \text{ (Approx)}$$

vi. $\sqrt[5]{31}$ **Multan 2007**

Sol. $$\sqrt[5]{31}=(31)^{1/5}=(32-1)^{1/5}=(32)^{1/5}\left(1-\frac{1}{32}\right)^{1/5}$$

$$=2^{5\times\frac{1}{5}}(1+(-0.0321))^{1/5}=2\left(1+\frac{1}{5}(-0.031)+\frac{\frac{1}{5}\left(\frac{1}{5}-1\right)}{2!}(-0.031)^2+.............\right)$$

$$=2(1-0.00625-0.000077+.........)=2(0.9936)=1.9873 \text{ (Approx)}$$

vii. $\dfrac{1}{\sqrt[3]{998}}$ **Federal**

Sol. $$\frac{1}{\sqrt[3]{998}}=\frac{1}{(998)^{1/3}}(998)^{-1/3}=(1000-2)^{-1/3}$$

$$=(1000)^{-1/3}\left(1-\frac{2}{1000}\right)^{-1/3}=\frac{1}{(1000)^{1/3}(1+(-0.002))^{-1/3}}$$

$$=\frac{1}{10^{3\times\frac{1}{3}}}\left[1+\frac{-1}{3}(-0.002)+\frac{\left(-\frac{1}{3}\right)\left(\frac{-1}{3}-1\right)}{2!}(-0.002)^2+.........\right]$$

$$=\frac{1}{10}[1+0.00066+0.00000088+........]=\frac{1.00066}{10}$$

$$=0.100 \text{ (Approx)}$$

viii. $\dfrac{1}{\sqrt[5]{252}}$

Sol. $\dfrac{1}{\sqrt[5]{252}}=\dfrac{1}{(252)^{1/5}}(252)^{-1/5}=(243+9)^{-1/5}$

$=(243)^{-1/5}\left(1+\dfrac{9}{243}\right)^{-1/5}=\dfrac{1}{(243)^{1/5}}(1+0.037)^{-1/5}$

$=\dfrac{1}{3^{5\times\frac{1}{5}}}\left[1+\left(-\dfrac{1}{5}\right)(0.037)+\dfrac{\left(-\dfrac{1}{5}\right)\left(\dfrac{-1}{5}-1\right)}{2!}(0.037)^2+\ldots\ldots\right]$

$=\dfrac{1}{3}(1-0.0074+0.000167+\ldots\ldots$

$=0.331$ (Approx)

ix. $\dfrac{\sqrt{7}}{\sqrt{8}}$

Sol. $\dfrac{\sqrt{7}}{\sqrt{8}}=\dfrac{\sqrt{7}}{\sqrt{8}}=\left(\dfrac{7}{8}\right)^{1/2}=\left(1-\dfrac{1}{8}\right)^{1/2}$

$=1+\dfrac{1}{2}\left(-\dfrac{1}{8}\right)=\dfrac{\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}-1\right)}{2!}\left(-\dfrac{1}{8}\right)^2+\ldots\ldots\ldots$

$=1-0.0625+\dfrac{1}{2}\left(\dfrac{1}{2}\right)\left(-\dfrac{1}{2}\right)\left(\dfrac{1}{64}\right)+\ldots\ldots\ldots$

$=1-0.0625-0.001953$

$=0.935$ (Approx)

x. $(.998)^{-1/3}$

Sol. $(.998)^{-1/3}=(1-0.002)^{-1/3}=[1+(-.002)]^{-1/3}$

$=1+\left(-\dfrac{1}{3}\right)(-0.002)+\dfrac{\left(-\dfrac{1}{3}\right)\left(-\dfrac{1}{3}-1\right)}{2!}(-0.002)^2+\ldots\ldots\ldots$

$=1+0.00066+0.00000088+\ldots\ldots\ldots=1.0006$ (Approx)

$=1.001$ (Approx)

xi. $\dfrac{1}{\sqrt[6]{486}}$

Sol. $\dfrac{1}{\sqrt[6]{486}}=(486)^{-1/6}=(480+6)^{-1/6}(1+0.0104)^{-1/6}$

$$=\frac{1}{(480)^{1/6}}\left(1+\left(\frac{-1}{6}\right)(0.0104)+\frac{\left(\frac{-1}{6}\right)\left(-\frac{1}{6}-1\right)}{2!}(0.0104)^2+\dots\dots\dots\right)$$

$$=\frac{1}{2.7981}(1-0.0017+0.00001051+\dots\dots\dots)$$

$=0.3573(0.99831051)=0.3566$ (Approx)

xii. $(1280)^{1/4}$

Sol. $(1280)^{1/4}=(1296-16)^{1/4}=(1296)^{1/4}\left[1+\left(\dfrac{-16}{1296}\right)\right]^{1/4}$

$=6^{4\times\frac{1}{4}}[1+(-0.0123)]^{1/4}$

$$=6\left(1+\left(\frac{1}{4}\right)(-0.0123)+\frac{\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)}{2!}(-0.0123)^2+\dots\dots\dots\right)$$

$=6(1-0.00208-0.0000142-\dots\dots\dots)=5.981$ (Approx)

3. Find the coefficient of x^n in the expansion of:

i. $\dfrac{1+x^2}{(1+x)^2}$

Sol. $\dfrac{1+x^2}{(1+x)^2}=(1+x^2)(1+x)^{-2}$

$$=(1+x^2)\left[1+(-2)x+\frac{(-2)(-2-1)}{2!}x^2+\frac{(-2)(-2-1)(-2-2)}{3!}x^3+\dots\dots\right]$$

$$=(1+x^2)\left(1+(-2)x+\frac{(-2)(-3)}{2!}x^2+\frac{(-2)(-3)(-4)}{3.2.1}x^3+\dots\dots\right)$$

$=(1+x^2)(1+(-1)^1(2)x^1+(-1)^2.3x^2+(-1)^3(4)x^3+\dots\dots\dots(-1)^{n-2}(n-1)$

$$x^{n-2}+(-1)^{n-1}nx^{n-1}+(-1)^n(n+1)x^n)$$

$$=1+(-1)2x+........+(-1)^n(n+1)x^n+x^2+(-1)2x^3+........+(-1)^{n-2}$$
$$(n-1)x^n+(-1)^{n-1}nx^{n+1}+(-1)^n(n+1)x^{n+2}$$

Coefficient of x^n are $=(-1)^n(n+1)+(-1)^{n-2}(n-1)$

$$=(-1)^n(n+1)+(-1)^n(-1)^{-2}(n-1)$$

$$=(-1)^n(n+1)+(-1)^n\frac{1}{(-1)^2}(n-1)$$

$$=(-1)^n(n+1)+\frac{(-1)^n}{1}(n-1)=(-1)^n\left[n+1+n-1\right]=(-1)^n(2n)$$

ii. $$\frac{(1+x)^2}{(1-x)^2}$$

Sol. $$\frac{(1+x)^2}{(1-x)^2}=(1+2x+x^2)(1-x)^{-2}$$

$$=(1+2x+x^2)\left[1+(-2)(-x)+\frac{(-2)(-2-1)}{2!}(-x)^2+\frac{-2(-2-1)(-2-2)}{3!}(-x^3)+......\right]$$

$$=(1+2x+x^2)\left[1+(-1)^2 2x+(-1)^4\frac{(2)(3)}{2}x^2+\frac{(-1)^6(2)(3)(4)}{3.2.1}x^3+......\right]$$

$$=(1+2x+x^2)\left[1+(-1)^2.2x+(-1)^4 3x^2+(-1)^6 4x^3+........+(-1)^{2(n-2)}(n-1)\right.$$
$$\left.x^{n-2}+(-1)^{2(n-1)}nx^{n-1}+(-1)^{2n}(n+1)x^n\right]$$

$$1+(-1)^2 2x+..........+(-1)^{2n}(n+1)x^n+2x+(-1)^2 4x^2+.......+(-1)^{2(n-1)}2nx^n$$
$$+x^2+(-1)^2 2x^3+..........+(-1)^{2(2-1)}(n-1)x^n$$

Coefficient of x^n are $=(-1)^{2n}(n+1)+(-1)^{2(n-2)}2n+(-1)^{2(n-1)}(n-1)$

$$=(-1)^{2n}(n+1)+(-1)^{2n-2}2n+(-1)^{2n-4}(n-1)$$

$$=(-1)^{2n}(n+1)+(-1)^{2n}(-1)^{-2}2n+(-1)^{2n}(-1)^{-4}(n-1)$$

$$=(-1)^{2n}(n+1)+(-1)^{-2n}\frac{1}{(-1)^2}2n+(-1)^{2n}\frac{1}{(-1)^4}(n-1)$$

$$=(-1)^{2n}(n+1)+(-1)^{-2n}\frac{2n}{1}+(-1)^{2n}\frac{1}{1}(n-1)$$

$$=(-1)^{2n}\left[n+\cancel{1}+2n+n-\cancel{1}\right]=(-1)^{2n}4n=4n\ \ (Because\,(-1)^{2n}=1)$$

iii. $\dfrac{(1+x)^3}{(1-x)^2}$

Sol. $=(1+x)^3(1-x)^{-2}$

$$=(1+3x^2+3x+x^3)\left(1+(-2)(-x)+\frac{(-2)(-2-1)}{2!}(-x)^2+\frac{(-2)(-2-1)(-2-2)}{3!}(-x^3)+\ldots\right)$$

$$=(1+3x^2+3x+x^3)\left(1+2x+3x^2+4x^3+\ldots..+(n-2)x^{n-3}+(n-1)x^{n-2}+nx^{n-1}+(n+1)x^n\right)$$

$$=1+2x+\ldots...+(n+1)x^n+3x+6x^2+\ldots.+3nx^n+3x^2+6x^3+\ldots.+3(n-1)x^n$$

$$+x^3+2x^4+\ldots.+(n-2)x^n$$

$$co-efficient\ of\ x^n=(n+1)+3n+3(n-1)+(n-2)=n+1+3n+3n-3+n-2$$

$$=8n-4=4(2n-1)$$

iv. $\dfrac{(1+x)^2}{(1-x)^3}$

Sol. $=(1+x)^2(1-x)^{-3}$

$$=(1+2x+x^2)\left(1+(-3)(-x)+\frac{(-3)(-3-1)}{2!}(-x)^2+\frac{(-3)(-3-1)(-3-2)}{3!}(-x)^3+..\right)$$

$$=(1+2x+x^2)\left(1+3x+\frac{3.4}{2!}x^2+\frac{3.4.5}{3!}x^3+\ldots...\right)$$

$$=(1+2x+x^2)\left(1+\frac{2.3}{2}x+\frac{2.3.4}{2.2!}x^2+\frac{2.3.4.5}{2.3!}x^3+\ldots....\right)\qquad(\times\ \&\ \div\ by\ 2)$$

$$=(1+2x+x^2)\left(1+\frac{3!}{2.1!}x+\frac{4!}{2.2!}x^2+\frac{5!}{2.3!}x^3+\ldots....+\frac{n!.x^{n-2}}{2.(n-2)}+\frac{(n+1)!x^{n-1}}{2.(n-1)!}+\frac{(n+2)!x^n}{2.n!}\right)$$

$$=1+\frac{3!}{2!.1!}x+\ldots.....+\frac{(n+2)!.x^n}{2.n!}+2x+\frac{3!.x^2}{1!}+\ldots....+\frac{(n+1)!.x^n}{(n-1)!}$$

$$+x^2+\frac{3!.x^3}{2!.1!}+\ldots..+\frac{n!.x^n}{2(n-2)!}$$

$$Co.effecents\ of\ x^n=\frac{(n+2)!}{2.n!}+\frac{(n+1)!}{(n-1)!}+\frac{n!}{2(n-2)!}$$

$$=\frac{(n+2)(n+1).\cancel{n!}}{2.\cancel{n!}}+\frac{(n+1)(n)\cancel{(n-1)!}}{\cancel{(n-1)!}}+\frac{n(n-1)\cancel{(n-2)!}}{2\cancel{(n-2)!}}$$

$$=\frac{n^2+n+2n+2}{2}+n^2+n+\frac{n^2-n}{2}$$

$$=\frac{n^2+3n+2+2n^2+2n+n^2-n^2}{2}=\frac{4n^2+4n+2}{2}$$

$$=\frac{\cancel{2}(2n^2+2n+1)}{\cancel{2}}=2n^2+2n+1$$

v. $\mathbf{(1-x+x^2-x^3+..........)^2}$

Sol. $(1-x+x^2-x^3+..........)^2$

Suppose

$$(1+x)^{-1}=1+(-1)x+\frac{(-1)(-1-1)}{2!}x^2+\frac{(-1)(-1-1)(-1-2)}{3!}x^3+..........$$

$$(1+x)^{-1}=1-x+\frac{(-1)(-2)}{2}x^2+\frac{(-1)(-2)(-3)}{3.2.1}x^3+..........$$

$$(1+x)^{-1}=1-x+x^2-x^3+..........$$

Squaring both sides

$$(1+x)^{-2}=(1-x+x^2-x^3+..........)^{-2}$$

So $(1-x+x^2-x^3+..........)^{-2}=(1+x)^{-2}$

$$=1+(-2)x+\frac{(-2)(-2-1)}{2!}x^2+\frac{(-2)(-2-1)(-2-2)}{3!}x^3+..........$$

$$=1+(-2)x+\frac{(-2)(-3)}{2.1}x^2+\frac{(-2)(-3)(-4)}{3.2.1}x^3+..........$$

$$=1+(-2)x+(-1)^2 3x^2+(-1)^3 4x^3+..........+(-1)^n(n+1)x^n$$

Coefficient of $x^n=(-1)^n(n+1)$

4. If x is so small that its square and higher powers can be neglected, then show that:

i. $\mathbf{\dfrac{1-x}{\sqrt{1+x}}\approx 1-\dfrac{3}{2}x}$ **Multan 2008, Sargodha 2010**

Sol. L.H.S $=\dfrac{1-x}{\sqrt{1+x}}=(1-x)(1+x)^{-1/2}$

$$=(1-x)\left[1+\left(\frac{-1}{2}\right)x+\textit{neglecting } x^2 \ \& \ \textit{higher power}\right]$$

$$\approx (1-x)\left[1-\frac{x}{2}\right]=1-\frac{x}{2}-x+\frac{x^2}{2} \text{ (neglect)}$$

$$\approx 1-\frac{x}{2}-x=1-\frac{3x}{2}= \text{R.H.S}$$

Hence $\dfrac{1-x}{\sqrt{1+x}}\approx 1-\dfrac{3}{2}x$

ii. $$\frac{\sqrt{1+2x}}{\sqrt{1-x}}\approx 1+\frac{3}{2}x$$

Sol. $$\text{L.H.S}=\frac{(1+2x)^{1/2}}{(1-x)^{1/2}}=(1+2x)^{1/2}(1-x)^{-1/2}$$

$$=\left(1+\frac{1}{2}(2x)+\textit{neglecting } x^2 \textit{ \& higher power}\right)\left(1+\left(\frac{-1}{2}\right)(-x)+\textit{neglect}\right)$$

$$\approx (1+x)\left(1+\frac{x}{2}\right)=1+x+\frac{x}{2}+\frac{x^2}{2} \text{ (neglect)}$$

$$\approx 1+\frac{3x}{2}= \text{R.H.S}$$

Hence $\dfrac{\sqrt{1+2x}}{\sqrt{1-x}}\approx 1+\dfrac{3}{2}x$

iii. $$\frac{(9+7x)^{1/2}-(16+3x)^{1/4}}{4+5x}\approx \frac{1}{4}-\frac{17}{384}x$$ **Lahore 2009**

Sol. $$\text{L.H.S}\frac{(9+7x)^{1/2}-(16+3x)^{1/4}}{4+5x}=\left[(9+7x)^{1/2}-(16+3x)^{1/4}\right](4+5x)^{-1}$$

$$=\left[9^{1/2}\left(1+\frac{7}{9}x\right)^{1/2}-16^{1/4}\left(1+\frac{3}{16}x\right)^{1/4}\right]\times 4^{-1}\left(1+\frac{5x}{4}\right)^{1}$$

$$=\left[3^{2\times\frac{1}{2}}\left(1+\frac{1}{2}\left(\frac{7}{9}x\right)+\textit{neglect}\right)-2^{4\times\frac{1}{4}}\left(1+\frac{1}{4}\right)\left(\frac{3}{16}x\right)\right]\times\frac{1}{4}\left(1+(-1)\left(\frac{5}{4}x\right)+\textit{neglect}\right)$$

$$\approx\left[3\left(1+\frac{7x}{18}\right)-2\left(1+\frac{3x}{64}\right)\right]\frac{1}{4}\left(1-\frac{5x}{4}\right)$$

$$=\left[\left(3+\frac{7x}{6}\right)-\left(2+\frac{3x}{32}\right)\right]\left(\frac{1}{4}-\frac{5x}{16}\right)$$

$$=\left(3+\frac{7x}{6}-2-\frac{3x}{32}\right)\left(\frac{1}{4}-\frac{5x}{16}\right)$$

$$=\left(3-2+\frac{7x}{6}-\frac{3x}{32}\right)\left(\frac{1}{4}-\frac{5x}{16}\right)$$

$$=\left(1+\frac{224x-18x}{192}\right)\left(\frac{1}{4}-\frac{5x}{16}\right)$$

$$=\left(1+\frac{206x}{192}\right)\left(\frac{1}{4}-\frac{5x}{16}\right)=\left(1+\frac{103x}{96}\right)\left(\frac{1}{4}-\frac{5x}{16}\right)$$

$$=\frac{1}{4}-\frac{5x}{16}+\frac{103x}{384}-neglect\ x^2$$

$$\approx\frac{1}{4}-\left(\frac{5x}{16}-\frac{103x}{384}\right)$$

$$=\frac{1}{4}-\left(\frac{120x-103x}{384}\right)=\frac{1}{4}-\frac{17x}{384}$$

iv. $$\frac{\sqrt{4+x}}{(1-x)^3}\approx 2+\frac{25}{4}x$$ **Multan 2007**

Sol. L.H.S $$=\frac{\sqrt{4+x}}{(1-x)^3}=(4+x)^{1/2}(1-x)^{-3}=4^{1/2}\left(1+\frac{x}{4}\right)^{1/2}(1-x)^{-3}$$

$$=2^{2\times\frac{1}{2}}\left(1+\frac{x}{4}\right)^{1/2}(1-x)^{-3}$$

$$=2\left[1+\frac{1}{2}\left(\frac{x}{4}\right)+neglect\right]\left[1+(-3)(-x)+neglect\right]$$

$$\approx 2\left[1+\frac{x}{8}\right]\left[1+3x\right]=2\left[1+3x+\frac{x}{8}+\frac{3x^2}{8}\right]$$

$$=2\left[1+\frac{24x+x}{8}+neglect\right]\approx 2\left[1+\frac{25x}{8}\right]=2+\frac{25x}{4}=\text{R.H.S}$$

v. $\dfrac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} \approx 4\left(1-\dfrac{5x}{6}\right)$ Federal

Sol. $\text{L.H.S} = \dfrac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} = (1+x)^{1/2}.4^{3/2}\left(1-\dfrac{3x}{4}\right)^{3/2}.8^{-1/3}\left(1+\dfrac{5x}{8}\right)^{-1/3}$

$$= (1+x)^{1/2}.2^{2\times\frac{3}{2}}\left(1-\frac{3x}{4}\right)^{3/2}.2^{3\times\frac{-1}{3}}\left(1+\frac{5x}{8}\right)^{-1/3}$$

$$= \left[1+\frac{1}{2}(x)+neglect\right].2^3\left[1+\frac{3}{2}\left(-\frac{3x}{4}\right)+neglect\right].2^{-1}\left[1+\left(-\frac{1}{3}\right)\left(\frac{5x}{8}\right)+neglect\right]$$

$$\approx \left(1+\frac{x}{2}\right).8\left(1-\frac{9x}{8}\right).\frac{1}{2}\left(1-\frac{5x}{24}\right)$$

$$= 4\left(1+\frac{x}{2}\right)\left(1-\frac{9x}{8}\right)\left(1-\frac{5x}{24}\right)$$

$$= 4\left(1+\frac{x}{2}\right)\left(1-\frac{5x}{24}-\frac{9x}{8}+neglect\right)$$

$$\approx 4\left(1+\frac{x}{2}\right)\left(1-\frac{5x+27x}{24}\right) = 4\left(1+\frac{x}{2}\right)\left(1-\frac{32x}{24}\right)$$

$$= 4\left(1-\frac{32x}{24}+\frac{x}{2}-\frac{32x^2}{72}(neglect)\right)$$

$$\approx 4\left(1+\frac{x}{2}-\frac{32x}{24}\right) = 4\left(1+\frac{12x-32x}{24}\right)$$

$$= 4\left(1-\frac{20x}{24}\right) = 4\left(1-\frac{5x}{6}\right) = \text{R.H.S}$$

vi. $\dfrac{(1-x)^{1/2}(9-4x)^{1/2}}{(8+3x)^{1/3}} \approx \dfrac{3}{2}-\dfrac{61}{48}x$

Sol. $\text{L.H.S} = (1-x)^{1/2}.9^{1/2}\left(1-\dfrac{4x}{9}\right)^{1/2}.8^{-1/3}\left(1+\dfrac{3x}{8}\right)^{-1/3}$

$$= \left[1+\frac{1}{2}(-x)+neglect\right]\times 3\left[1+\frac{1}{2}\left(\frac{-4x}{9}\right)+neglect\right]\times\frac{1}{2^{3\times\frac{1}{3}}}\left[1+\left(\frac{-1}{3}\right)\left(\frac{3x}{8}\right)+neglect\right]$$

$$=\left(1-\frac{x}{2}\right)\times 3\left(1-\frac{2x}{9}\right)\times\frac{1}{2}\left(1-\frac{x}{8}\right)=\left(1-\frac{x}{2}\right)\left(3-\frac{2x}{3}\right)\left(\frac{1}{2}-\frac{x}{16}\right)$$

$$=\left(1-\frac{x}{2}\right)\left(\frac{3}{2}-\frac{3x}{16}-\frac{x}{3}+\frac{x^2}{24}\right)=\left(1-\frac{x}{2}\right)\left(\frac{3}{2}+\frac{-9x-16x}{48}+neglect\right)$$

$$=\left(1-\frac{x}{2}\right)\left(\frac{3}{2}-\frac{25x}{48}\right)=\frac{3}{2}-\frac{25x}{48}-\frac{3x}{4}+neglect$$

$$=\frac{3}{2}+\frac{-25x-36x}{48}\approx\frac{3}{2}-\frac{61x}{48}$$

vii. $$\frac{\sqrt{4-x}+(8-x)^{1/3}}{(8-3x)^{1/3}}\approx 2-\frac{1}{12}x$$ **Multan 2008**

Sol. $$\frac{\sqrt{4-x}+(8-x)^{1/3}}{(8-3x)^{1/3}}\approx 2-\frac{x}{12}$$

$$\text{L.H.S}=\frac{\sqrt{4-x}+(8-x)^{1/3}}{(8-3x)^{1/3}}=\frac{(4-x)^{1/2}+(8-x)^{1/3}}{(8-3x)^{1/3}}$$

$$=\frac{4^{1/2}\left(1-\frac{x}{4}\right)^{1/2}+8^{1/3}\left(1-\frac{x}{8}\right)^{1/3}}{(8-x)^{1/3}}=\frac{2^{2\times\frac{1}{2}}\left(1-\frac{x}{4}\right)^{1/2}+2^{3\times\frac{1}{3}}\left(1-\frac{x}{8}\right)^{1/3}}{(8-x)^{1/3}}$$

$$=2\left[1+\frac{1}{2}\left(-\frac{x}{4}\right)+neglect\right]+2\left[1+\frac{1}{3}\left(-\frac{x}{8}\right)+neglect\right](8-x)^{-1/3}$$

$$=\left[2\left(1-\frac{x}{8}\right)+2\left(1-\frac{x}{24}\right)\right]8^{-1/3}\left(1-\frac{x}{8}\right)^{-1/3}$$

$$=\left[\left(2-\frac{x}{4}\right)+\left(2-\frac{x}{12}\right)\right]2^{3\times\frac{-1}{3}}\left(1+\left(-\frac{1}{3}\right)\left(-\frac{x}{8}\right)+neglect\right)$$

$$=\left[2-\frac{x}{4}+2-\frac{x}{12}\right]2^{-1}\left(1+\frac{x}{24}\right)=\left(4+\frac{-3x-x}{12}\right)\frac{1}{2}\left(1+\frac{x}{24}\right)$$

$$=\left(4-\frac{4x}{12}\right)\cdot\left(\frac{1}{2}+\frac{x}{48}\right)=\left(4-\frac{x}{3}\right)\left(\frac{1}{2}+\frac{x}{48}\right)$$

$$=4\times\frac{1}{2}+\frac{4x}{48}-\frac{x}{6}-\frac{x^2}{144}(neglect)=2+\frac{x}{12}-\frac{x}{6}=2+\frac{x-2x}{2}=2-\frac{x}{12}=\text{R.H.S}$$

5. If x is so small that its cube and higher power can be neglected, then show that:

i. $$\sqrt{1-x-2x^2} \approx 1-\frac{1}{2}x-\frac{9}{8}x^2$$

Sol. $$\text{L.H.S} = \sqrt{1-x-2x^2} = \left[1-(x+2x^2)\right]^{1/2}$$

$$= 1+\frac{1}{2}(-(x+2x^2))+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(-(x+2x^2))^2 + neglect\ x^3 + higher\ power$$

$$\approx 1-\frac{1}{2}(x+2x^2)+\frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(x^2+4x^3+4x^4)$$

$$= 1-\frac{x}{2}-x^2-\frac{x^2}{8}+neglect$$

$$\approx 1-\frac{x}{2}+\frac{-8x^2-x^2}{8} = 1-\frac{x}{2}-\frac{9x^2}{8} = \text{R.H.S}$$

ii. $$\sqrt{\frac{1+x}{1-x}} = 1+x+\frac{1}{2}x^2$$

Sol. $$\sqrt{\frac{1+x}{1-x}} = (1+x)^{1/2}(1-x)^{-1/2}$$

$$= \left(1+\frac{1}{2}x+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}x^2 + neglect\right)\left(1+\left(-\frac{1}{2}\right)(-x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-x)^2 + neglect\right)$$

$$= \left(1+\frac{x}{2}+\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{-1}{2}x^2\right)\right)\times\left(1+\frac{x}{2}+\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)(-x^2)\right)$$

$$= \left(1+\frac{x}{2}-\frac{x^2}{8}\right)\left(1+\frac{x}{2}+\frac{3x^2}{8}\right) = 1+\frac{x}{2}+\frac{3x^2}{8}+\frac{x}{2}+\frac{x^2}{4}+\frac{3x^3}{16}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{3x^4}{64}$$

$$= 1+\frac{x}{2}+\frac{x}{2}+\frac{3x^2}{8}+\frac{x^2}{4}-\frac{x^2}{8}+ \text{ Ignore}$$

$$= 1+\frac{x+x}{2}+\frac{3x^2+2x^2-x^2}{8} = 1+\frac{2x}{2}+\frac{4x^2}{8} = 1+x+\frac{x^2}{2} = \text{R.H.S}$$

6. **If x is very nearly equal 1, then prove that $px^p - qx^q \approx (p-q)x^{p+q}$**

Sol. $px^p - qx^q \approx (p-q)x^{p+q}$ **Lahore 2009**

Take $x = 1+h$ where h is very small

L.H.S= $p(1+h)^p - q(1+h)^q$

$$= p[1+ph+ignore] - q[1+qh+ignore]$$

$$= p[1+ph] - q[1+qh]$$

$$= p + p^2h - q - q^2h$$

$$= (p-q) + (p^2-q^2)h$$

$$= (p-q) + (p-q)(p+q)h$$

$$= (p-q)[1+(p+q)h]$$

R.H.S $= (p-q)(1+h)^{p+q}$

$$= (p-q)[1+(p+q)h + Ignore]$$

$$= (p-q)[1+(p+q)h]$$

So L.H.S = R.H.S

7. **If $p-q$ is small when compared with p or q show that**

$$\frac{(2n+1)p+(2n-1)q}{(2n-1)p+(2n+1)q} \approx \left(\frac{p+q}{2q}\right)^{1/n}$$ **Gujranwala 2009**

Sol. $$\frac{(2n+1)p+(2n-1)q}{(2n-1)p+(2n+1)q} \approx \left(\frac{p+q}{2q}\right)^{1/n}$$

Take $p-q = h \Rightarrow p = q+h$ where h is very small

L.H.S= $\dfrac{(2n+1)(q+h)+(2n-1)q}{(2n-1)(q+h)+(2n+1)q}$

$$= \frac{2nq+2nh+\cancel{q}+h+2nq-\cancel{q}}{2nq+2nh-\cancel{q}-h+2nq+\cancel{q}}$$

$$= \frac{4nq+2nh+h}{4nq+2nh-h} = \frac{\cancel{4nq}\left(1+\dfrac{2nh+h}{4nq}\right)}{\cancel{4nq}\left(1+\dfrac{2nh-h}{4nq}\right)}$$

$$=\frac{1+\left(\frac{2n+1}{4nq}\right)h}{1+\left(\frac{2n-1}{4nq}\right)h}=\left[1+\left(\frac{2n+1}{4nq}\right)h\right]\left[1+\left(1+\frac{2n-1}{4nq}\right)h\right]^{-1}$$

$$=\left[1+\left(\frac{2n+1}{4nq}\right)h\right]=\left[1+(-1)\left(\frac{2n-1}{4nq}\right)h+\text{neglect } h^2 \text{ higher power}\right]$$

$$\approx\left[1+\left(\frac{2n+1}{4nq}\right)h\right].\left[1-\left(\frac{2n-1}{4nq}\right)h\right]$$

$$=1+\left(\frac{2n+1}{4nq}\right)h-\left(\frac{2n-1}{4nq}\right)h-\left(\frac{2n+1}{4nq}\right)\left(\frac{2n-1}{4nq}\right)h^2 \text{ (neglect)}$$

$$=1+\left(\frac{2n+1}{4nq}\right)h-\left(\frac{2n-1}{4nq}\right)$$

$$=1+\left(\frac{2n+1-2n+1}{4nq}\right)h$$

$$=1+\frac{2}{4nq}h=1+\frac{h}{2nq}$$

$$\text{Now R.H.S}=\left(\frac{p+q}{2q}\right)^{1/n}=\left(\frac{q+h+q}{2q}\right)^{1/n}$$

$$=\left(\frac{2q+h}{2q}\right)^{1/n}=\left(\frac{2q}{2q}+\frac{h}{2q}\right)^{1/n}=\left(1+\frac{h}{2q}\right)^{1/n}$$

$$=\left[1+\frac{1}{n}\left(\frac{h}{2q}\right)+neglect\ h^2\right]=1+\frac{h}{2nq} \quad \text{So L.H.S = R.H.S}$$

8. **Show that** $\left[\frac{n}{2(n+N)}\right]^{1/2}\approx\frac{8n}{9n-N}-\frac{n+N}{4n}$ **Where n and N are nearly equal.**

Sol. $\left[\frac{n}{2(n+N)}\right]^{1/2}\approx\frac{8n}{9n-N}-\frac{n+N}{4n}$ **Federal**

L.H.S= $\left[\dfrac{n}{2(n+N)}\right]^{1/2}$ Take $N=n+h$ where h is very small.

$$=\left[\frac{n}{2(n+n+h)}\right]^{1/2}=\left[\frac{n}{2(2n+h)}\right]^{1/2}=\left(\frac{n}{2}\right)^{1/2}(2n+h)^{-1/2}$$

$$=\left(\frac{n}{2}\right)^{1/2}\left(1+\frac{h}{2n}\right)^{-1/2}(2n)^{-1/2}$$

$$=\left(\frac{n}{2}\right)^{1/2}\left[1+\left(\frac{-1}{2}\right)\left(\frac{h}{2n}\right)+neglect\ h^2\ \&\ higher\ power\right]\times\frac{1}{(2n)^{1/2}}$$

$$\approx\frac{1}{(2n)^{1/2}}\left(\frac{n}{2}\right)^{1/2}\left[1-\frac{h}{4n}\right]=\frac{1}{2^{1/2}\cancel{n^{1/2}}}\frac{\cancel{n^{1/2}}}{2^{1/2}}\left(1-\frac{h}{4n}\right)$$

$$=\frac{1}{2}\left(1-\frac{h}{4n}\right)=\frac{1}{2}-\frac{h}{8n}$$

R.H.S $=\dfrac{8n}{9n-N}-\dfrac{n+N}{4n}$

$$=\frac{8n}{9n-(n+h)}-\frac{n+n+h}{4n}=\frac{8n}{9n-n-h}-\frac{2n+h}{4n}$$

$$=\frac{8n}{8n-h}-\frac{2n+h}{4n}=\frac{(8n)(4n)-(2n+h)(8n-h)}{(8n-h)(4n)}$$

$$=\frac{32n^2-16n^2+2nh-8nh+h^2}{4n(8n-h)}\ neglect\ h^2$$

$$=\frac{16n^2-6nh}{2(8n-h)}=\frac{2n(8n-3h)}{4n(8n-h)}=\frac{8n-3h}{2(8n-h)}$$

$$=\frac{8n-h-2h}{2(8n-h)}=\frac{1}{2}\left[\frac{8n-h}{8n-h}-\frac{2h}{8n-h}\right]$$

$$=\frac{1}{2}\left[1-\frac{2h}{8n-h}\right]=\frac{1}{2}-\frac{1}{2}\frac{2h}{(8n-h)}$$

$8n-h=8n$ because h is very small.

$=\dfrac{1}{2}-\dfrac{h}{8n}$ So L.H.S = R.H.S

9. Indentify the following series as binomial expansion and find the sum in each case.

i. $1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1.3}{2!.4}\times\left(\frac{1}{4}\right)^2-\frac{1.3.5}{3!.8}\left(\frac{1}{4}\right)^3+\ldots\ldots\ldots\ldots$

Sol. $1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1.3}{2!.4}\left(\frac{1}{4}\right)^2-\frac{1.3.5}{3!.8}\left(\frac{1}{4}\right)^3+\ldots\ldots\ldots\ldots$

Comparing with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots\ldots\ldots\ldots$$

$$nx=\left(-\frac{1}{2}\right)\left(\frac{1}{4}\right)=-\frac{1}{8}\text{———}I\Rightarrow n^2x^2=\frac{1}{64}\text{———}II\ and$$

$$\frac{n(n-1)x^2}{2!}x^2=\frac{1.3}{2!.4}\left(\frac{1}{4}\right)^2\text{———}III$$

$$\text{Dividing } III \text{ by } II\ \frac{n(n-1)\cancel{x^2}}{n^2\cancel{x^2}}=\frac{\frac{3}{64}}{\frac{1}{63}}$$

$$\Rightarrow\frac{n-1}{n}=\frac{3}{64}\times\frac{64}{1}\Rightarrow n-1=3n\Rightarrow -1=3n-n$$

$\Rightarrow -1=2n\Rightarrow \boxed{n=-\frac{1}{2}}$ put value of n in I.

$$\left(-\frac{1}{2}\right)x=-\frac{1}{8}\Rightarrow x=\left(-\frac{1}{8}\right)\left(-\frac{2}{1}\right)\Rightarrow\boxed{x=\frac{1}{4}}$$

So $(1+x)^n=\left(1+\frac{1}{4}\right)^{-1/2}=\left(\frac{4+1}{4}\right)^{-1/2}=\left(\frac{5}{4}\right)^{-1/2}=\left(\frac{4}{5}\right)^{1/2}=\frac{2}{\sqrt{5}}$

ii. $1-\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1.3}{2.4}\left(\frac{1}{2}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{2}\right)^3+\ldots\ldots\ldots\ldots$

Sol. Comparing with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots\ldots\ldots\ldots$$

$$nx=\left(\frac{-1}{2}\right)\left(\frac{1}{2}\right)=-\frac{1}{4}\text{———}I\Rightarrow n^2x^2=\frac{1}{16}\text{———}II$$

$$\frac{n(n-1)}{2!}x^2=\frac{1.3}{2.4}\left(\frac{1}{2}\right)^2=\frac{3}{16}\text{———}III$$

$$\text{Dividing } III \text{ by } II\ \frac{n(n-1)x^2}{n^2x^2}=\frac{\frac{3}{16}}{\frac{1}{16}}$$

$$\Rightarrow\frac{n-1}{n}=\frac{3}{16}\times\frac{16}{1}\Rightarrow n-1=3n\Rightarrow-1=3n-n=2n$$

$$\Rightarrow-1=2n\Rightarrow\boxed{n=-\frac{1}{2}}$$

Put $n=-\frac{1}{2}$ in I $\Rightarrow\left(-\frac{1}{2}\right)x=\frac{-1}{4}$

$$\Rightarrow x=\left(-\frac{1}{4}\right)\left(-\frac{2}{1}\right)=\frac{1}{2}\Rightarrow\boxed{x=\frac{1}{2}}$$

So $(1+x)^n=\left(1+\frac{1}{2}\right)^{-1/2}=\left(\frac{2+1}{2}\right)^{-1/2}=\left(\frac{3}{2}\right)^{-1/2}\left(\frac{2}{3}\right)^{1/2}=\sqrt{\frac{2}{3}}$

iii. $\mathbf{1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\ldots\ldots\ldots\ldots}$ **Faisalabad 2008**

Sol. Comparing with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots\ldots\ldots\ldots$$

$$nx=\frac{3}{4}\text{———}I\Rightarrow n^2x^2=\frac{9}{16}\text{———}II\ and\ \frac{n(n-1)x^2}{2!}=\frac{3.5}{4.8}$$

$$\Rightarrow n(n-1)x^2=\frac{15}{32}\times2=\frac{15}{16}\text{———}III$$

Dividing III by II

$$\Rightarrow\frac{n(n-1)\cancel{x^2}}{n^2\cancel{x^2}}=\frac{\frac{15}{16}}{\frac{9}{16}}=\frac{15}{16}\times\frac{16}{9}=\frac{5}{3}$$

$$\Rightarrow\frac{n-1}{n}=\frac{5}{3}\Rightarrow3(n-1)=5n\Rightarrow3n-3=5n\Rightarrow-3=5n-3n\Rightarrow-3=2n\Rightarrow\boxed{n=-\frac{3}{2}}$$

Put $n=\frac{-3}{2}$ *in I*

$$-\frac{3}{2}x=\frac{3}{4}\Rightarrow x=\left(\frac{3}{4}\right)\left(-\frac{2}{3}\right)\boxed{x=-\frac{1}{2}}$$

Now $(1+x)^n=\left(1-\frac{1}{2}\right)^{-3/2}=\left(\frac{2-1}{2}\right)^{-3/2}=\left(\frac{1}{2}\right)^{-3/2}=(2)^{3/2}=2^{1+\frac{1}{2}}$

$=2^1.2^{1/2}=2\sqrt{2}$

iv. $$1-\frac{1}{2}.\frac{1}{3}+\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{3}\right)^3+..........$$

Sol. Comparing with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+..........$$

$$nx=\frac{-1}{2}.\frac{1}{3}=-\frac{1}{6}\text{———}I\Rightarrow n^2x^2=\frac{1}{36}\text{———}II$$

and $\frac{n(n-1)}{2!}x^2=\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2$

$$\Rightarrow n(n-1)x^2=\frac{1}{12}\text{———}III$$

Dividing *III by II*

$$\Rightarrow\frac{n(n-1)\cancel{x^2}}{n^2\cancel{x^2}}=\frac{\frac{1}{12}}{\frac{1}{36}}=\frac{1}{12}\times\frac{36}{1}$$

$$\Rightarrow\frac{n-1}{n}=3\Rightarrow n-1=3n\Rightarrow -1=3n-n=2n\Rightarrow\boxed{n=-\frac{1}{2}}$$

Put in I $-\frac{1}{2}x=-\frac{1}{6}\Rightarrow x=\left(-\frac{1}{6}\right)\left(-\frac{2}{1}\right)\Rightarrow\boxed{x=\frac{1}{3}}$

So $(1+x)^n=\left(1+\frac{1}{3}\right)^{-1/2}=\left(\frac{3+1}{3}\right)^{-1/2}=\left(\frac{4}{3}\right)^{-1/2}$

$$(1+x)^n=\left(\frac{3}{4}\right)^{1/2}=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$$

10. **Use binomial theorem to show that** $1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+.........=\sqrt{2}$

Sol. $1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+.........=\sqrt{2}$ **Faisalabad 2009, Sargodha 2008**

Comparing with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+..........$$

$$nx=\frac{1}{4}\text{————}I\Rightarrow n^2x^2=\frac{1}{16}\text{————}II$$

$$\text{and } \frac{n(n-1)}{2!}x^2=\frac{13}{4.8}\Rightarrow n(n-1)x^2=\frac{1.3}{4.8}\times 2=\frac{3}{16}\text{————}III$$

$$\text{Dividing III by II}\quad \frac{n(n-1)\cancel{x^2}}{n^2\cancel{x^2}}=\frac{\frac{3}{16}}{\frac{1}{16}}$$

$$\frac{n-1}{n}=\frac{3}{16}\times\frac{16}{1}\Rightarrow n-1=3n\Rightarrow -1=3n-n=2n\Rightarrow \boxed{n=-\frac{1}{2}}$$

$$\text{(Put values of in I)}(-\frac{1}{2})x=\frac{1}{4}\Rightarrow x=\frac{1}{4}\left(-\frac{2}{1}\right)\Rightarrow \boxed{x=-\frac{1}{2}}$$

$$\text{so } (1+x)^n=\left(1-\frac{1}{2}\right)^{-1/2}=\left(\frac{2-1}{2}\right)^{-1/2}=\left(\frac{1}{2}\right)^{-1/2}=(2)^{1/2}=\sqrt{2}$$

11. **If** $y=\frac{1}{3}+\frac{1.3}{2.1}\left(\frac{1}{3}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3+...........$**, then prove that** $y^2+2y-2=0$

Sol. $y=\frac{1}{3}+\frac{1.3}{2.1}\left(\frac{1}{3}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3+...........,$ **Faisalabad 2007**

Adding both side 1.

$$1+y=1+3+\frac{1.3}{2!}\left(\frac{1}{3}\right)^2+............$$

Comparing with

$$(1+x)^n=1+nx+\frac{n(n-1)x^2}{2!}+.........\Rightarrow nx=\frac{1}{3}-I\Rightarrow n^2x^2=\frac{1}{9}-II$$

$$also\frac{n(n-1)}{2!}x^2=\frac{1.3.5}{2!}\left(\frac{1}{3}\right)^2=\frac{3}{9}=\frac{1}{3}-III$$

$$\text{Dividing III by II}\quad \frac{n(n-1)x^2}{n^2x^2}=\frac{\frac{1}{3}}{\frac{1}{9}}=\frac{1}{3}\times\frac{9}{1}$$

$$\frac{n-1}{n}=3\Rightarrow n-1=3n\Rightarrow -1=3n-n=2n$$

$$\Rightarrow\boxed{n=-\frac{1}{2}}\ (put\ n\ in\ 1)-\frac{1}{2}x=\frac{1}{3}\Rightarrow\boxed{x=-\frac{2}{3}}$$

$$1+y=(1+x)^n=\left(1-\frac{2}{3}\right)^{-1/2}=\left(\frac{3-2}{3}\right)^{1/2}=\left(\frac{1}{3}\right)^{-1/2}$$

$$1+y=(3)^{1/2}=\sqrt{3}$$

Square both sides

$$(1+y)^2=(\sqrt{3})^2\Rightarrow 1+2y+y^2=3\Rightarrow y^2+2y+1-3=0$$

$$y^2=2y-2=0$$

12. If $2y=\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\frac{1.3.5}{3!}.\frac{1}{2^6}+$......., then prove that $4y^2+4y-1=0$

Sol. $2y=\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\frac{1.3.5}{3!}.\frac{1}{2^6}+$........, **Federal**

Adding both side 1.

$$1+2y=1+\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\frac{1.3.5}{3!}.\frac{1}{2^6}+\ldots\ldots\ldots\ldots$$

Comparing with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots\ldots\ldots\ldots$$

$$nx=\frac{1}{2^2}=\frac{1}{4}\text{———}I\Rightarrow n^2x^2=\frac{1}{16}\text{———}II$$

$$and\ \frac{n(n-1)}{2!}x^2=\frac{1.3}{2!}.\frac{1}{2^4}\Rightarrow nn(n-1)x^2=\frac{3}{8}\text{———}III$$

Dividing III by II

$$\frac{n(n-1)\cancel{x^2}}{n^2\cancel{x^2}}=\frac{\frac{3}{16}}{\frac{1}{16}}\Rightarrow\frac{n-1}{n}=\frac{3}{16}\times\frac{16}{1}$$

$$\frac{n-1}{n}=3\Rightarrow n-1=3n\Rightarrow -1=3n-n\ -1=2n\Rightarrow\boxed{n=-\frac{1}{2}}$$

$$(\text{Put } n \text{ in I})-\frac{1}{2}x=\frac{1}{4}\Rightarrow x=\frac{1}{4}\left(-\frac{2}{1}\right)=-\frac{1}{2}\Rightarrow\boxed{x=-\frac{1}{2}}$$

$$1+2y=(1+x)^n=\left(1-\frac{1}{2}\right)^{-1/2}=\left(\frac{2-1}{2}\right)^{-1/2}=\left(\frac{1}{2}\right)^{-1/2}$$

$$1+2y=(2)^{1/2}=\sqrt{2}$$

Squaring both sides

$$1+4y+4y^2=2\Rightarrow 4y^2+4y+1-2=0$$

$$4y^2+4y-1=0$$

13. $$y=\frac{2}{5}+\frac{1.3}{2!}\left(\frac{2}{5}\right)^2+\frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3+\text{.......,}$$

Sol. Adding both side 1 we get

$$1+y=1+\frac{2}{5}+\frac{1.3}{2!}\left(\frac{2}{5}\right)^2+\frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3+\text{.......,}$$

Comparing with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\text{..........}$$

$$nx=\frac{2}{5}\text{———}I\Rightarrow n^2x^2=\frac{4}{25}\text{———}II$$

$$also\ \frac{n(n-1)}{\cancel{2!}}x^2=\frac{1.3}{\cancel{2!}}\left(\frac{2}{5}\right)^2\Rightarrow n(n-1)x^2=3\left(\frac{4}{25}\right)$$

$$n(n-1)x^2=\frac{12}{25}\text{———}III$$

$$\textit{Dividing III by II}\quad \frac{n(n-1)\cancel{x^2}}{n^2\cancel{x^2}}=\frac{\frac{12}{25}}{\frac{4}{25}}=\frac{12}{25}\times\frac{25}{4}$$

$$\frac{(n-1)}{n}=3\Rightarrow n-1=3n\Rightarrow -1=3n-n-2n$$

$$\Rightarrow\boxed{n=-\frac{1}{2}}\ (\textit{Put in I})-\frac{1}{2}x=\frac{2}{5}$$

$$\Rightarrow x=\frac{2}{5}\left(\frac{-2}{5}\right)=-\frac{4}{5}\Rightarrow\boxed{x=-\frac{4}{5}}$$

$$\textit{So } 1+y=(1+x)^n=\left(1-\frac{4}{5}\right)^{-1/2}=\left(\frac{5-4}{5}\right)^{-1/2}$$

$$1+y=\left(\frac{1}{5}\right)^{-1/2}\Rightarrow 1+y=5^{1/2}=\sqrt{5}$$

Squaring both sides

$$1+2y+y^2=5\Rightarrow y^2+2y+1-5=0$$

$$\Rightarrow y^2+2y-4=0$$

TEST YOUR SKILLS — Marks: 50

Q # 1. Select the Correct Option **(10)**

i. General term in the expansion of $(a+x)^n$ is:

a) $\binom{n+1}{r}a^{n-r}x^r$ b) $\binom{n}{r-1}a^{n-r}x^r$

c) $\binom{n}{r+1}a^{n-r}x^r$ d) $\binom{n}{r}a^{n-r}x^r$

ii. $1-x+x^2-x^3+.......+(-1)^r x^r+....=$

a) $(1-x)^1$ b) $(1-x)^{-1}$

c) $(1+x)^{-1}$ d) $(1+x)^{1/2}$

iii. The expansion of $(1+2x)^{-2}$ is valid if

a) $|x|<1/2$ b) $|x|<1$

c) $|x|<2$ d) None

iv. The number of terms in the expansion of $\left(x^2-\frac{1}{x^2}\right)^7$ is:

a) 2 b) 7

c) 8 d) 14

v. The middle term in expansion of $(a+b)^6$ is:

a) T_3 b) T_4

c) T_5 d) T_6

vi. The method of induction was given by Francesco who lived from:

a) $1494-1575$ b) $1500-1575$

c) $1498-1575$ d) $1494-1570$

vii. $n^2>n+3$ is true for:

a) $n\geq 3$ b) $n\geq 0$

c) $n\geq 2$ d) $n\geq 1$

viii. $2^n>2(n+1)$ is true for all:

a) $n\geq 1$ b) $n\geq 2$

c) $n=2$ d) $n>4$

ix. The sum of exponent a & b in every term in the expansion of $(a+b)^n$ is:

a) Zero b) 1

c) $n+1$ d) n

x. $3+6+9+.........+3n=$ (when n is +ve)

a) $3n(n+1)$ b) $\frac{3n(n+1)}{2}$

c) $\frac{3n(n+1)}{3}$ d) $\frac{3n(n+1)}{4}$

Q # 2. Short Questions: **(10 X 2 = 20)**

i. Show that $(x-y)$ is a factor of $x^3-y^3;\ n=1,2$

ii. Find 6^{th} term in the expansion of $\left(x^2-\frac{3}{2x}\right)^{10}$

iii. State Binomial Theorem for Positive integer n.

iv. If x is so small that its square and higher power can be neglected then show that

$$\frac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1+\frac{3}{2}x$$

v. Find the value of $\sqrt[4]{17}$ to three places of decimal by using binomial theorems:

vi. Expand $(1-2x)^{1/3}$ up to 3 terms.

vii. Show that n^3-n is divided by 6 for $n=2,3$

viii. Prove that $2+4+6........+2n=n(n+1)$ for $n=1,2$

ix. Evaluate $(9.98)^{1/2}$ by Binomial Theorem:

x. For what value of x, the expansion $(4-3x)^{1/2}$ is valid:

Long Questions: **(2 X 10 = 20)**

Q # 3. (a) Find the term independent of x in the expansion of $\left(x-\frac{2}{x}\right)^{10}$

(b) Find the Co-efficient o x^5 in the expansion of $\left(x^2-\frac{3}{2x}\right)^{10}$

Q # 4. (a) Use the Mathematical induction to show that

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+........+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

(b) Use Binomial to show that $1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+.......=\sqrt{2}$

Fundamentals of Trigonometry

9

Trigonometry: The word trigonometry has been derived form three Greek words trei (three) Goni (angles) and metron (measurement). It means measurement of triangle.

Angle: Two rays with common starting point form an angle.

Degree: If the circumference of circle is divided into 360 equal parts in length, the angle subtented by one part at the centre of the circle is called a degree.

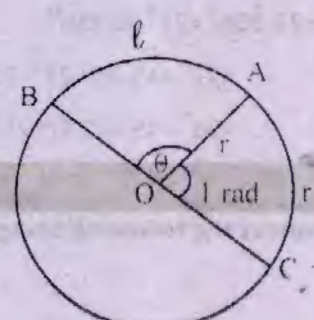
Radian: **Faisalabad 2008**

A radian is the measure of the central angle of an arc of a circle whose length is equal to the radius of the circle.

Sexagesimal system: **Sargodha 2011**

As this system of measurement of angle owes its origin to the English and because 90,60 are multiples of 6 and 10 so it is known as English or sexagesimal system.

Relation: π radian = 180° $\Rightarrow$ 1 radian = $\dfrac{180^\circ}{\pi}$



Theorem 1: Prove that $\ell = r\,\theta$

Proof: Here ℓ = arc length

θ = Central angle

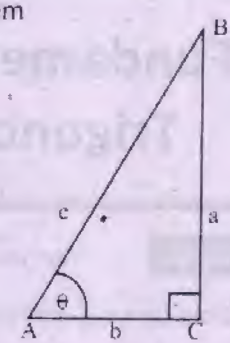
r = radius

We know by elementary geometry that measure of central angles of arcs of a circle are proportional to the length of their arcs.

$$\Rightarrow \frac{m\angle AOB}{m\angle AOC} = \frac{m\widehat{AB}}{m\widehat{AC}} \Rightarrow \frac{\theta rad}{1 rad} = \frac{\ell}{r} \Rightarrow \ell = r\theta \text{ or } \theta = \frac{\ell}{r}$$

Theorem 2: **Prove the $\sin^2\theta + \cos^2\theta = 1$** **Faisalabad 2008, Lahore 2009**

Proof: If ABC is right angle triangle then by Pythagoras theorem



$a^2 + b^2 = c^2$ (Divide both sides by c^2.)

$$\frac{a^2}{c^2}+\frac{b^2}{c^2}=\frac{c^2}{c^2}$$

or $$\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2=1 \qquad \boxed{\frac{a}{c}=Sin\theta\ ,\ \frac{b}{c}=Cos\theta}$$

$$\Rightarrow \sin^2\theta + \cos^2\theta = 1$$

Theorem 3: **Prove that $1 + \tan^2\theta = \sec^2\theta$**

Proof: We know that $\sin^2\theta + \cos^2\theta = 1$ divide both sides by $\cos^2\theta$.

$$\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$

or $\tan^2\theta + 1 = \sec^2\theta \Rightarrow 1 + \tan^2\theta = \sec^2\theta$

Example 2: **Convert 21.256° to the $D^\circ m'S''$ form**

Sol: $0.256^\circ = 0.256(1^\circ) = 0.256\,(60')$

$= 15.36'$

and $0.36' = 0.36(1') = 0.36(60'') = 21.6''$

Therefore $21.256^\circ = 21^\circ + 0.256^\circ$

$= 21^\circ + 15.36' = 21^\circ + 15' + 0.36'$

$= 21^\circ + 15' + 21.6'' = 21^\circ 15' 22''$

Exercise 9.1

1. **Express the following sexagesimal measures of angles in radians.**

i. 30°

Sol. $= 30\times 1^\circ = 30\times\frac{\pi}{180}$ rad$= \frac{\pi}{6}$ rad

ii. 45°

Sol. $45^\circ = 45\times 1^\circ = 45\times\frac{\pi}{180}$ rad$= \frac{\pi}{4}$ rad

iii. 60°

Sol. $60^\circ = 60\times 1^\circ = 60\times\frac{\pi}{180}$ rad$=\frac{\pi}{3}$ rad

iv. 75°

Sol. $75^\circ = 75\times 1^\circ = 75\times\frac{\pi}{180}$ rad $= \frac{5\pi}{12}$ rad

v. 90°

Sol. $90^\circ = 90 \times 1^\circ = 90 \times \frac{\pi}{180}$ rad $= \frac{\pi}{2}$ rad

vii. 135°

Sol. $135^\circ = 135 \times 1^\circ = 135 \times \frac{\pi}{180}$ rad

$= \frac{3\pi}{4}$ rad

ix. $10^\circ 15'$

Sol. $10^\circ 15' = \left(10 + \frac{15}{60}\right)^{\circ}$

$= \left(10 + \frac{1}{4}\right)^{\circ} = \left(\frac{41}{4}\right)^{\circ}$

$= \frac{41}{4} \times \frac{\pi}{180} rad = \frac{41\pi}{720} rad$

xi. $120'20''$

Sol. $120'40'' = \left(\frac{120}{60} + \frac{40}{60 \times 60}\right)^{\circ}$

$= \left(2 + \frac{1}{90}\right)^{\circ} = \left(\frac{181}{90}\right)^{\circ}$

$= \frac{181}{90} \times \frac{\pi}{180} rad = \frac{181}{18200} rad$

xiii. 0°

Sol. $0^\circ = 0 \times \frac{\pi}{180} rad = 0\ rad$

vi. 105°

Sol. $105^\circ = 105 \times 1^\circ = 105 \times \frac{\pi}{180}$ rad $= \frac{7\pi}{12}$ rad

viii. 150°

Sol. $150^\circ = 150 \times 1^\circ = 150 \times \frac{\pi}{180}$ rad $= \frac{5\pi}{6}$ rad

x. $35^\circ 20'$

Sol. $35^\circ 20' = \left(35 + \frac{20}{60}\right)^{\circ}$

$= \left(35 + \frac{1}{3}\right)^{\circ} = \left(\frac{105+1}{3}\right)^{\circ}$

$= \left(\frac{106}{3}\right)^{\circ} = \frac{106}{3} \times 1^{\circ}$

$= \frac{106}{3} \times \frac{\pi}{180} rad = \frac{53\pi}{270} rad$

xii. $154^\circ 20''$

Sol. $154^\circ 20'' = \left(154 + \frac{20}{360}\right)^{\circ}$

$= \left(154 + \frac{1}{180}\right)^{\circ} = \left(\frac{27721}{180}\right)^{\circ}$

$= \frac{27721}{180} \times 1^{\circ}$

$= \frac{27721}{180} \times \frac{\pi}{180} rad = \frac{27721\pi}{32400} rad$

xiv. $3''$

Sol. $3'' = \left(\frac{3}{60 \times 60}\right)^{\circ} = \left(\frac{1}{1200}\right)^{\circ}$

$= \frac{1}{1200} \times \frac{\pi}{180} rad$

$= \frac{\pi}{21600} rad$

2. **Convert the following radian measures of angles into the measures of sexagesimal system**

i. $\frac{\pi}{8}$

Sol. $\frac{\pi}{8} = \frac{\pi}{8} \times \frac{180}{\pi} \, degree$

$= \frac{180}{8} \, degree = 22.5 \, degree$

$= 22°30'$

iii. $\frac{\pi}{4}$

Sol. $\frac{\pi}{4} = \frac{\pi}{4} \times \frac{180}{\pi}$ = 45 degree = 45°

v. $\frac{\pi}{2}$

Sol. $\frac{\pi}{2} = \frac{\pi}{2} \times \frac{180}{\pi} \, degree$

= 90 degree = 90°

vii. $\frac{3\pi}{4}$

Sol. $\frac{3\pi}{4} = \frac{3\pi}{4} \times \frac{180}{\pi} \, degree$

= 135 degree = 135°

ix. $\frac{7\pi}{12}$

Sol. $\frac{7\pi}{12} = \frac{7\pi}{12} \times \frac{180}{\pi} \, degree$

= 105 degree = 105°

ii. $\frac{\pi}{6}$

Sol. $\frac{\pi}{6} = \frac{\pi}{6} \times \frac{180}{\pi} \, degree$

= 30 degree = 30°

iv. $\frac{\pi}{3}$

Sol. $\frac{\pi}{3} = \frac{\pi}{3} \times \frac{180}{\pi} \, degree$

= 60 degree = 60°

vi. $\frac{2\pi}{3}$

Sol. $\frac{2\pi}{3} = \frac{2\pi}{3} \times \frac{180}{\pi} \, degree$

= 120 degree = 120°

viii. $\frac{5\pi}{6}$

Sol. $\frac{5\pi}{6} = \frac{5\pi}{6} \times \frac{180}{\pi} \, degree$

= 150 degree = 150°

x. $\frac{9\pi}{5}$

Sol. $\frac{9\pi}{5} = \frac{9\pi}{5} \times \frac{180}{\pi} \, degree$

= 324 degree = 324°

xi. $\dfrac{11\pi}{27}$

Sol. $\dfrac{11\pi}{27} = \dfrac{11\pi}{27} \times \dfrac{180}{\pi}\, degree$

$= 73.333$ degree

$= 73°20'$

xiii. $\dfrac{17\pi}{24}$ **Faisalabad 2007**

Sol. $\dfrac{17\pi}{24} = \dfrac{17\pi}{24} \times \dfrac{180}{\pi}\, degree$

$= 127.5$ degree $= 127°\ 30'$

xii. $\dfrac{13\pi}{16}$

Sol. $\dfrac{13\pi}{16} = \dfrac{13\pi}{16} \times \dfrac{180}{\pi}\, degree$

$= 146.25$ degree $= 146°15'$

xiv. $\dfrac{25\pi}{36}$

Sol. $\dfrac{25\pi}{36} = \dfrac{25\pi}{36} \times \dfrac{180}{\pi}\, degree$

$= 125$ degree $= 125°$

xv. $\dfrac{19\pi}{32}$

Sol. $\dfrac{19\pi}{32} = \dfrac{19\pi}{32} \times \dfrac{180}{\pi}\, degree$

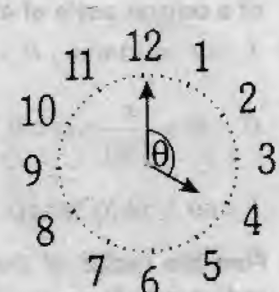
$= 106.875$ degree $= 106°\ 52'\ 30''$

3. What is the circular measure of the angle between the hands of a watch at 4 O'clock?

Sol. $Angle\ in\ 12\ hours = 2\pi$

$Angle\ in\ one\ hour = \dfrac{2\pi}{12} = \dfrac{\pi}{6}\, rad$

$Angle\ at\ 4, O\ clock = 4 \times \dfrac{\pi}{6}\, rad = \dfrac{2\pi}{3}\, rad$



4. Find θ, when: Sargodha 2006, Multan 2008

i. ℓ = 1.5 cm, r = 2.5 cm

Sol. $\theta = ?$, $\ell = 1.5$ cm, r = 2.5 cm.

$\ell = r\theta \Rightarrow \theta = \dfrac{\ell}{r} = \dfrac{1.5}{2.5}$

$\Rightarrow \theta = 0.6$ rad

ii. ℓ = 3.2 cm, r = 2cm,

Sol. $\ell = 3.2$m, r = 2m , $\theta = ?$,

$\theta = \dfrac{\ell}{r} = \dfrac{3.2}{2} = 1.6$ rad

5. Find ℓ, when: **Multan 2008**

i. $\theta = \pi$ **radians, r = 6 cm**

Sol. ℓ = ?, $\theta = \pi$ rad, r = 6cm

$\ell = r\theta = 6\pi = 6\,(3.1416)$

ℓ = 18.84 cm

6. Find r, when: **Faisalabad 09**

i. ℓ **= 5cm,** $\theta = \frac{1}{2}$ **radian**

Sol. r = ?, ℓ = 5, $\theta = \frac{1}{2}$ rad.

$$r = \frac{\ell}{\theta} = \frac{5}{1/2} \Rightarrow r = 5 \times \frac{2}{1}$$

r = 10cm

ii. $\theta = 65°20'$**, r = 18mm** **Sargodha2010**

Sol. $\theta = 65°20'$, r = 18, ℓ = ?

$$\theta = \left(65 + \frac{20}{60}\right)^{o} = \left(65 + \frac{1}{3}\right)^{o}$$

$$= \left(\frac{95+1}{3}\right)^{o} = \left(\frac{196}{3}\right)^{o}$$

$$= \frac{196}{3} \times \frac{\pi}{180}\, rad = \frac{49\pi}{135}\, rad$$

Now $\ell = r\theta = 18\left(\frac{49\pi}{135}\right) = 20.5$ mm

ii. ℓ **= 56cm,** θ **= 45°** **Rawalpindi 2009**

Sol. ℓ = 56 cm, $\theta = 45°$, r = ?

$$\theta = 45 \times \frac{\pi}{180}\, rad = 0.7854\, rad$$

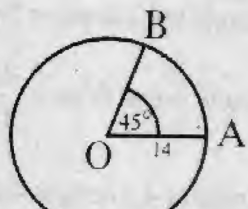
$$r = \frac{\ell}{\theta} = \frac{56}{0.7854} = 71.30\, cm$$

7. **What is the length of the arc intercepted on a circle of radius 14 cms by the arms of a central angle of 45°?**

Sol. ℓ = ?, r = 14cm, θ = 45°

$$\theta = 45 \times \frac{\pi}{180}\, rad = 0.7854\, rad$$

$\ell = r\theta$ = 14 (0.7854) = 10.99cm



8. **Find the radius of the circle, in which the arms of a central angle of measure 1 radian cut off an are of length 35 cm.**

Sol. r = ?, $\theta = 1\, rad$, ℓ = 35

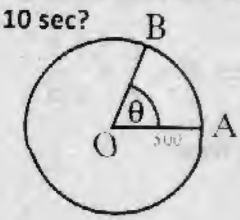
$$r = \frac{\ell}{\theta} = \frac{35}{1} = 35 \text{ cm}$$

9. **A railway train is running on a circle track of radius 500 meters at the rate of 30 km per hour. Through what angle will it turn in 10 sec?**

Sol. V = (30km/h), r = 500, θ = ?

$S = \ell = vt$

ℓ = (30km/h) × t

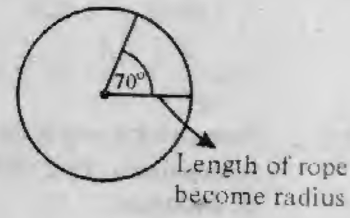


$$\ell = \frac{30 \times 1000}{60 \times 60}(10)\,m/\sec = \frac{250}{3}\,m/s$$

$$\theta = \frac{\ell}{r} = \frac{250/3}{500} = \frac{250}{3} \times \frac{1}{500} \Rightarrow \theta = \frac{1}{6}\,rad$$

10. A horse is tethered to a peg by a rope of 9 meters length and it can move in a circle with the peg as centre. If the horse moves along the circumference of the circle, keeping the rope tight, how far will it have gone when the rope has turned through an angle of 70°?

Sol. $r = 9$, $\theta = 70^\circ$, $\ell = ?$



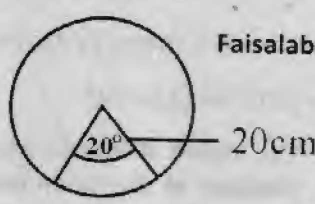
$$\theta = 70 \times \frac{\pi}{180}\,rad = 1.22$$

$$\ell = r\theta = 8\,(1.22) = 10.99\text{ m}$$

11. The pendulum of a clock is 20 cm long and it swings through an angle of 20° each second. How far does the tip of the pendulum move in 1 second?

Sol. $\theta = 20^\circ = 20 \times \frac{\pi}{180}\,rad = \frac{\pi}{9}\,rad$, $r = 20$, $\ell = ?$

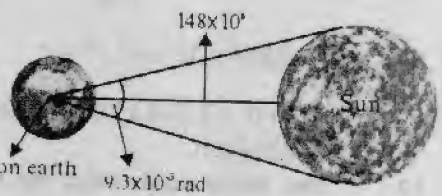
Faisalabad 2008



$$\ell = r\theta = 20\left(\frac{\pi}{9}\right) = 6.98\text{ cm}$$

12. Assuming the average distance of the earth from the sun to be 148×10^6 km and the angle subtended by the sun at the eye of a person on the earth of measure 9.3×10^{-3} radian. Find the diameter of the sun.

Sol. $r = 148 \times 106$, $\theta = 9.3 \times 10^{-3}$ rad



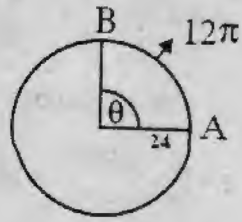
$$\ell = r\theta$$

$$= 148 \times 10^6 \times 9.3 \times 10^{-3} = 1376.4 \times 10^3$$

$$= 1376400\text{ km}$$

13. A circular wire of radius 6 cm is cut straightened and then bent so as to lie along the circumference of a hoop of radius 24 cm. Find the measure of the angle which it subtends at the centre of the hoop.

Sol. $r = 24$ (of Hoop), $\theta = ?$ $r = 6$ (of circle)



$$\ell = 2\pi r = 2\pi(6) = 12\pi$$

$$\theta = \frac{\ell}{r} = \frac{12\pi}{24} = \frac{\pi}{2}\,rad$$

14. **Show that the area of a sector of a circular region of radius r is $1/2r^2\theta$, where θ is the circular measure of the central angle of the sector.**

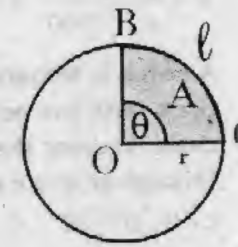
Sol. Let A = Area of sector

θ = Centrel angle

r = radius

we know by elementary geometry that

$Area\ of\ sector : Area\ of\ circle = \theta : 2\pi$

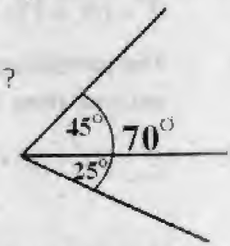


$$\Rightarrow \frac{Area\ of\ sector}{Area\ of\ circle} = \frac{\theta}{2\pi}$$

$$\Rightarrow \frac{A}{\pi r^2} = \frac{\theta}{2\pi} \Rightarrow A = \frac{\theta}{2\pi} \times \pi r^2 \Rightarrow A = \frac{1}{2} r^2 \theta$$

15. **Two cities A and B lie on the equator such that their longitudes are 45°E and 25°W respectively. Find the distance between the two cities, taking radius of the earth as 6400 kms.**

Sol. $\theta = 45^\circ + 25^\circ = 70^\circ = 70 \times \frac{\pi}{180} rad = 1.2217$, r = 6400, ℓ = ?

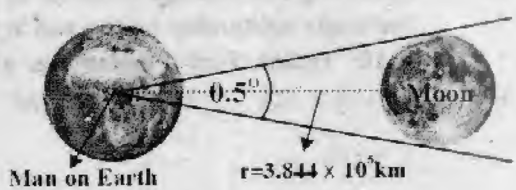


$\ell = r\theta$ = (6400) (1.2217) = 7818.8 km

= 7819 km (approx.)

16. **The moon subtends an angle of 0.5° at the eye of an observer on earth. The distance of the moon from the earth is 3.844 x 10^5 km approx. what is the length of the diameter of the moon?**

Sol. $\theta = 0.5^\circ = 0.5 \times \frac{\pi}{180} = 0.008726$ rad



r = 3.844 x 10^5 km, ℓ = ?

$\ell = r\theta$ = 3.844 x 10^5 x 0.008726

= 3354 km

17. **The angle subtended by the earth at the eye of a spacemen, landed on the moon, is 1° 54′. The radius of the earth is 6400 km. Find the approximat distance between the moon and the earth.**

Sol. $\theta = 1^\circ 54' = \left(1 + \frac{54}{60}\right)^\circ = \left(\frac{114}{60}\right)^\circ = \frac{114}{60} \times \frac{\pi}{180} rad = 0.033 rad$

ℓ = 2r = 2 (6400) = 12800, r = ?

$r = \frac{\ell}{\theta} = \frac{12800}{0.033}$ = 385992.6 km = 3895993 km

Exercise 9.2

1. Find the signs of the following. Sargodha 2011

(i) $\sin 160^\circ$

Sol. $\sin 160^\circ$ + ve

(ii) $\cos 190^\circ$

Sol. $\cos 190^\circ$ – ve

(iii) $\tan 115^\circ$

Sol. $\tan 115^\circ$ – ve

(iv) $\sec 245^\circ$

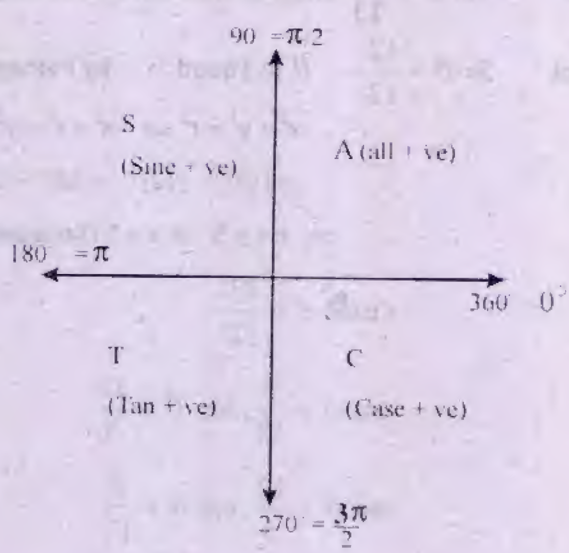
Sol. $\sec 245^\circ$ – ve

(v) $\cot 80^\circ$

Sol. $\cot 80^\circ$ + ve

(vi) $\operatorname{cosec} 297^\circ$

Sol. $\operatorname{cosec} 297^\circ$ – ve



For remember Read (CAST) start from IV quad

2. Fill in the blanks.

i. $\sin(-310^\circ) = \ldots\ \sin 310^\circ$

Sol. $\sin(-310^\circ) = -\sin 310^\circ$

ii. $\cos(-75^\circ) = \ldots\ \cos 75^\circ$

Sol. $\cos(-75^\circ) = +\cos 75^\circ$

iii. $\tan(-180^\circ) = \ldots\ \tan 182^\circ$

Sol. $\tan(-182^\circ) = -\tan 182^\circ$

iv. $\cot(-173^\circ) = \ldots\ \cot 137^\circ$

Sol. $\cot(-137^\circ) = -\cot 137^\circ$

v. $\sec(-216^\circ) = \ldots\ \sec 216^\circ$

Sol. $\sec(-216^\circ) = +\sec 216^\circ$

vi. $\operatorname{cosec}(-15^\circ) = \ldots\ \operatorname{cosec} 15^\circ$

Sol. $\operatorname{cosec}(-15^\circ) = -\operatorname{cosec} 15^\circ$

3. In which quadrant are the terminal arms of the angle lie when

i. $\sin\theta < 0$ and $\cos\theta > 0$

Sol. lies in quadrant IV

ii. $\cos\theta > 0$ and $\operatorname{cosec}\theta > 0$

Sol. lies in quadrant I

iii. $\tan\theta < 0$ and $\cos\theta > 0$

Sol. lies in quadrant IV

iv. $\sec\theta < 0$ and $\sin\theta < 0$ Sargodha 2008

Sol. lies in quadrant III

v. $\cot\theta > 0$ and $\sin\theta < 0$

Sol. Lies in quadrant III

vi. $\cos\theta < 0$ and $\tan\theta < 0$ Fsd 2008, Sgd 2009

Sol. lies in quadrant II

4. Find the values of the remaining trigonometric functions.

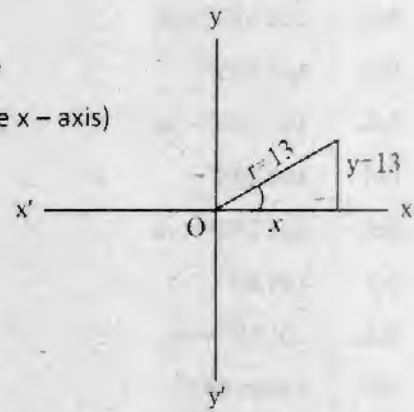
(i) $\sin\theta = \frac{12}{13}$ **and the terminal arm of the angle is in quad. I** **Sargodha 2010**

Sol. $\sin\theta = \frac{12}{13}$ θ in I quad. By Pythagoras

$x^2 + y^2 = r^2 \Rightarrow x^2 = r^2 - y^2$

$= (13)^2 - (12)^2 = 169 - 144 = 25$

$\Rightarrow x = \pm 5 \Rightarrow x = 5$ (Because on + ve x – axis)



$\text{Cosec}\,\theta = \frac{13}{12}$

$\text{Cos}\,\theta = \frac{5}{13}$, $\text{Sec}\,\theta = \frac{13}{5}$

$\tan\theta = \frac{12}{5}$, $\cot\theta = \frac{5}{12}$

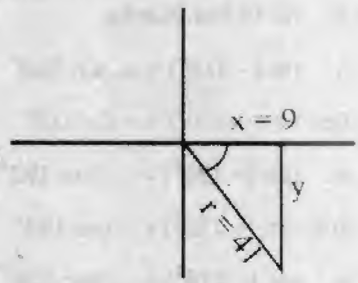
(ii) $\cos\theta = \frac{9}{41}$ **and the terminal arm of the angle is in quad. IV.**

Sol. $\text{Cos}\,\theta = \frac{9}{41}$ θ in IV quad by Pythagoras

$x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2$

$y^2 = (41)^2 - (9)^2 = 1681 - 81 = 1600 \Rightarrow y = \pm 40$

$y = -40$ (Because on – ve y – axis)

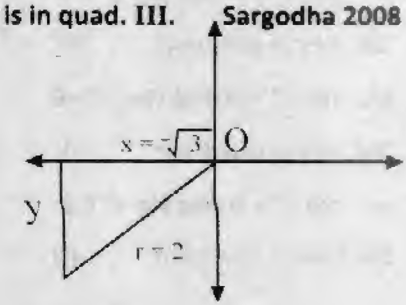


$\text{Sec}\,\theta = \frac{41}{9}$, $\text{Sin}\,\theta = \frac{-40}{41}$, $\text{Cosec}\,\theta = \frac{41}{-40}$

$\text{Tan}\,\theta = \frac{-40}{9}$, $\text{Cot}\,\theta = \frac{-9}{40}$

(iii) $\cos\theta = -\frac{\sqrt{3}}{2}$ **and the terminal arm of the angle is in quad. III.** **Sargodha 2008**

Sol. $\text{Cos}\,\theta = -\frac{\sqrt{3}}{2}$ (θ in III quad) by Pythagoras,



$x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2$

$y^2 = (2)^2 - (-\sqrt{3})^2 = 4 - 3 = 1$

$y = \pm 1 \Rightarrow y = -1$ (Because on negative y – axis)

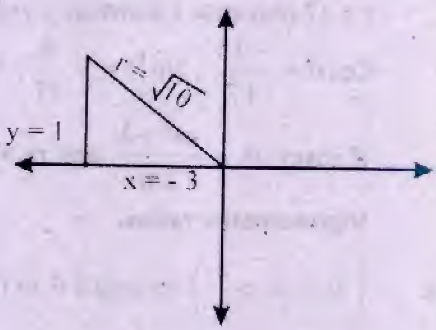
$\text{Sec}\,\theta = -\dfrac{2}{\sqrt{3}}$ $\quad$ $\text{Tan}\,\theta = \dfrac{-1}{-\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\text{Sin}\,\theta = \dfrac{-1}{2}$ $\quad$ $\text{Cot}\,\theta = \sqrt{3}$

$\text{Cosec}\,\theta = -2$

(iv) **$\tan\theta = -\dfrac{1}{3}$ and the terminal arm of the angle is in quad. II.** **Multan 2008**

Sol. $\tan\theta = -\dfrac{1}{3}$ (θ in II quad) By Pythagoras



$r^2 = x^2 + y^2$

$= (-3)^2 + (1)^2 \Rightarrow r^2 = 9 + 1 = 10$

$r = \pm\sqrt{10} = \sqrt{10}$ (always + ve)

$\text{Cot}\,\theta = -\dfrac{3}{1} = -3$

$\text{Sin}\,\theta = \dfrac{1}{\sqrt{10}}$, $\text{Conec}\,\theta = \sqrt{10}$

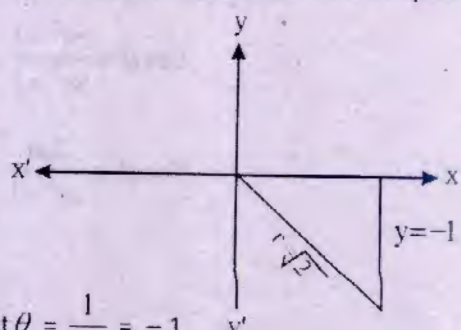
$\text{Cos}\,\theta = \dfrac{-3}{\sqrt{10}}$, $\text{Sec}\,\theta = \dfrac{-\sqrt{10}}{3}$

(v) **$\text{Sin}\,\theta = -\dfrac{1}{\sqrt{2}}$ and the terminal arm of the angle is not in quad. III.**

Sol. $\text{Sin}\,\theta = -\dfrac{1}{\sqrt{2}}$ (θ *not in III quad*)

$\text{Sin}\,\theta$ = – ve given and $\text{Sin}\,\theta$ is – ve in III and IV but given not in III. Its means $\text{Sin}\,\theta$ is in IV quad

By Pythagoras



$x^2 + y^2 = r^2 \Rightarrow x^2 = r^2 - y^2$

$x^2 = (\sqrt{2})^2 - (-1)^2 = 2 - (1) = 2 - 1 = 1$

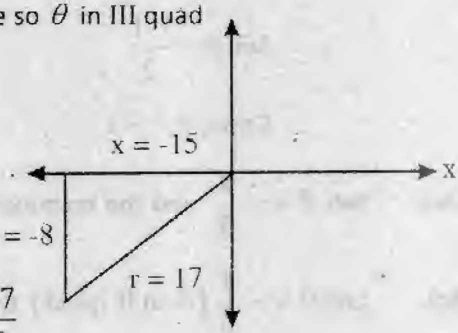
$x = \pm 1 \Rightarrow x = 1$ (Because on + ve x – axis)

$\text{Cos}\,\theta = \dfrac{1}{\sqrt{2}}$, $\text{Sec}\,\theta = \sqrt{2}$

$\tan\theta = \dfrac{-1}{1} = -1$, $\text{Cosec}\,\theta = \dfrac{-\sqrt{2}}{1} = -\sqrt{2}$, $\cot\theta = \dfrac{1}{-1} = -1$

5. **If cot $\theta = \frac{15}{8}$ and the terminal arm of the angle is not is quad. I , find the values of cos θ and cosec θ .** **Multan 2007**

Sol. $\text{Cot}\,\theta = \frac{15}{8}$ $(\theta \text{ not in I})$ Because cot θ is + ve so θ in III quad



$\cot\theta = \frac{15}{8} \Rightarrow \tan\theta = \frac{8}{15}$

$r^2 = x^2 + y^2 = (-15)^2 + (-8)^2$

$= 225 + 64 = 289$

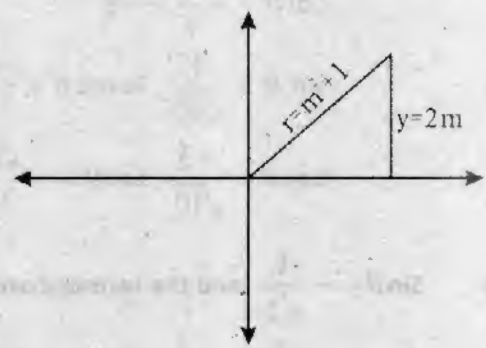
$r = \pm 17$

r = 17 (Because r is always + ve)

$\text{Cos}\,\theta = \frac{-15}{17}$, $\text{Sin}\,\theta = -\frac{8}{17}$, $\text{Cosec}\,\theta = -\frac{17}{8}$

6. **If cosec $\theta = \frac{m^2+1}{2m}$ and m > 0 $\left(0 < \theta < \frac{\pi}{2}\right)$, find the values of the remaining trigonometric ratios.** **Sargodha 2008, 2010, 2011**

Sol. $\left(0 < \theta < \frac{\pi}{2}\right)$ Its mean θ in I quad.



$\text{Cosec}\,\theta = \frac{m^2+1}{2m}$

$\Rightarrow \text{Sin}\,\theta = \frac{2m}{m^2+1}$

$X^2 + y^2 = r^2$

$X^2 = r^2 - y^2$

$X^2 = (m^2+1)^2 - (2m)^2 = m^4 + 2m^2 + 1 - 4m^2$

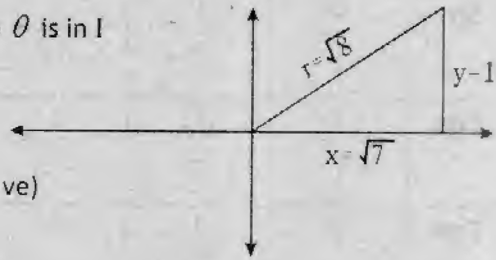
$X^2 = m^4 + 1 - 2m^2 \Rightarrow x^2 = (m^2-1)^2$

$X = \pm(m^2-1) \Rightarrow x = (m^2-1)$ (Because on + ve – axis)

$\text{Cos}\,\theta = \frac{m^2-1}{m^2+1}$, $\quad \text{Sec}\,\theta = \frac{m^2+1}{m^2-1}$

$\text{Tan}\,\theta = \frac{2m}{m^2-1}$ $\quad \text{Cot}\,\theta = \frac{m^2-1}{2m}$

7. **If tan $\theta = 1/\sqrt{7}$ and the terminal arm of the angle is not the III quad, find the values of $\dfrac{\cos^2\theta - \sec^2\theta}{\cos^2\theta + \sec^2\theta}$**



Sol. $\tan\theta = \dfrac{1}{\sqrt{7}}$ $(\theta \text{ not in III})$ tan is + ve so θ is in I

$r^2 = x^2 + y^2 = (\sqrt{7})^2 + (1)^2 = 7 + 1 = 8$

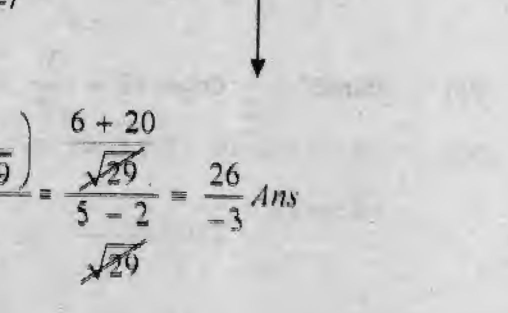
$r = \pm\sqrt{8} \Rightarrow r = \sqrt{8}$ (always + ve)

$\text{Sin}\,\theta = \dfrac{1}{\sqrt{8}} \Rightarrow \text{cosec}\,\theta = \sqrt{8}$

$\text{Cos}\,\theta = \dfrac{\sqrt{7}}{\sqrt{8}} \Rightarrow \text{Sec}\,\theta = \dfrac{\sqrt{8}}{\sqrt{7}}$

Now $\dfrac{\text{cosec}^2\theta - \text{Sec}^2\theta}{\text{cosec}^2\theta + \text{Sec}^2\theta} = \dfrac{(\sqrt{8})^2 - \left(\dfrac{\sqrt{8}}{\sqrt{7}}\right)^2}{(\sqrt{8})^2 + \left(\dfrac{\sqrt{8}}{\sqrt{7}}\right)^2} = \dfrac{8 - \dfrac{8}{7}}{8 + \dfrac{8}{7}} = \dfrac{\dfrac{56-8}{\cancel{7}}}{\dfrac{56+8}{\cancel{7}}} = \dfrac{48}{64} = \dfrac{3}{4}$

8. **If cot $\theta = 5/2$ and the terminal arm of the angle is in the I quad, find the values of $\dfrac{3\sin\theta + 4\cos\theta}{\cos\theta - \sin\theta}$** **Multan 2009, Lahore 2009, Faisalabad 2009**



Sol. $\text{Cot}\,\theta = 5/2\,(\theta \textit{ in quadrant I})$

$\Rightarrow \tan\theta = \dfrac{2}{5}$, $r^2 = x^2 + y^2$

$r^2 = (5)^2 + (4)^2 = 25 + 4 = 29$

$r = \pm\sqrt{29} \Rightarrow r = \sqrt{29}$ (always + ve)

$\text{Sin} = \dfrac{2}{\sqrt{29}}, \text{Cos}\,\theta = \dfrac{5}{\sqrt{29}}$

Now $\dfrac{3\,Sin\theta + 4\,Cos\theta}{\cos\theta - Sin\theta} = \dfrac{3\left(\dfrac{2}{\sqrt{29}}\right) + 4\left(\dfrac{5}{\sqrt{29}}\right)}{\dfrac{5}{\sqrt{29}} - \dfrac{2}{\sqrt{29}}} = \dfrac{\dfrac{6+20}{\cancel{\sqrt{29}}}}{\dfrac{5-2}{\cancel{\sqrt{29}}}} = \dfrac{26}{-3}\,Ans$

Exercise 9.3

θ	0°	30°	45°	60°	90°	180°	270°
Sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
Tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞

1. Verify the following:

(i) $\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ = \sin 30^\circ$ **Sargodha 2009**

Sol. L.H.S = $\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ$

$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{4} - \frac{1}{4} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$$

R.H.S = $\sin 30^\circ = \frac{1}{2}$

Hence L.H.S = R.H.S

(ii) $\sin^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{3} + \tan^2 \frac{\pi}{4} = 2$

Sol. L.H.S = $\sin^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{3} + \tan^2 \frac{\pi}{4}$

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + (1)^2 = \frac{1}{4} + \frac{3}{4} + 1 = \frac{1+3+4}{4} = \frac{8}{4} = 2 = \text{R.H.S}$$

(iii) $2\sin 45^\circ + \frac{1}{2} \operatorname{Cosec} 45^\circ = \frac{3}{\sqrt{2}}$ **Faisalabad 2008**

Sol. L.H.S = $2 \sin 45^\circ + 1/2 \operatorname{Cosec} 45^\circ$

$$= 2 \sin 45^\circ + \frac{1}{2 \sin 45^\circ} = 2.\frac{1}{\sqrt{2}} + \frac{1}{2.\frac{1}{\sqrt{2}}} = \frac{2}{\sqrt{2}} + \frac{1}{\frac{2}{\sqrt{2}}}$$

$$= \frac{2}{\sqrt{2}} + \frac{\sqrt{2}}{2} = \frac{4+2}{2\sqrt{2}} = \frac{6}{2\sqrt{2}} = \frac{3}{\sqrt{2}} = R.H.S$$

(iv) $\sin^2\frac{\pi}{6} : \sin^2\frac{\pi}{4} : \sin^2\frac{\pi}{3} : \sin^2\frac{\pi}{2} = 1:2:3:4$ **Faisalabad 2009**

Sol. L.H.S = $\sin^2\frac{\pi}{6} : \sin^2\frac{\pi}{4} : \sin^2\frac{\pi}{3} : \sin^2\frac{\pi}{2}$

$$= \left(\frac{1}{2}\right)^2 : \left(\frac{1}{\sqrt{2}}\right)^2 : \left(\frac{\sqrt{3}}{2}\right)^2 : (1)^2$$

$$= \frac{1}{4} : \frac{1}{2} : \frac{3}{4} : 1$$

Multiplying by 4

$$= \cancel{4} \times \frac{1}{\cancel{4}} : 4 \times \frac{1}{2} : \cancel{4} \times \frac{3}{\cancel{4}} : 4 \times 1 = 1:2:3:4$$

2. Evaluate the following

i. $\dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{6}}$

Sol. $\dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{6}} = \dfrac{\sqrt{3} - \frac{1}{\sqrt{3}}}{1 + \sqrt{3}.\frac{1}{\sqrt{3}}}$

$$= \frac{\frac{3-1}{\sqrt{3}}}{1+1} = \frac{2}{\sqrt{3}} \times \frac{1}{2} = \frac{1}{\sqrt{3}}$$

ii. $\dfrac{1 - \tan^2\frac{\pi}{3}}{1 + \tan^2\frac{\pi}{3}}$

Sol. $\dfrac{1 - \tan^2\pi/3}{1 + \tan^2\pi/3} = \dfrac{1 - (\sqrt{3})^2}{1 + (\sqrt{3})^2} = \dfrac{1-3}{1+3}$

$$= \frac{-2}{4} = -\frac{1}{2}$$

3. Verify the following when $\theta = 30°, 45°$

i. $\sin 2\theta = 2\sin\theta\cos\theta$

Sol. When $\theta = 30°$

L.H.S = $\sin 2\theta = \sin 2(30°)$

$= \sin 60° = \frac{\sqrt{3}}{2}$

R.H.S = $2\sin\theta\cos\theta$

ii. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

Sol. When $\theta = 30°$

L.H.S = $\cos 2\theta = \cos 2(30°) = \cos 60° = \frac{1}{2}$

R.H.S = $\cos^2\theta - \sin^2\theta$

$= \cos^2 30° - \sin^2 30°$

$= 2\sin 30^\circ \cos 30^\circ$

$= 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$

L.H.S = R.H.S

When $\theta = 45^\circ$

L.H.S = $\sin 2\theta = \sin 2(45^\circ)$

$= \sin 90^\circ$

$= 1$

R.H.S = $2\sin\theta\cos\theta$

$= 2\sin 45^\circ \cos 45^\circ$

$= 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 2\left(\frac{1}{2}\right) = 1$

L.H.S = R.H.S

$= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{1}{4}$

$= \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$

L.H.S = R.H.S

When $\theta = 45^\circ$

L.H.S = $\cos 2\theta = \cos 2(45^\circ) = \cos 90^\circ = 0$

R.H.S = $\cos^2\theta - \sin^2\theta = \cos^2 45^\circ - \sin^2 45^\circ$

$= \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} - \frac{1}{2} = 0$

L.H.S = R.H.S

iii $\mathbf{\cos 2\theta = 2\cos^2\theta - 1}$

Sol. when $\theta = 30^\circ$

L.H.S = $\cos 2\theta = \cos 2(30^\circ) = \cos 60^\circ = 1/2$

R.H.S = $2\cos^2\theta - 1 = 2\cos^2 30^\circ - 1$

$$= 2\left(\frac{\sqrt{3}}{2}\right)^2 - 1 = 2\left(\frac{3}{4}\right) - 1 = \frac{3}{2} - 1 = \frac{3-2}{2} = \frac{1}{2}$$

L.H.S = R.H.S

When $\theta = 45^\circ$

L.H.S = $\cos 2\theta = \cos 2(45^\circ) = \cos 90^\circ = 0$

R.H.S = $2\cos^2\theta - 1 = 2\cos^2(45^\circ) - 1 = 2\left(\frac{1}{\sqrt{2}}\right)^2 - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$

L.H.S = R.H.S

iv. $\mathbf{\cos 2\theta = 1 - 2\sin^2\theta}$

Sol. when $\theta = 30^\circ$

L.H.S = $\cos 2\theta = \cos 2(30^\circ) = \cos 60^\circ = 1/2$

R.H.S = $1 - 2\sin^2\theta = 1 - 2\sin^2 30^\circ = 1 - 2\left(\frac{1}{2}\right)^2 = 1 - 2\left(\frac{1}{4}\right) = 1 - \frac{1}{2} = \frac{1}{2}$

L.H.S = R.H.S

When $\theta = 45^\circ$

L.H.S = $\cos 2\theta = \cos^2(45^\circ) = \cos 90^\circ = 0$

R.H.S = $1 - 2\sin^2\theta = 1 - 2\sin^2 45^\circ = 1 - 2\left(\frac{1}{\sqrt{2}}\right)^2 = 1 - 2\left(\frac{1}{2}\right) = 1 - 1 = 0$

L.H.S = R.H.S

v. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Sol. When $\theta = 30^\circ$

L.H.S = $\tan 2\theta = \tan 2(30^\circ) = \tan 60^\circ = \sqrt{3}$

$$\text{R.H.S} = \frac{2\tan 30^\circ}{1-\tan^2 30^\circ} = \frac{2\left(\frac{1}{\sqrt{3}}\right)}{1-\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{\frac{2}{\sqrt{3}}}{1-\frac{1}{3}} = \frac{\frac{2}{\sqrt{3}}}{\frac{2}{3}} = \frac{\cancel{2}}{\cancel{\sqrt{3}}} \times \frac{\cancel{\sqrt{3}} \times \sqrt{3}}{\cancel{2}} = \sqrt{3}$$

L.H.S = R.H.S

When $\theta = 45^\circ$

L.H.S = $\tan 2\theta = \tan 2(45^\circ) = \tan 90^\circ = \infty$

$$\text{R.H.S} = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\tan 45^\circ}{1-\tan^2 45^\circ} = \frac{2(1)}{1-(1)^2} = \frac{2}{1-1} = \frac{2}{0} = \infty$$

4. **Find x, if $\tan^2 45^\circ - \cos^2 60^\circ = x \sin 45^\circ \cos 45^\circ \tan 60^\circ$** **Sargodha 2008, 2009, 2010**

Sol. $\tan^2 45^\circ - \cos^2 60^\circ = x \sin 45^\circ \cos 45^\circ \tan 60^\circ$ **Multan 2009, Faisalabad 08**

$(1)^2 - \left(\frac{1}{2}\right)^2 = x \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \sqrt{3}$

$$1 - \frac{1}{4} = x.\frac{\sqrt{3}}{2} \Rightarrow \frac{4-1}{4} = x.\frac{\sqrt{3}}{2} \Rightarrow x = \frac{3}{4} \times \frac{2}{\sqrt{3}} = \frac{\sqrt{3}\,\cancel{\sqrt{3}}}{\cancel{4}_2} \times \frac{\cancel{2}}{\cancel{\sqrt{3}}} = \frac{\sqrt{3}}{2}$$

5. Find the values of the trigonometric functions of the following quadrantal angles:

i. $-\pi$

Sol. $\because -\pi = \pi + (-1)\,2\pi = \pi$, $k = -1$

Values of Trigonometric functions at $-\pi$ and π are same

$\text{Sin}(-\pi) = \text{Sin}\,\pi = 0$

$\text{Cos}(-\pi) = \text{Cos}\,\pi = -1$

$\text{Tan}(-\pi) = \tan\pi = 0$

$$\text{Cot}(-\pi) = \cot\pi = \frac{1}{\tan\pi} = \frac{1}{0} = \infty$$

$$\text{Sec}(-\pi) = \text{See}\,\pi = \frac{1}{\cos\pi} = \frac{1}{-1} = -1$$

$$\text{Cosec}(-\pi) = \text{Cosec}\,\pi = \frac{1}{Sin\pi} = \frac{1}{0} = \infty$$

ii. -3π

Sol. $-3\pi = -4\pi + \pi = (-2)\,2\pi + \pi = \pi$

Values of Trigonometric functions at -3π and π are same

$\text{Sin}(-3\pi) = \text{Sin}\,\pi = 0$

$\text{Cos}(-3\pi) = \cos\pi = -1$

$\tan(-3\pi) = \tan\pi = 0$

$$\text{Cot}(-3\pi) = \text{Cot}\,\pi = 1/\tan\pi = \frac{1}{0} = \infty$$

$$\text{Sec}(-3\pi) = \text{Sec}\,\pi = 1/\cos\pi = \frac{1}{-1} = -1$$

$$\text{Cosec}(-3\pi) = \text{Cosec}\,\pi = 1/\text{Sin}\,\pi = \frac{1}{0} = \infty$$

iii. $\frac{5}{2}\pi$

Sol. $\frac{5\pi}{2} = 2\pi + \frac{\pi}{2} = \pi/2$

Values of Trigonometric functions at $\frac{5\pi}{2}$ and $\pi/2$ are same

$\text{Sin}\,\frac{5\pi}{2} = \text{Sin}\,\frac{\pi}{2} = 1$, $\text{Cos}\,\frac{5\pi}{2} = \text{Cos}\,\frac{\pi}{2} = 0$

$\tan\frac{5\pi}{2} = \tan\frac{\pi}{2} = \infty$

$$\text{Cot}\,\frac{5\pi}{2} = \text{Cot}\,\frac{\pi}{2} = \frac{1}{\tan\pi/2} = \frac{1}{\infty} = 0$$

$$\text{Cosec}\,\frac{5\pi}{2} = \text{cosec}\,\frac{\pi}{2} = \frac{1}{\sin\pi/2} = \frac{1}{1} = 1$$

$$\text{Sec}\ \frac{5\pi}{2} = Sec\ \pi/2 = \frac{1}{\cos e\ \pi/2} = \frac{1}{0} = \infty$$

iv. $-\frac{9}{2}\pi$

Sol. $-9\pi/2 = -6\pi + \frac{3\pi}{2}$

$$= (-3)\ 2\pi + \frac{3\pi}{2} = \frac{3\pi}{2}$$

Values of Trigonometric functions at $-9\pi/2$ and $3\pi/2$ are same

$\text{Sin}\ (-9\pi/2) = \text{Sin}\ 3\pi/2 = -1$

$\text{Cos}\ (-9\pi/2) = \text{Cos}\ 3\pi/2 = 0$

$\tan\ (-9\pi/2) = \tan\ 3\pi/2 = \infty$

$$\text{Cot}\ (-9\pi/2) = \cot\ 3\pi/2 = \frac{1}{\tan 3\pi/2} = \frac{1}{\infty} = 0$$

$$\text{Sec}\ (-9\pi/2) = \text{Sec}\ 3\pi/2 = \frac{1}{\cos 3\pi/2} = \frac{1}{0} = \infty$$

$$\text{Cosec}\ (-9\pi/2) = \text{Cosec}\ 3\pi/2 = \frac{1}{\sin 3\pi/2} = \frac{1}{-1} = -1$$

v. -15π

Sol. $-15\pi = -16\pi + \pi = (-8)\ 2\pi + \pi = \pi$, $k = -8$

Values of Trigonometric functions at -15π and π are same

$\text{Sin}\ (-15\pi) = \text{Sin}\ \pi = 0$
$\text{Cos}\ (-15\pi) = \text{Cos}\ \pi = -1$
$\text{Tan}\ (-15\pi) = \tan\pi = 0$

$$\text{Cot}\ (-15\pi) = \cot\ \pi = \frac{1}{\tan\pi} = \frac{1}{0} = \infty$$

$$\text{Sec}\ (-15\pi) = \text{Sec}\ \pi = 1/\cos\pi = \frac{1}{-1} = -1$$

$$\text{Cosec}\ (-15\pi) = \text{cosec}\ \pi = 1/\text{Sin}\ \pi = \frac{1}{0} = \infty$$

vi. **1530°**

Sol. $1530^\circ = (4 \times 360^\circ) + 90^\circ) = 90^\circ$, $k = 4$

Values of Trigonometric functions at 1530° and 90° are same

$\text{Sin}\,(1530^\circ) = \text{Sin}\,(90^\circ) = 1$

$\text{Cos}\,(1530^\circ) = \cos\,(90^\circ) = 0$

$\text{Tan}\,(1530^\circ) = \tan 90^\circ = \infty$

$\text{Cot}\,(1530^\circ) = \cot 90^\circ = \dfrac{1}{\tan 90^o} = \dfrac{1}{\infty} = 0$

$\text{Sec}\,(1530^\circ) = \sec 90^\circ = \dfrac{1}{\cos 90^o} = \dfrac{1}{0} = \infty$

$\text{Cos}\,(1530^\circ) = \text{cosec}\, 90^\circ = \dfrac{1}{\sin 90^o} = \dfrac{1}{1} = 1$

vii. **-2430°**

Sol. $-2430^\circ = -7 \times 360^\circ + 90^\circ = 90^\circ$, $K = -7$
Values of Trigonometric functions at -2430° and 90° are same
$\text{Sin}\,(-2430^\circ) = \text{Sin}\, 90^\circ = 1$
$\text{Cos}\,(-2430^\circ) = \text{Sin}\, 90^\circ = 0$
$\text{Tan}\,(-2430^\circ) = \tan 90^\circ = \infty$

$\text{Cot}\,(-2430^\circ) = \cot 90^\circ = \dfrac{1}{\tan 90^o} = \dfrac{1}{\infty} = 0$

$\text{Sec}\,(-2430^\circ) = \text{Sec}\, 90^\circ = \dfrac{1}{\cos 90^o} = \dfrac{1}{0} = \infty$

$\text{Cosec}\,(-2430^\circ) = \text{Cosec}\, 90^\circ = \dfrac{1}{Sin 90^o} = \dfrac{1}{1} = 1$

viii. $\dfrac{235}{2}\pi$ **Faisalabad 2008**

Sol. $\dfrac{235}{2}\pi = 116\pi + \dfrac{3\pi}{2} = 58 \times 2\pi + \dfrac{3\pi}{2} \doteq \dfrac{3\pi}{2}$, $k = 58$

Values of Trigonometric functions at $\dfrac{235\pi}{2}$ and $\dfrac{3\pi}{2}$ are same

$\text{Sin}\left(\frac{235\pi}{2}\right) = \text{Sin}\left(\frac{3\pi}{2}\right) = -1$

$\text{Cos}\left(\frac{235\pi}{2}\right) = \cos\frac{3\pi}{2} = 0$

$\text{Tan}\left(\frac{235\pi}{2}\right) = \tan\frac{3\pi}{2} = \infty$

$\text{Cot}\left(\frac{235\pi}{2}\right) = \cot\frac{3\pi}{2} = \frac{1}{\tan 3\pi/2} = \frac{1}{\infty} = 0$

$\text{Sec}\left(\frac{235\pi}{2}\right) = \text{Sec}\frac{3\pi}{2} = \frac{1}{\cos 3\pi/2} = \frac{1}{0} = \infty$

$\text{Cosec}\left(\frac{235\pi}{2}\right) = \text{Cosec}\frac{3\pi}{2} = \frac{1}{Sin 3\pi/2} = \frac{1}{-1} = -1$

ix. $\frac{\mathbf{407}}{\mathbf{2}}\pi$

Sol. $\frac{407\pi}{2} = 202\pi + \frac{3\pi}{2} = 101 \times 2\pi + \frac{3\pi}{2} = \frac{3\pi}{2}$, K=101

Values of Trigonometric functions at $\frac{407\pi}{2}$ and $\frac{3\pi}{2}$ are same

$\text{Sin}\left(\frac{407\pi}{2}\right) = \text{Sin}\frac{3\pi}{2} = -1$

$\text{Cos}\left(\frac{407\pi}{2}\right) = \cos\frac{3\pi}{2} = 0$

$\tan\left(\frac{407\pi}{2}\right) = \tan\frac{3\pi}{2} = \infty$

$\text{Cot}\left(\frac{407\pi}{2}\right) = \cot\left(\frac{3\pi}{2}\right) = \frac{1}{\tan 3\pi/2} = \frac{1}{\infty} = 0$

$\text{Sec}\frac{407\pi}{2} = \text{Sec}\frac{3\pi}{2} = \frac{1}{\cos 3\pi/2} = \frac{1}{0} = \infty$

$\text{Cosec}\frac{407\pi}{2} = \text{cosec}\frac{3\pi}{2} = \frac{1}{Sin 3\pi/2} = \frac{1}{-1} = -1$

6. Find the values of the trigonometric functions of the following angles:

i. 390°

Sol. $390^\circ = (1) \times 360^\circ + 30^\circ = 30^\circ$, $K=1$

Values of Trigonometric functions at 390° and 30° are same

$\text{Sin}(390^\circ) = \text{Sin } 30^\circ = \dfrac{1}{2}$

$\text{Cos}(390^\circ) = \cos 30^\circ = \sqrt{3}/2$

$\text{Tan}(390^\circ) = \tan 30^\circ = 1/\sqrt{3}$

$\text{Cot}(390^\circ) = \cot 30^\circ = \dfrac{1}{\tan 30^\circ} = \dfrac{1}{1/\sqrt{3}} = \sqrt{3}$

$\text{Sec}(390^\circ) = \text{Sec } 30^\circ = \dfrac{1}{Cos\,30^\circ}\ \dfrac{1}{\sqrt{3}/2} = \dfrac{2}{\sqrt{3}}$

$\text{Cosec}(390^\circ) = \text{cosec } 30^\circ = \dfrac{1}{Sin\,30^\circ}\ \dfrac{1}{1/2} = 2$

ii. -330°

Sol. $-330^\circ = -360^\circ + 30 = (-1) \times 360^\circ + 30^\circ = 30^\circ$, $k = -1$

Value of Trigonometric functions at -330° and 30° are same

$\text{Sin}(-330^\circ) = \text{Sin } 30^\circ = \dfrac{1}{2}$

$\text{Cos}(-330^\circ) = \text{Cos } 30^\circ = \sqrt{3}/2$

$\text{Tan}(-330^\circ) = \tan 30^\circ = 1/\sqrt{3}$

$\text{Cot}(-330^\circ) = \cot 30^\circ = \dfrac{1}{\tan 30^\circ} = \dfrac{1}{1/\sqrt{3}} = \sqrt{3}$

$\text{Sec}(-330^\circ) = \text{Sec } 30^\circ = \dfrac{1}{Cos\,30^\circ} = \dfrac{1}{\sqrt{3}/2} = \dfrac{2}{\sqrt{3}}$

$\text{Cosec}(-330^\circ) = \text{cosec } 30^\circ = \dfrac{1}{Sin\,30^\circ} = \dfrac{1}{1/2} = 2$

iii. 765°

Sol. $765^\circ = 2 \times 360^\circ + 45^\circ = 45^\circ$, $k = 2$

Value of Trigonometric functions at 765° and 45° are same

$\text{Sin } 765^\circ = \text{Sin } 45^\circ = \dfrac{1}{\sqrt{2}}$

$\text{Cos } 765^\circ = \text{Cos } 45^\circ = \dfrac{1}{\sqrt{2}}$

$\text{Tan } 765^\circ = \text{Tan } 45^\circ = 1$

$\text{Cot } 765^\circ = \text{Cot } 45^\circ = 1/\tan 45^\circ = \dfrac{1}{1} = 1$

$\text{Sec } 765^\circ = \text{Sec } 45^\circ = \dfrac{1}{Cos 45^\circ} = \dfrac{1}{1/\sqrt{2}} = \sqrt{2}$

$\text{Cosec } 765^\circ = \text{Cosce } 45^\circ = \dfrac{1}{Sin 45^\circ} = \dfrac{1}{1/\sqrt{2}} = \sqrt{2}$

iv. -675°

Sol. $-675^\circ = (-2) \times 360^\circ + 45^\circ = 45^\circ$, $k = -2$

Values of Trigonometric functions at -675° and 45° are same

$\text{Sin }(-675^\circ) = \text{Sin } 45^\circ = \dfrac{1}{\sqrt{2}}$

$\text{Cos }(-675^\circ) = \text{Cos } 45^\circ = \dfrac{1}{\sqrt{2}}$

$\text{Tan }(-675^\circ) = \text{Tan } 45^\circ = 1$

$\text{Cot }(-675^\circ) = \text{Cot } 45^\circ = 1/\tan 45^\circ = \dfrac{1}{1} = 1$

$\text{Sec }(-675^\circ) = \text{Sec } 45^\circ = \dfrac{1}{Cos 45^\circ} = \dfrac{1}{1/\sqrt{2}} = \sqrt{2}$

$\text{Cosec }(-675^\circ) = \text{Cosec } 45^\circ = \dfrac{1}{Sin 45^\circ} = \dfrac{1}{1/\sqrt{2}} = \sqrt{2}$

v. $-\dfrac{17}{3}\pi$ **Federal**

Sol. $-\dfrac{17}{3}\pi = (-6\pi) + \dfrac{\pi}{3} = -3 \times 2\pi + \pi/3$, $k = -3$

Value of Trigonometric functions at $\frac{-17\pi}{3}$ and $\frac{\pi}{3}$ are same

$$\text{Sin}\left(\frac{-17\pi}{3}\right) = \text{Sin}\ \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\text{Cos}\left(\frac{-17\pi}{3}\right) = \text{Cos}\ \frac{\pi}{3} = \frac{1}{2}$$

$$\text{Tan}\left(\frac{-17\pi}{3}\right) = \text{Tan}\ \frac{\pi}{3} = \sqrt{3}$$

$$\text{Cot}\left(\frac{-17\pi}{3}\right) = \text{Cot}\ \frac{\pi}{3} = \frac{1}{\tan\ \pi/3} = \frac{1}{\sqrt{3}} = = \frac{1}{\sqrt{3}}$$

$$\text{Sec}\left(\frac{-17\pi}{3}\right) = \text{Sec}\ \frac{\pi}{3} = \frac{1}{\cos\pi/3} = \frac{1}{1/2} = 2$$

$$\text{Cosec}\left(\frac{-17\pi}{3}\right) = \text{Cosec}\ \frac{\pi}{3} = \frac{1}{Sin\,\pi/3} = \frac{1}{\sqrt{3}/2} = 2/\sqrt{3}$$

vi. $\frac{13}{3}\pi$

Sol. $\frac{13\pi}{3} = 4\pi + \frac{\pi}{3} = 2\,(2\pi) + \pi/3 = \frac{\pi}{3}$, k = 2

Value of Trigonometric functions at $\frac{13\pi}{3}$ and $\frac{\pi}{3}$ are same

$$\text{Sin}\ \frac{13\pi}{3} = \text{Sin}\ \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\text{Cos}\ \frac{13\pi}{3} = \text{Cos}\ \frac{\pi}{3} = 1/2$$

$$\text{Tan}\ \frac{13\pi}{3} = \text{Tan}\ \frac{\pi}{3} = \sqrt{3}$$

$$\text{Cot}\ \frac{13\pi}{3} = \text{Cot}\ \frac{\pi}{3} = \frac{1}{\tan\pi/3} = \frac{1}{\sqrt{3}}$$

$$\text{Sec}\ \frac{13\pi}{3} = \text{Sec}\ \frac{\pi}{3} = \frac{1}{\cos\pi/3} = \frac{1}{1/2} = 2$$

$$\text{Cosec}\ \frac{13\pi}{3} = \text{Cosec}\ \frac{\pi}{3} = \frac{1}{\text{Sin}\ \pi/3} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$$

vii. $\frac{25}{6}\pi$ **Sargodha 2008, Multan 2009**

Sol. $\frac{25\pi}{6} = 4\pi + \frac{\pi}{6} = 2 \times 2\pi + \pi/6 = \frac{\pi}{6}$, k=2

Value of Trigonometric functions at $\frac{25\pi}{6}$ and $\frac{\pi}{6}$ are same

$\text{Sin}\ \frac{25\pi}{6} = \text{Sin}\ \frac{\pi}{6} = \frac{1}{2}$

$\text{Cos}\ \frac{25\pi}{6} = \text{Cos}\ \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\text{Tan}\ \frac{25\pi}{6} = \text{Tan}\ \frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$\text{Cot}\ \frac{25\pi}{6} = \text{Cot}\ \frac{\pi}{6} = \frac{1}{\tan\pi/6} = \frac{1}{1/\sqrt{3}} = \sqrt{3}$

$\text{Sec}\ \frac{25\pi}{6} = \text{Sec}\ \frac{\pi}{6} = \frac{1}{Cos\,\pi/6} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

$\text{Cosec}\ \frac{25\pi}{6} = \text{Cosec}\ \frac{\pi}{6} = \frac{1}{Sin\ \pi/6} = \frac{1}{1/2} = 2$

viii. $\frac{-71}{6}\pi$ **Faisalabad 2009, Federal**

Sol. $\frac{-71\pi}{6} = -12\pi + \frac{\pi}{6} = (-6)\,2\pi + \frac{\pi}{6} = \frac{\pi}{6}$, k = −6

Value of Trigonometric functions at $\frac{-71\pi}{6}$ and $\frac{\pi}{6}$ are same

$\text{Sin}\ (\frac{-71\pi}{6}) = \text{Sin}\ \frac{\pi}{6} = \frac{1}{2}$

$\text{Cos}\ (\frac{-71\pi}{6}) = \text{Cos}\ \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$$\text{Tan}\left(\frac{-71\pi}{6}\right) = \text{Tan}\,\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$\text{Cot}\left(\frac{-71\pi}{6}\right) = \text{Cot}\,\frac{\pi}{6} = \frac{1}{\tan\pi/6} = \frac{1}{1/\sqrt{3}} = \sqrt{3}$$

$$\text{Sec}\left(\frac{-71\pi}{6}\right) = \text{Sec}\,\frac{\pi}{6} = \frac{1}{Cos\,\pi/6} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

$$\text{Cosec}\left(\frac{-71\pi}{6}\right) = \text{Cosec}\,\frac{\pi}{6} = \frac{1}{Sin\,\pi/6} = \frac{1}{1/2} = 2$$

ix. -1035° **Multan 2007**

Sol. $-1035^\circ = (-3) \times 360^\circ + 45^\circ = 45^\circ$, $k=-3$

Value of Trigonometric functions at -1035° and 45° are same

$$\text{Sin}(-1035^\circ) = \text{Sin}\,45^\circ = \frac{1}{\sqrt{2}}$$

$$\text{Cos}(-1035^\circ) = \text{Cos}\,45^\circ = \frac{1}{\sqrt{2}}$$

$$\text{Tan}(-1035^\circ) = \text{Tan}\,45^\circ = 1$$

$$\text{Cot}(-1035^\circ) = \text{Cot}\,45^\circ = \frac{1}{\tan 45^\circ} = \frac{1}{1} = 1$$

$$\text{Sec}(-1035^\circ) = \text{Sec}\,45^\circ = \frac{1}{Cos\,45^\circ} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$$

$$\text{Cosec}(-1035^\circ) = \text{Cosec}\,45^\circ = \frac{1}{Sin\,45^\circ} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$$

Exercise 9.4

Formulas

(i) $\sin^2\theta + \cos^2\theta = 1$

(ii) $\sin^2\theta = 1 - \cos^2\theta$

(iii) $Co^2\theta = 1 - \sin^2\theta$

(iv) $1 + \tan^2\theta = \sec^2\theta$

(v) $1 + \cot^2\theta = Cosc^2\theta$

(vi) $Cosec = \dfrac{1}{Sin\theta}$

(vii) $Sec = \dfrac{1}{Cos\theta}$

(viii) $\cot\theta = \dfrac{1}{\tan\theta}$

(ix) $\tan\theta = \dfrac{Sin\theta}{Cos\theta}$

Example 4: $\cot^4\theta + \cot^2\theta = \cos ec^4\theta - \cos ec^2\theta$ *Mul* tan 2009

Sol:
$$L.H.S = \cot^4\theta + \cot^2\theta = \cot^2\theta(\cot^2\theta + 1)$$
$$= (\cos ec^2\theta - 1)(\cos ec^2\theta) = \cos ec^4\theta - \cos ec^2\theta = R.H.S$$

Prove the following identities, state the domain of θ in each case:

1. $\tan\theta + \cot\theta = \text{cosec}\,\theta \sec\theta$

Sol. L.H.S = $\tan\theta + \text{Cot}\,\theta$

$$= \frac{Sin\theta}{Cos\theta} + \frac{Cos\theta}{Sin\theta} = \frac{Sin^2\theta + Co^2\theta}{Sin\theta\, Cos\theta}$$, **Domain** = $\theta \in$ R but $\theta \neq \dfrac{n\pi}{2}$

$$= \frac{1}{Sin\theta\, Cos\theta} = \frac{1}{Sin\theta}\,\frac{1}{Cos\theta}$$

= $\text{Cosec}\,\theta\ \text{Sec}\,\theta$ = R.H.S

2. $\sec\theta\ \text{cosec}\,\theta\ \sin\theta\ \cos\theta = 1$ **Multan 2008**

Sol. L.H.S = $\text{Sec}\,\theta\ \text{Cosec}\,\theta\ \text{Sin}\,\theta\ \text{Cos}\,\theta$, **Domain** = $\theta \in$ R but $\theta \neq \dfrac{n\pi}{2}$

$$= \frac{1}{\cancel{Cos\theta}}\,\frac{1}{\cancel{Sin\theta}}\,\cancel{Sin\theta}\,\cancel{Cos\theta} = 1 = R.H.S$$

3. $\cos\theta + \tan\theta\ \sin\theta = \sec\theta$

Sol. L.H.S = $\text{Cos}\,\theta + \text{Tan}\,\theta\ \text{Sin}\,\theta$, **Domain** = $\theta \in$ R but $\theta \neq \dfrac{(2n+1)\pi}{2}$

$$= Cos\theta + \frac{Sin\theta}{Cos\theta} Sin\theta = Cos\theta + \frac{Sin^2\theta}{Cos\theta}$$

$$= \frac{Cos^2\theta + Sin^2\theta}{Cos\theta} = \frac{1}{Cos\theta} = Sec\theta \ R.H.S$$

4. **$\text{cosec}\,\theta + \tan\theta\ \sec\theta = \text{cosec}\,\theta\ \sec^2\theta$** **Faisalabad 2008**

Sol. L.H.S = $\text{Cosec}\,\theta + \text{Tan}\,\theta\ \text{Sec}\,\theta$, **Domain** = $\theta \in$ R but $\theta \neq \frac{n\pi}{2}$

$$= \text{Cosec}\,\theta + \frac{Sin\theta}{Cos\theta}\frac{1}{Cos\theta} = \frac{1}{Sin\theta} + \frac{Sin\theta}{Cos^2\theta} = \frac{Cos^2\theta + Sin^2\theta}{Sin\theta Cos^2\theta}$$

$$= \frac{1}{Sin\theta\ Cos^2\theta} = \frac{1}{Sin\theta}\frac{1}{Cos^2\theta} = Co\sec Sec^2\theta = \text{R.H.S}$$

5. **$\sec^2\theta - \text{cosec}^2\theta = \tan^2\theta - \cot^2\theta$**

Sol. L.H.S = $\text{Sec}^2\theta - \text{Cosec}^2\theta$, **Domain** = $\theta \in$ R but $\theta \neq \frac{(2n+1)\pi}{2}$

$= 1 + \tan^2\theta - (1 + \text{Cot}^2\theta) = 1 + \tan^2\theta - 1 - \text{Cot}^2\theta$

$= \tan^2\theta - \text{Cot}^2\theta$ =R.H.S

6. **$\cot^2\theta - \cos^2\theta = \cot^2\theta\ \cos^2\theta$**

Sol. L.H.S = $\text{Cot}^2\theta - \text{Cos}^2\theta$, **Domain** = $\theta \in$ R but $\theta \neq n\pi$

$$= \frac{Cos^2\theta}{Sin^2\theta} - Cos^2\theta = \frac{Cos^2\theta - Cos^2\theta Sin^2\theta}{Sin^2\theta}$$

$$= \frac{Cot^2\theta(1 - Sin^2\theta)}{Sin^2\theta} = \frac{Cos^2\theta}{Sin^2\theta} Cos^2\theta$$

$= \text{Cot}^2\theta . \text{Cos}^2\theta$ =R.H.S

7. **$(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$**

Sol. L.H.S = $(\text{Sec}\,\theta + \tan\theta)(\text{Sec}\,\theta - \tan\theta)$, **Domain** = $\theta \in$ R but $\theta \neq \frac{(2n+1)\pi}{2}$

$= \text{Sec}^2\theta - \tan^2\theta$

$= 1 + \tan^2\theta - \tan^2\theta = 1 =$ R.H.S

8. $2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

Sol. L.H.S $= 2Cos^2\theta - 1 = 2(1 - Sin^2\theta) - 1$, **Domain** $= \theta \in R$

$= 2 - 2Sin^2\theta - 1 = 1 - 2Sin^2\theta =$ R.H.S

9. $\cos^2\theta - \sin^2\theta - = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$ **Faisalabad 2009, Sargodha 2011**

Sol. R.H.S $= \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{1-\tan^2\theta}{Sec^2\theta}$, **Domain** $= \theta \in R$ but $\theta \neq \dfrac{(2n+1)\pi}{2}$

$$= \frac{1}{Sec^2\theta} - \frac{\tan^2\theta}{Sec^2\theta} = Cos^2\theta - \frac{Sin^2\theta}{\cancel{Cos^2\theta}} \times \cancel{Cos^2\theta}$$

$= Cos^2\theta - Sin^2\theta =$ L.H.S

10. $\dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \dfrac{\cot\theta - 1}{\cot\theta + 1}$ **Multan 2007**

Sol. R.H.S $= \dfrac{Cot\theta - 1}{Cot\theta + 1} = \dfrac{\dfrac{Cos\theta}{Sin\theta} - 1}{\dfrac{Cos\theta}{Sin\theta} + 1}$, **Domain** $= \theta \in R$ but $\theta \neq n\pi$

$$= \frac{Cos\theta - Sin\theta}{Sin\theta} \times \frac{Sin\theta}{Cos\theta + Sin\theta} = \frac{Cos\theta - Sin\theta}{Cos\theta + Sin\theta} = \text{R.H.S}$$

11. $\dfrac{\sin\theta}{1+\cos\theta} + \cot\theta = \operatorname{cosec}\theta$ **Multan 2008**

Sol. L.H.S $= \dfrac{Sin\theta}{1+Cos\theta} + Cot\theta$, **Domain** $= \theta \in R$ but $\theta \neq n\pi$

$$= \frac{Sin\theta}{1+Cos\theta} + \frac{Cos\theta}{Sin\theta} = \frac{Sin^2\theta + Cos^2\theta + Cos\theta}{Sin\theta(1+Cos\theta)}$$

$$= \frac{\cancel{(1+Cos\theta)}}{Sin\theta\ \cancel{(1+Cos\theta)}} = \frac{1}{Sin\theta} = Cosec\,\theta\ R.H.S$$

12. $\dfrac{\cot^2\theta-1}{1+\cot^2\theta}=2\cos^2\theta-1$ **Sargodha 2011**

Sol. L.H.S = $\dfrac{Cot^2\theta-1}{1+Cot^2\theta}=\dfrac{Cot^2\theta-1}{Co\sec^2\theta}$, **Domain** = $\theta\in R$ but $\theta\neq n\pi$

$$=\frac{Cot^2\theta}{Co\sec^2\theta}-\frac{1}{Co\sec^2\theta}$$

$$=\frac{Cos^2\theta}{\cancel{Sin^2\theta}}\times\cancel{Sin^2\theta}-Sin^2\theta=\text{Cos}^2\theta-(1-\text{Cos}^2\theta)$$

$$=\text{Cos}^2\theta-1+\text{Cos}^2\theta=2\text{Cos}^2\theta-1=\text{R.H.S}$$

13. $\dfrac{1+\cos\theta}{1-\cos\theta}=(\cos ec\theta+\cot\theta)^2$

Sol. R.H.S = $(\text{Cosec}\,\theta+\text{Cot}\,\theta)^2$, **Domain** = $\theta\in R$ but $\theta\neq n\pi$

$$=\left(\frac{1}{Sin\theta}+\frac{Cos\theta}{Sin\theta}\right)^2=\frac{(1+Cos\theta)^2}{Sin^2\theta}$$

$$=\frac{(1+Cos\theta)^2}{1-Cos^2\theta}=\frac{(1+Cos\theta)^2}{\cancel{(1+Cos\theta)}\,(1-Cos\theta)}=\frac{1+Cos\theta}{1-Cos\theta}=\text{L.H.S}$$

14. $(\sec\theta-\tan\theta)^2=\dfrac{1-\sin\theta}{1+\sin\theta}$ **Sargodha 2008, 2011**

Sol. L.H.S = $(\text{Sec}\,\theta-\tan\theta)^2$, **Domain** = $\theta\in R$ but $\theta\neq\dfrac{(2n+1)\pi}{2}$

$$=\left(\frac{1}{Cos\theta}-\frac{Sin\theta}{Cos\theta}\right)^2=\left(\frac{1-Sin\theta}{Cos\theta}\right)^2$$

$$=\frac{(1-Sin\theta)^2}{Cos^2\theta}=\frac{(1-Sin\theta)^2}{1-Sin^2\theta}$$

$$\frac{(1-Sin\theta)^2}{(1-Sin\theta)(1+Sin\theta)}=\frac{1-Sin\theta}{1+Sin\theta}=\text{R.H.S}$$

15. $\dfrac{2\tan\theta}{1+\tan^2\theta} = 2\sin\theta\cos\theta$ **Multan 2007**

Sol. L.H.S = $\dfrac{2\tan\theta}{1+\tan^2\theta} = \dfrac{2\tan\theta}{Sec^2\theta}$, Domain = $\theta \in R$ but $\theta \neq \dfrac{(2n+1)\pi}{2}$

$= 2\tan\theta\ Cos^2\theta = \dfrac{2\,Sin\theta}{Cos\theta}\ Cos^2\theta = 2\ Sin\theta\ .\ \cos\theta$ = R.H.S

16. $\dfrac{1-\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1+\sin\theta}$

Sol. L.H.S = $\dfrac{1-Sin\theta}{Cos\theta} = \dfrac{1-Sin\theta}{Cos\theta} \times \dfrac{1+Sin\theta}{1+Sin\theta}$, Domain = $\theta \in R$

$= \dfrac{1-Sin^2\theta}{Cos\theta(1+Sin\theta)} = \dfrac{Cos^2\theta}{Cos\theta(1+Sin\theta)} = \dfrac{Cos\theta}{1+Sin\theta}$ =R.H.S

17. $(\tan\theta + \cot\theta)^2 = \sec^2\theta\ \text{cosec}^2\theta$ **Multan 2008, Lahore 2009**

Sol. L.H.S = $(\tan\theta + Cot\theta)^2$, Domain = $\theta \in R\ but\ \theta \neq \dfrac{n\pi}{2}$

$= \left(\dfrac{Sin\theta}{Cos\theta} + \dfrac{Cos\theta}{Sin\theta}\right)^2 = \left(\dfrac{Sin^2\theta + Cos^2\theta}{Cos\theta\, Sin\theta}\right)^2$

$= \left(\dfrac{1}{Cos\theta\ Sin\theta}\right)^2 = \dfrac{1}{Cos^2\theta}\ \dfrac{1}{Sin^2\theta}$

$= Sec^2\theta\ Cosec^2\theta$ = R.H.S

18. $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = \tan\theta + \sec\theta$ **Faisalabad 2007**

Sol. L.H.S = $\dfrac{\tan\theta+Sec\theta-1}{\tan\theta-Sec\theta+1}$, Domain = $\theta \in R\ but\ \theta \neq \dfrac{(2n+1)\pi}{2}$

$= \dfrac{(\tan\theta+Sec\theta) - (Sec^2\theta-\tan^2\theta)}{(\tan\theta - Sec\theta+1)}$

$= \dfrac{(\tan\theta+Sec\theta) - (Sec\theta + \tan\theta)(Sec - \tan\theta)}{(\tan\theta - Sec\theta+1)}$

$= \dfrac{((\tan\theta+Sec\theta)\,[1-(Sec - \tan\theta)])}{(\tan\theta - Sec\theta+1)}$

$$= \frac{(\tan\theta + Sec\theta)\ \cancel{(1 - Sec + \tan\theta)}}{\cancel{(\tan\theta - Sec\theta + 1)}} \quad = \tan\theta + Sec\theta = \text{R.H.S}$$

19. $\dfrac{1}{Cosec\,\theta - Cot\theta} - \dfrac{1}{Sin\theta} = \dfrac{1}{Sin\theta} - \dfrac{1}{Cosec\,\theta + Cot\theta}$ **Multan 2007,2008**

Sol. L.H.S $= \dfrac{1}{Cosec\,\theta - Cot\theta} - \dfrac{1}{Sin\theta}$

$$= \frac{1}{\frac{1}{\sin\theta} - \frac{Cos\theta}{Sin\theta}} - \frac{1}{\sin\theta}$$

$$= \frac{1}{\frac{1 - Cos\theta}{Sin\theta}} - \frac{1}{Sin\theta} = \frac{Sin\theta}{1 - Cos\theta} - \frac{1}{Sin\theta} = \frac{Sin^2\theta - 1 + Cos\theta}{Sin\theta(1 - Cos\theta)}$$

$$= \frac{1 - Cos^2\theta - 1 + Cos\theta}{Sin\theta(1 - Cos\theta)} = \frac{Cos\theta - Cos^2\theta}{Sin\theta(1 - Cos\theta)} = \frac{Cos\theta(1 - \cancel{Cos\theta})}{Sin\theta(1 - \cancel{Cos\theta})} = Cot\theta$$

R.H.S $= \dfrac{1}{Sin\theta} - \dfrac{1}{Cosce\theta + Cot\theta}$

$$= \frac{1}{Sin\theta} - \frac{1}{\frac{1}{Sin\theta} + \frac{Cos\theta}{Sin\theta}}$$

$$= \frac{1}{Sin\theta} - \frac{1}{\frac{1 + Cos\theta}{Sin\theta}} = \frac{1}{Sin\theta} - \frac{Sin\theta}{1 + Cos\theta}$$

$$= \frac{1 + Cos\theta - Sin^2\theta}{Sin\theta(1 + Cos\theta)} = \frac{1 + Cos\theta - (1 - Cos^2\theta)}{Sin\theta(1 + Cos\theta)}$$

$$= \frac{1 + Cos\theta - 1 + Cos^2\theta}{Sin\theta(1 + Cos\theta)} = \frac{Cos\theta\ (1 + Cos\theta)}{Sin\theta(1 + Cos\theta)} = \text{Cot}\,\theta$$

L.H.S=R.H.S

20. $\sin^3\theta - \cos^3\theta = (\sin\theta - \cos\theta)(1 + \sin\theta\ \cos\theta)$

Sol. L.H.S $= \text{Sin}^3\theta - \text{Cos}^3\theta$, **Domain** $= \theta \in R$

$$= (\text{Sin}\,\theta - \text{Cos}\,\theta)(\text{Sin}^2\theta + \text{Cos}^2\theta + \text{Sin}\,\theta\ \text{Cos}\,\theta)$$

$$= (\text{Sin}\,\theta - \text{Cos}\,\theta)(1 + \text{Sin}\,\theta\ \text{Cos}\,\theta) = \text{R.H.S}$$

21. $\sin^6\theta - \cos^6\theta = (\sin^2\theta - \cos^2\theta)(1 - \sin^2\theta\cos^2\theta)$ **Fsd 2008, Sgd2009, Lhr 2009**

Sol. L.H.S = $\text{Sin}^6\theta - \text{Cos}^6\theta = (\text{Sin}^2\theta)^3 - (\text{Cos}^2\theta)^3$, **Domain** = $\theta \in R$

$= (\text{Sin}^2\theta - \text{Cos}^2\theta)((\text{Sin}^2\theta)^2 + (\text{Cos}^2\theta)^2 + \text{Sin}^2\theta\,\text{Cos}^2\theta))$

$= (\text{Sin}^2\theta - \text{Cos}^2\theta)[(\text{Sin}^2\theta)^2 + (\text{Cos}^2\theta)^2 + 2\text{Sin}^2\theta\,\text{Cos}^2\theta - 2\text{Sin}^2\theta$
$\text{Cos}^2\theta + \text{Sin}^2\theta\,\text{Cos}^2\theta]$

$= (\text{Sin}^2\theta - \text{Cos}^2\theta)[(\text{Sin}^2\theta + \text{Cos}^2\theta)^2 - \text{Sin}^2\theta\,\text{Cos}^2\theta]$

$= (\text{Sin}^2\theta - \text{Cos}^2\theta)(1 - \text{Sin}^2\theta\,\text{Cos}^2\theta)$ = R.H.S

22. $\text{Sin}^6\theta + \cos^6\theta = 1 - 3\text{Sin}^2\theta\,\text{Cos}^2\theta \quad \theta \in R$ **Rawalpindi 2009**

Sol. L.H.S = $\text{Sin}^6\theta + \cos^6\theta = (\text{Sin}^2\theta)^3 + (\text{Cos}^2\theta)^3$

$= (\text{Sin}^2\theta + \text{Cos}^2\theta)[(\text{Sin}^2\theta)^2 + (\text{Cos}^2\theta)^2 - \text{Sin}^2\theta\,\text{Cos}^2\theta]$

$= 1.[(\text{Sin}^2\theta)^2 + (\text{Cos}^2\theta)^2 + 2\text{Sin}^2\theta\,\text{Cos}^2\theta - 2\,\text{Sin}^2\theta\,\text{Cos}^2\theta - \text{Sin}^2\theta\,\text{Cos}^2\theta]$

$= (\text{Sin}^2\theta + \text{Cos}^2\theta)^2 - 3\text{Sin}^2\theta\,\text{Cos}^2\theta = 1 - 3\text{Sin}^2\theta\,\text{Cos}^2\theta$ = R.H.S

23. $\dfrac{1}{1+Sin\theta} + \dfrac{1}{1-Sin\theta} = \cancel{2Sin^2\theta}$ $2\sec^2\theta$

Sol. L.H.S = $\dfrac{1}{1+Sin\theta} + \dfrac{1}{1-Sin\theta} = \dfrac{1-\cancel{Sin\theta}+1+\cancel{Sin\theta}}{(1+Sin\theta)(1-Sin\theta)}$, **Domain** = $\theta \in R$

$= \dfrac{2}{1-Sin^2\theta} = \dfrac{2}{Cos^2\theta} = 2\left(\dfrac{1}{Cos^2\theta}\right) = 2\text{Sec}^2\theta$ = R.H.S

24. $\dfrac{Cos\theta+Sin\theta}{Cos\theta-Sin\theta} + \dfrac{Cos\theta-Sin\theta}{Cos\theta+Sin\theta} = \dfrac{2}{1-2Sin^2\theta}$ **Faisalabad 2007, Sargodha 2009**

Sol. L.H.S = $\dfrac{Cos\theta+Sin\theta}{Cos\theta-Sin\theta} + \dfrac{Cos\theta-Sin\theta}{Cos\theta+Sin\theta}$, **Domain** = $\theta \in R$ but $\theta \neq \dfrac{(2n+1)\pi}{2}$

$= \dfrac{(Cos\theta+Sin\theta)^2 + (Cos\theta-Sin\theta)^2}{(Cos\theta-Sin\theta)(Cos\theta+Sin\theta)}$

$= \dfrac{Cos^2 + Sin^2\theta + 2\,Sin\theta\,Cos\theta + Cos^2\theta + Sin^2\theta - 2Sin\theta\,Cos\theta}{(Cos\theta-Sin\theta)\,(Cos\theta+Sin\theta)}$

$= \dfrac{2Cos^2\theta+2Sin^2\theta}{1-Sin^2\theta-Sin^2\theta} = \dfrac{2(Cos^2\theta+Sin^2\theta)}{1-2Sin^2\theta} = \dfrac{2}{1-2Sin^2\theta} = R.H.S$

TEST YOUR SKILLS Marks: 50

Q # 1. Select the Correct Option **(10)**

i. In one hour, the hour hand of a clock turns through radians:

a) $\pi/8$ b) $\pi/4$

c) $\pi/6$ d) $\pi/2$

ii. If $Tan\theta < 0$ and $Cos\theta > 0$ then terminal arm is in quadrant:

a) I b) II

c) III d) IV

iii. If the terminal side lies on $x-axis$ or $y-axis$ then angle is called:

a) Central angle b) Quadrantal angle

c) Co-terminal angle d) Acute angle

iv. Domain of $Sin\theta$ and $Cos\theta$ is set of ________

a) Integers b) Natural numbers

c) Real numbers d) None

v. $\dfrac{3\pi}{2}$ radian equal to:

a) 270^{o} b) 90^{o}

c) 180^{o} d) 60^{o}

vi. $Co\sec^2\theta - Cot^2\theta$ equals:

a) 1 b) 0

c) 2 d) -1

vii. Which one is true:

a) $1\ radian < 1^{o}$ b) $1\ radian > 1^{o}$

c) $1\ radian = 1^{o}$ d) $5\ radian = 2^{o}$

viii. $Sin390^{o}$ is equal to:

a) $1/2$ b) $\sqrt{3}/2$

c) 1 d) $1/\sqrt{2}$

ix. The value of $Sin420^{o}$ is equal to

a) $2/\sqrt{3}$ b) $1/2$

c) $\sqrt{3}$ d) $\sqrt{3}/2$

x. The 60th part of one degree is called one:

a) Second b) Radian

c) Minute d) Degree

Q # 2. Short Questions: (10 X 2 = 20)

i. Write the Sign of Trigonometric functions in II and IV quadrant:

ii. Prove that $Cos^2\theta - Sin^2\theta = \dfrac{1-Tan^2\theta}{1+Tan^2\theta}$

iii. Find x if $Tan^2 45^o - Cos^2 60^o = xSin45^o.Cos45^o.Tan60^0$

iv. Find l when $\theta = 60^o 20'$ and $r = 18mm$

v. Verify that $Sin60^o Cos30^o - Cos60^o Sin30^o = Sin30^o$

vi. In which quadrant terminal arm lie if $Cos\theta < 0$ and $Tan\theta < 0$

vii. $Cos\theta = \dfrac{\sqrt{3}}{2} (0 < \theta < \pi/2)$ Find remaining trigonometric functions.

viii. Prove that $(Sec\theta - Tan\theta)^2 = \dfrac{1-Sin\theta}{1+Sin\theta}$

ix. Define Radian

x. Prove that $Co\sec\theta + Tan\theta Sec\theta = Co\sec\theta.Sec^2\theta$

Long Questions: (2 X 10 = 20)

Q # 3. (a) Show that $(Sec\theta - Tan\theta)^2 = \dfrac{1-Sin\theta}{1+Sin\theta}$

(b) $Co\sec\theta = \dfrac{m^2+1}{2m}, m > 0$ and $0 < \theta < \pi/2$ Find value of the remaining trigonometric ratios:

Q # 4. (a) Prove that $\dfrac{Cos\theta + Sin\theta}{Cos\theta - Sin\theta} + \dfrac{Cos\theta - Sin\theta}{Cos\theta + Sin\theta} = \dfrac{2}{1-2Sin^2\theta}$

(b) Prove that $Sin^6\theta - Cos^6\theta = (Sin^2\theta - Cos^2\theta)(1 - Sin^2\theta Cos^2\theta)$

Trigonometry Identities

10

Fundamental Law

Theorem:

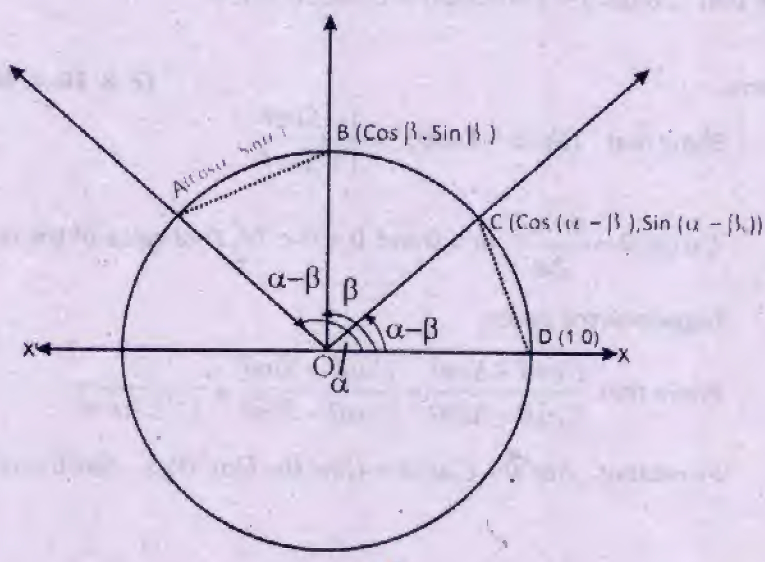
Cos ($\alpha - \beta$) = Cos α Cos β + Sin α Sin β **Sargodha 2011(only statement)**

Proof:

Consider a unit circle with centre at O.

Where < AOD = α , < BOD = β

< AOB = <COD = $\alpha - \beta$



Now Δ AOB and Δ COD are congruent then |AB| = |CD| $\Rightarrow$ $|AB|^2 = |CD|^2$

Use distance formula, we have

$(\text{Cos}\,\alpha - \text{Cos}\,\beta)^2 + (\text{Sin}\,\alpha - \text{Sin}\,\beta)^2 = (\text{Cos}\,(\alpha - \beta) - 1)^2 + (\text{Sin}(\alpha - \beta) - 0)^2$

$\text{Cos}^2\alpha + \text{Cos}^2\beta - 2\,\text{Cos}\,\alpha\,\text{Cos}\,\beta + \text{Sin}^2\alpha + \text{Sin}^2\beta - 2\,\text{Sin}\,\alpha\,\text{Sin}\,\beta$

$= \text{Cos}^2(\alpha - \beta) + 1 - 2\text{Cos}\,((\alpha - \beta) + \text{Sin}^2(\alpha - \beta)$

$\text{Cos}^2\alpha + \text{Sin}^2\alpha + \text{Cos}^2\beta + \text{Sin}^2\beta - 2\,(\text{Cos}\,\alpha\,\text{Cos}\,\beta + \text{Sin}\,\alpha\,\text{Sin}\,\beta) = \text{Cos}^2(\alpha - \beta) + \text{Sin}^2$

$(\alpha - \beta) + 1 - 2\text{Cos}(\alpha - \beta)$

$1 + 1 - 2\,(\text{Cos}\,\alpha\,\text{Cos}\,\beta + \text{Sin}\,\alpha\,\text{Sin}\,\beta) = 1 + 1 - 2\text{Cos}(\alpha - \beta)$

$2 - 2\,(\text{Cos}\,\alpha\,\text{Cos}\,\beta + \text{Sin}\,\alpha\,\text{Sin}\,\beta) = 2 - 2\text{Cos}(\alpha - \beta)$

Subtract 2 from both sides

$$-2\,(\text{Cos}\,\alpha\,\text{Cos}\,\beta + \text{Sin}\,\alpha\,\text{Sin}\,\beta) = 2\text{Cos}(\alpha - \beta)$$

Divide by – 2 from both sides

$$\text{Cos}\,\alpha\,\text{Cos}\,\beta + \text{Sin}\,\alpha\,\text{Sin}\,\beta = \text{Cos}(\alpha - \beta)$$

or $\quad \text{Cos}(\alpha - \beta) = \text{Cos}\,\alpha\,\text{Cos}\,\beta + \text{Sin}\,\alpha\,\text{Sin}\,\beta$

Hence Proved

Distance formula

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points. If d denotes distance between them.

$$d = |PQ| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$or = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Sargodha 2011

CHAPTER. 10

Note Sign of trigonometric ratio depends in which quadrant θ exists. Important Formulas.

1. $\sin(\alpha+\beta) = \sin\alpha\ \cos\beta + \cos\alpha\ \sin\beta$

2. $\sin(\alpha-\beta) = \sin\alpha\ \cos\beta - \cos\alpha\ \sin\beta$

3. $\cos(\alpha+\beta) = \cos\alpha\ \cos\beta - \sin\alpha\ \sin\beta$

4. $\cos(\alpha-\beta) = \cos\alpha\ \cos\beta + \sin\alpha\ \sin\beta$

5. $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

6. $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

7. (Even number) $\left(\left(\dfrac{\pi}{2}\right)\pm\theta\right)$ then no change of trigonometric function.

   Example $\sin\left(4\dfrac{\pi}{2} + \theta\right) = \sin\theta$

8. $\left(\text{Odd number } \dfrac{\pi}{2}\pm\theta\right)$ then change trigonometric function as given below

$\sin\theta \rightleftarrows \cos\theta$

$\tan\theta \rightleftarrows \cot\theta$

$\sec\theta \rightleftarrows \csc\theta$

$\pi-\theta$ —— *II quadrant*

Also $\pi+\theta$ —— *III quadrant*

$2\pi-\theta$ —— *IV quadrant*

EXERCISE. 10.1

1. Without using calculator. Find the values of

i Sin (– 780°)

Sol. $-\text{Sin}780^\circ = -\text{Sin}(2\times360^\circ+60^\circ) = -\text{Sin}60^\circ = -\dfrac{\sqrt{3}}{2}$

ii. Cot (– 855°)

Sol. $= -\text{Cot}(2\times360^\circ) + 135^\circ) = -\text{Cot}\,135^\circ = -\text{Cot}(180^\circ - 45^\circ) = -(-\text{Cot}\,45^\circ)$

$= \text{Cot}\,45^\circ = \dfrac{1}{\tan 45^\circ} = \dfrac{1}{1} = 1$

iii. Cosec 2040°

Sol. $= \text{Cosec}(5\times360^\circ + 240^\circ) = \text{Cosec}\,240^\circ = \text{Cosec}(180^\circ + 60^\circ)$

$= -\text{Cosec}\,60^\circ = -\dfrac{1}{Sin\,60^\circ} = \dfrac{-1}{\sqrt{3}/2} = \dfrac{-2}{\sqrt{3}}$

iv. Sec (– 960°)

Sol. $= \text{Sec}\,960^\circ = \text{Sec}(2\times360^\circ + 240^\circ) = \text{Sec}\,240^\circ = \text{Sec}(180^\circ + 60^\circ) = -\text{Sec}60^\circ$

$= -\dfrac{1}{Cos\,60^\circ} = -\dfrac{1}{1/2} = -2$

v. tan (1110°)

Sol. $= \tan(3\times360^\circ + 30^\circ) = \tan 30^\circ = \dfrac{1}{\sqrt{3}}$

vi. Sin (– 300°)

Sol. $= -\text{Sin}\,300^\circ = -\text{Sin}(360^\circ - 60^\circ) = -(-\text{Sin}\,60^\circ) = \text{Sin}\,60^\circ = \dfrac{\sqrt{3}}{2}$

2. Express each of the following as a trigonometric function of an angle positive degree measure of less than 45°

i. Sin 196° = Sin (180° + 16°)

Sol. **Method I** $= \text{Sin}\,180^\circ\,\text{Cos}\,16^\circ + \text{Cos}\,180^\circ\,\text{Sin}\,16^\circ$

$= 0.\text{Cos}\,16^\circ + (-1)\,\text{Sin}\,16^\circ = -\text{Sin}\,16^\circ$

Method II $\text{Sin}\,196^\circ = \text{Sin}(180^\circ + 16^\circ) = -\text{Sin}\,16^\circ$

ii. **Cos 147°**

Sol. $= \cos(180° - 33°) = -\cos 33°$

iii. **Sin 319°**

Sol. $= \sin(360° - 41°) = -\sin 41°$

iv. **Cos 254°**

Sol. $= \cos(270° - 16°) = -\sin 16°$

v. **tan 294°**

Sol. $= \tan(270° + 24°) = -\cot 24°$

vi. **Cos 728°**

Sol. $= \cos(2 \times 360° + 8°) = \cos 8°$

vii. **Sin (– 625°)**

Sol. $\sin(-625°) = -\sin 625°$

$= \sin(2 \times 360° - 95°) = -(-\sin 95°)$

$= \sin 95° = \sin(90° + 5°) = \cos 5°$

viii. **Cos (– 435°)**

Sol. $= \cos 435° = \cos(360° + 75°) = \cos 75° = \cos(90° - 15°) = \sin 15°$

ix. **Sin 150°**

Sol. $= \sin(180° - 30°) = \sin 30°$

3. i. **Prove that Sin (180° + α) Sin (90° – α) = – Sinα Cosα**

Sol. L.H.S $= \sin(180° + \alpha)\sin 90° - \alpha) = \sin[2 \times 90 + \alpha]\sin[1 \times 90 - \alpha] = (-\sin\alpha)(\cos\alpha)$

$= -\sin\alpha \cos\alpha =$ R.H.S **Sargodha 2006, 2008, 2009, Multan 2009**

ii. **Sin 780° Sin 480° + Cos 120° Sin 30° = 1/2**

Sol. L.H.S $= \sin 780° \sin 480° + \cos 120° \sin 30°$

$= \sin(2 \times 360° + 60°)\sin(360° + 120°) + \cos 120° \sin 30°$

$= \sin 60° \sin 120° + \cos 120° \sin 30°$

$= \sin 60° \sin(180° - 60°) + \cos(180° - 60°)\sin 30°$

$= \sin 60° \sin 60° + (-\cos 60°)\sin 30°$

$$= \frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2} - \frac{1}{2}.\frac{1}{2} = \frac{3}{4} - \frac{1}{4} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} = R.H.S$$

iii. **Cos 306° + Cos 234° + Cos 162° + Cos18° = 0**

Sol. L.H.S = Cos 306° + Cos 234° + Cos 162° + Cos 18°

$= \cos(360° - 54°) + \cos(180° + 54°) + \cos(180° - 18°) + \cos 18°$

$= \cancel{\cos 54°} - \cancel{\cos 54°} - \cancel{\cos 18°} + \cancel{\cos 18°} = 0 = \text{R.H.S}$

iv. **Cos 330° Sin 600° + Cos 120° Sin 150° = – 1**

Sol. L.H.S = Cos 330° Sin 600° + Cos 120° Sin 150°

$= \cos(360° - 30°)\sin(360° + 240°) + \cos 120° \sin 150°$

$= \cos 30° \sin 240° + \cos 120° \sin 150°$

$= \cos 30° \sin(180° + 60°) + \cos(180° - 60°)\sin(180° - 30°)$

$= \cos 30°(-\sin 60°) + (-\cos 60°)\sin 30°$

$$= \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{3}{4} - \frac{1}{4} = -\frac{4}{4} = -1 = \text{R.H.S}$$

4. **Prove that**

i.
$$\frac{Sin^2(\pi+\theta)\tan(\frac{3\pi}{2}+\theta)}{Cot^2(\frac{3\pi}{2}-\theta)Cos^2(\pi-\theta)Co\sec(2\pi-\theta)} = Cos\theta$$

Sol.
$$\text{L.H.S} = \frac{Sin^2(\pi+\theta)\tan(\frac{3\pi}{2}+\theta)}{Cot^2(\frac{3\pi}{2}-\theta)Cos^2(\pi-\theta)Co\sec(2\pi-\theta)} = \frac{(-Sin\theta)^2(\cancel{-}Cot\theta)}{\tan^2\theta(-Cos)^2(\cancel{-}Co\sec\theta)}$$

$$= \frac{Sin^2\theta\, Cot\theta}{\frac{Sin^2\theta}{\cancel{Cos^2\theta}}\cancel{Cos^2\theta}\,\frac{1}{Sin\theta}} = \cancel{Sin^2\theta} \times \frac{Cos\theta}{\cancel{Sin\theta}} \times \frac{1}{\cancel{Sin^2\theta}} \times \cancel{Sin\theta} = Cos\theta = R.H.S$$

ii.
$$\frac{Cos(90°+\theta)Sec(-\theta)\tan(180°-\theta)}{Sec(360°-\theta)\ Sin\left(180°+\theta)Cot(90°-\theta)\right)} = -1$$

Sol.
$$\text{L.H.S} = \frac{Cos(90°+\theta)Sec(-\theta)\tan(180°-\theta)}{Sec(360°-\theta)\ Sin\left(180°+\theta)Cot(90°-\theta)\right)} = \frac{-Sin\theta\, Sec\theta(-\tan\theta)}{Sec\theta(-Sin\theta)\tan\theta}$$

$$= \frac{\cancel{Sin\theta}\ \cancel{Sec\theta}\ \cancel{\tan\theta}}{-\cancel{Sin\theta}\ \cancel{Sec\theta}\ \cancel{\tan\theta}} = -1 = \text{R.H.S}$$

5. If α, β, γ are angle of Triangle ABC, then prove that

i. $\text{Sin}(\alpha+\beta) = \text{Sin}\,\alpha$ Faisalabad 2008, 2009

Sol. let $\alpha+\beta+\gamma = 180^\circ$ (sum of angles of triangle=180°)

$\alpha+\beta = 180^\circ - \gamma$

$\text{Sin}(\alpha+\beta) = \sin(2 \times 90^\circ - \gamma)$

$\text{Sin}(\alpha+\beta) = \text{Sin}\,\gamma$ Hence proved

ii. $\text{Cos}\,\dfrac{(\alpha+\beta)}{2} = \sin\alpha/2$ Lahore 2009

Sol. Let $\alpha+\beta+\gamma = 180^\circ \Rightarrow \alpha+\beta = 180^\circ - \gamma$

$$\frac{\alpha+\beta}{2} = \frac{180^\circ - \gamma}{2}$$

$$\text{Cos}\left(\frac{\alpha+\beta}{2}\right) = \text{Cos}\left(\frac{180^\circ}{2} - \frac{\gamma}{2}\right)$$

$$\text{Cos}\left(\frac{\alpha+\beta}{2}\right) = \text{Cos}\left(90^\circ - \frac{\gamma}{2}\right)$$

$$\text{Cos}\left(\frac{\alpha+\beta}{2}\right) = \text{Sin}\frac{\gamma}{2}$$ Hence Proved

iii. $\text{Cos}(\alpha+\beta) = -\text{Cos}\,\gamma$ Faisalabad 2009

Sol. let $\alpha+\beta+\gamma = 180^\circ$

$\alpha+\beta = 180^\circ - \gamma$

$\text{Cos}(\alpha+\beta) = \text{Cos}(2 \times 90^\circ - \gamma)$

$\text{Cos}(\alpha+\beta) = -\text{Cos}\,\gamma$ Hence Proved

iv. $\tan(\alpha+\beta) + \tan\gamma = 0$ Multan 2007, Faisalabad 2009

Sol. let $\alpha+\beta+\gamma = 180^\circ$

$\alpha+\beta = 180^\circ - \gamma$

$\tan(\alpha+\beta) = \tan(2 \times 90^\circ - \gamma)$

$\tan(\alpha+\beta) = -\tan\gamma \Rightarrow \tan(\alpha+\beta) + \tan\gamma = 0$ Hence Proved

EXERCISE. 10.2

Example 2. **Without using tables. Find the values of all trigonometric functions of 75°**

Sol. As $75^\circ = 45^\circ + 30^\circ$ **Sargodha 2009, Faisalabad 2009**

$\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$\cos 75^\circ = \cos(54^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$\tan 75^\circ = \tan(45^\circ + 30^\circ) = \dfrac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$ **Multan 2007, Sargodha 2009**

$$= \frac{1+\frac{1}{\sqrt{3}}}{1-(1)\frac{1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\cancel{\sqrt{3}}} \times \frac{\cancel{\sqrt{3}}}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$\text{Cosec}75^\circ = \dfrac{1}{\sin 75^\circ} = \dfrac{2\sqrt{2}}{\sqrt{3}+1}$

$\text{Sec}75^\circ = \dfrac{1}{\cos 75^\circ} = \dfrac{2\sqrt{2}}{\sqrt{3}-1}, \cot 75^\circ = \dfrac{1}{\tan 75^\circ} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

Example3. **Prove that** $\dfrac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ} = \tan 56^\circ$ **Faisalabad 2008,**

Sol. R.H.S $= \tan 56^\circ = \tan(45^\circ + 11^\circ)$ **Sargodha 2009**

$$= \frac{\tan 45^\circ + \tan 11^\circ}{1 - \tan 45^\circ \tan 11^\circ} = \frac{1+\frac{\sin 11^\circ}{\cos 11^\circ}}{1-\frac{\sin 11^\circ}{\cos 11^\circ}} = \frac{\frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ}}{\frac{\cos 11^\circ - \sin 11^\circ}{\cos 11^\circ}}$$

$$= \frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ} \times \frac{\cos 11^\circ}{\cos 11^\circ - \sin 11^\circ} = \frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ} = \text{L.H.S}$$

1. Prove that

I $\quad$ **$\text{Sin}(180^\circ + \theta) = -\text{Sin}\,\theta$**

Sol. $\quad$ L.H.S $= \text{Sin}(180^\circ + \theta)$

$= \text{Sin}\,180^\circ \text{Cos}\,\theta + \text{Cos}\,180^\circ \text{Sin}\,\theta$

$= (0).\text{Cos}\,\theta + (-1)\,\text{Sin}\,\theta = -\text{Sin}\,\theta$ = R.H.S

ii. $\quad$ **$\text{Cos}(180^\circ + \theta) = -\text{Cos}\,\theta$** $\quad$ **Sargodha 2008**

Sol. $\quad$ L.H.S $= \text{Cos}(180^\circ + \theta)$

$= \text{Cos}\,180^\circ \text{Cos}\,\theta - \text{Sin}\,180^\circ \text{Sin}\,\theta$

$= (-1)\,\text{Cos}\,\theta - (0.)\text{Sin}\,\theta = -\text{Cos}\,\theta$ = R.H.S

iii. $\quad$ **$\tan(270^\circ - \theta) = \text{Cot}\,\theta$** $\quad$ **Multan 2008**

Sol. $\quad$ L.H.S $= \tan(270^\circ - \theta) = \dfrac{Sin(270^\circ - \theta)}{Cos(270^\circ - \theta)}$

$$= \frac{Sin\,270^\circ\, Cos\theta - Cos\,270^\circ\, Sin\theta}{Cos\,270^\circ\, Cos\theta + Sin\,270^\circ\, Sin\theta} = \frac{(-1)Cos\theta - (0)Sin\theta}{(0)Cos\theta + (-1)Sin\theta} = \frac{-Cos\theta}{-Sin\theta} = \text{Cot}\,\theta = \text{R.H.S}$$

iv. $\quad$ **$\text{Cos}(\theta - 180^\circ) = -\text{Cos}\,\theta$**

Sol. $\quad$ L.H.S $= \text{Cos}(\theta - 180^\circ)$

$= \text{Cos}\,\theta\, \text{Cos}\,180^\circ + \text{Sin}\,\theta\, \text{Sin}\,180^\circ$

$= \text{Cos}\,\theta\,(-1) + \text{Sin}\,\theta\,(0)$

$= -\text{Cos}\,\theta$ = R.H.S

v. $\quad$ **$\text{Cos}(270^\circ + \theta) = \text{Sin}\,\theta$** $\quad$ **Lahore 2009**

Sol. $\quad$ L.H.S $= \text{Cos}(270^\circ + \theta) = \text{Cos}\,270^\circ \text{Cos}\,\theta - \text{Sin}\,270^\circ \text{Sin}\,\theta$

$= 0.\text{Cos}\,\theta - (-1)\,\text{Sin}\,\theta$

$= 0 + \text{Sin}\,\theta = \text{Sin}\,\theta$ = R.H.S

vi. $\quad$ **$\text{Sin}(\theta + 270^\circ) = -\text{Cos}\,\theta$**

Sol. $\quad$ L.H.S $= \text{Sin}(\theta + 270^\circ) = \text{Sin}\,\theta\, \text{Cos}\,270^\circ + \text{Cos}\,\theta\, \text{Sin}\,270^\circ$

$= \text{Sin}\,\theta\,(0) + \text{Cos}\,\theta\,(-1)$

$= -\text{Cos}\,\theta$ = R.H.S

vii. $\tan(180° + \theta) = \tan\theta$

Sol. L.H.S $= \tan(180° + \theta)$

$$= \frac{\tan 180° + \tan\theta}{1 - \tan 180° \tan\theta} = \frac{0 + \tan\theta}{1 - (0).\tan\theta}$$

$$= \frac{\tan\theta}{1} = \tan\theta \text{ R.H.S}$$

viii. $\cos(360° - \theta) = \cos\theta$

Sol. L.H.S $= \cos(360° - \theta)$

$= \cos 360° \cos\theta + \sin 360° \sin\theta$

$= (1)(\cos\theta) + (0)\sin\theta$

$= \cos\theta$ = R.H.S

2. Find the values of

i **$\sin 15°$** $= \sin(45° - 30°)$

Sol. $= \sin 45° \cos 30° - \cos 45° \sin 30°$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2.\sqrt{2}}$$

ii. **$\cos 15°$** $= \cos(45° - 30°)$

Sol. $= \cos 45° \cos 30° + \sin 45° \sin 30°$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

iii. **$\tan 15°$** $= \tan(45° - 30°)$

Sol.
$$= \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1.\frac{1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}-1}{\cancel{\sqrt{3}}} \times \frac{\cancel{\sqrt{3}}}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

iv. **Sin105° = Sin (60° + 45°)** **Multan 2008, Gujranawala 2009**

Sol. = Sin60° Cos45° + Cos60° Sin45°

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

v. **Cos105° = Cos (60° + 45°)** **Faisalabad 2007**

Sol. = Cos60° Cos45° – Sin60° Sin45°

$$= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

vi. **tan105° = tan (60° + 45°)**

Sol. $$\frac{\tan 60^o + \tan 45^o}{1 - \tan 60^o \tan 45^o} = \frac{\sqrt{3}+1}{1-\sqrt{3}.1} = \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

3. **Prove that**

i. **Sin (45° + α) = $\frac{1}{\sqrt{2}}$ (Sin + Cos α)** **Multan 2009**

Sol. L.H.S = Sin (45° + α) = Sin 45° Cos α + Cos 45° Sin α

$$= \frac{1}{\sqrt{2}} Cos\alpha + \frac{1}{\sqrt{2}} Sin\alpha = \frac{1}{\sqrt{2}}(Sin\alpha + Cos\alpha)$$

ii. **Cos (α + 45°) = $\frac{1}{\sqrt{2}}$ (Cos α – Sin α)** **Faisalabad 2007**

Sol. L.H.S = Cos (α + 45°) = Cos α Cos 45° – Sin α Sin45°

$$= \text{Cos}\alpha \ \frac{1}{\sqrt{2}} - Sin\alpha\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \ (\text{Cos}\alpha - \text{Sin}\alpha) = \text{R.H.S}$$

4. **Prove that**

i **tan (45° + A) tan (45° – A) = 1** **Lahore 2009**

Sol. L.H.S = tan (45° + A). tan (45° – A)

$$= \left(\frac{\tan 45^o + \tan A}{1 - \tan 45^o \tan A}\right) . \left(\frac{\tan 45^o - \tan A}{1 + \tan 45^o \tan A}\right) = \left(\frac{1+\tan A}{1-1.\tan A}\right)\left(\frac{1-\tan A}{1+1.\tan A}\right)$$

$$= \left(\frac{\cancel{1+\tan A}}{\cancel{1-\tan A}}\right)\left(\frac{\cancel{1-\tan A}}{\cancel{1+\tan A}}\right) = 1 = R.H.S$$

ii. $\tan\left(\frac{\pi}{4}-\theta\right)+\tan\left(\frac{3\pi}{4}+\theta\right)=0$

Sol. L.H.S $=\tan\left(\frac{\pi}{4}-\theta\right)+\tan\left(\frac{3\pi}{4}+\theta\right)$

$$=\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}+\frac{\tan\frac{3\pi}{4}+\tan\theta}{1-\tan\frac{3\pi}{4}\tan\theta}=\frac{1-\tan\theta}{1+1.\tan\theta}+\frac{-1+\tan\theta}{1-(-1)\tan\theta}$$

$$=\frac{1-\tan\theta}{1+\tan\theta}+\frac{-1+\tan\theta}{1+\tan\theta}$$

$$=\frac{\cancel{1}-\cancel{\tan\theta}-\cancel{1}+\cancel{\tan\theta}}{1+\tan\theta}=\frac{0}{1+\tan\theta}=0=R.H.S$$

iii. $\sin\left(\theta+\frac{\pi}{6}\right)+\cos\left(\theta+\frac{\pi}{3}\right)=\cos\theta$ **Lahore 2009**

Sol. L.H.S $=\sin\left(\theta+\frac{\pi}{6}\right)+\cos\left(\theta+\frac{\pi}{3}\right)$

$$=\sin\theta\cos\frac{\pi}{6}+\cos\theta\sin\frac{\pi}{6}+\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}$$

$$=\cancel{\sin\theta\frac{\sqrt{3}}{2}}+\cos\theta\ \frac{1}{2}+\cos\theta\ \frac{1}{2}-\cancel{\sin\theta\frac{\sqrt{3}}{2}}$$

$$=\cos\theta\left(\frac{1}{2}+\frac{1}{2}\right)=\cos\theta\ (1)=\cos\theta=\text{R.H.S}$$

iv. $\frac{\sin\theta-\cos\theta.\tan\theta/2}{\cos\theta+\sin\theta.\tan\theta/2}=\tan\theta/2$

Sol. L.H.S $=\frac{\sin\theta-\cos\theta.\tan\theta/2}{\cos\theta+\sin\theta.\tan\theta/2}$

$$=\frac{\sin\theta-\cos\theta.\frac{\sin\theta/2}{\cos\theta/2}}{\cos\theta+\sin\theta.\frac{\sin\theta/2}{\cos\theta/2}}=\frac{\frac{\sin\theta\cos\theta/2-\cos\theta\sin\theta/2}{\cos\theta/2}}{\frac{\cos\theta\cos\theta+\sin\theta\sin\theta/2}{\cos\theta/2}}$$

$$=\frac{\frac{Sin(\theta-\theta/2)}{Cos\theta/2}}{\frac{Cos(\theta-\theta/2)}{Cos\theta/2}}=\frac{Sin\theta/2}{\cancel{Cos\theta/2}}\times\frac{\cancel{Cos\theta/2}}{Cos\theta/2}=\tan\theta/2=R.H.S$$

v. $$\frac{1-\tan\theta.\tan\varphi}{1+\tan\theta.\tan\varphi}=\frac{Cos(\theta+\varphi)}{Cos(\theta-\varphi)}$$

Sol. L.H.S $$=\frac{1-\tan\theta.\tan\phi}{1+\tan\theta.\tan\phi}=\frac{1-\frac{Sin\theta}{Cos\theta}\frac{Sin\phi}{Cos\phi}}{1+\frac{Sin\theta}{Cos\theta}\frac{Sin\phi}{Sin\phi}}$$

$$=\frac{\frac{Cos\theta Cos\phi-Sin\theta Sin\phi}{Cos\theta Cos\phi}}{\frac{Cos\theta Cos\phi+Sin\theta Sin\phi}{Cos\theta Cos\phi}}=\frac{\frac{Cos(\theta+\phi)}{Cos\theta Cos\phi}}{\frac{Cos(\theta-\phi)}{Cos\theta Cos\phi}}=\frac{Cos(\theta+\varphi)}{\cancel{Cos\theta Cos\varphi}}\times\frac{\cancel{Cos\theta Cos\varphi}}{Cos(\theta-\varphi)}$$

$$=\frac{Cos(\theta+\phi)}{Cos(\theta-\phi)}$$

5. $\mathbf{Cos(\alpha+\beta)\ Cos(\alpha-\beta)=Cos^2\alpha-Sin^2\beta=Cos^2\beta-Sin^2\alpha}$

Sol. L.H.S $=Cos(\alpha+\beta).Cos(\alpha-\beta)$ **Rawalpindi 2009, Sargodha 2009**

$=(Cos\alpha\ Cos\beta-Sin\alpha\ Sin\beta).(Cos\alpha\ Cos\beta+Sin\alpha\ Sin\beta)$

$=(Cos\alpha\ Cos\beta)^2-(Sin\alpha\ Sin\beta)^2=Cos^2\alpha\ Cos^2\beta-Sin^2\alpha\ Sin^2\beta$ ________ ***I***

$=Cos^2\alpha\ (1-Sin^2\beta)-(1-Cos^2\alpha)Sin^2\beta$

$=Cos^2\alpha-\cancel{Cos^2\alpha Sin^2\beta}-Sin^2\beta+\cancel{Cos^2\alpha Sin^2\beta}$

$=Cos^2 1-Sin^2\beta$ ***Result I***

Again from ***I*** $Cos(\alpha+\beta).Cos(\alpha-\beta)=Cos^2\alpha\ Cos^2\beta-Sin^2\alpha\ Sin^2\beta$

$=(1-Sin^2\alpha)Cos^2\beta-Sin^2\alpha(1-Cos^2\beta)$

$=Cos^2\beta-\cancel{Sin^2\alpha Cos^2\beta}-Sin^2\alpha+\cancel{Sin^2\alpha Cos^2\beta}$

$=Cos^2\beta-Sin^2\alpha$ ***Result II***

6. $\dfrac{Sin(\alpha+\beta)+Sin(\alpha-\beta)}{Cos(\alpha+\beta)+Cos(\alpha-\beta)} = \tan\alpha$

Sol. L.H.S $= \dfrac{Sin(\alpha+\beta)+Sin(\alpha-\beta)}{Cos(\alpha+\beta)+Cos(\alpha-\beta)}$

$$= \frac{Sin\alpha\, Cos\beta + \cancel{Cos\alpha\, Sin\beta} + Sin\alpha\, Cos\beta - \cancel{Cos\alpha\, Sin\beta}}{Cos\alpha\, Cos\beta - \cancel{Sin\alpha\, Sin\beta} + Cos\alpha\, Cos\beta + \cancel{Sin\alpha\, Sin\beta}}$$

$$= \frac{\cancel{2}\, Sin\alpha\, \cancel{Cos\beta}}{\cancel{2}\, Cos\alpha\, \cancel{Cos\beta}} = \tan\alpha = \text{R.H.S}$$

7. Show that

i. $\text{Cot}(\alpha+\beta) = \dfrac{Cot\alpha\, Cot\beta - 1}{Cot\,\alpha + Cot\,\beta}$

Sol. R.H.S $= \dfrac{Cot\alpha\, Cot\beta - 1}{Cot\,\alpha + Cot\,\beta} = \dfrac{\dfrac{1}{\tan\alpha\,\tan\beta} - 1}{\dfrac{1}{\tan\alpha} + \dfrac{1}{\tan\beta}} = \dfrac{\dfrac{1-\tan\alpha\tan\beta}{\tan\alpha\tan\beta}}{\dfrac{\tan\alpha+\tan\beta}{\tan\alpha\tan\beta}}$

$$= \frac{1-\tan\alpha\tan\beta}{\cancel{\tan\alpha\tan\beta}} \times \frac{\cancel{\tan\alpha\tan\beta}}{\tan\alpha+\tan\beta}$$

$$= \frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta} = \frac{1}{\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}} = \frac{1}{\tan(\alpha+\beta)} = \text{Cot}(\alpha+\beta) = \text{L.H.S}$$

ii. $\text{Cot}(\alpha-\beta) = \dfrac{Cot\alpha\, Cot\beta + 1}{Cot\,\alpha - Cot\,\beta}$ **Multan 2008**

Sol. Let R.H.S $= \dfrac{Cot\alpha\, Cot\beta + 1}{Cot\,\beta - Cot\alpha}$

$$= \frac{\dfrac{1}{\tan\alpha\,\tan\beta} + 1}{\dfrac{1}{\tan\beta} - \dfrac{1}{\tan\alpha}} = \frac{\dfrac{1+\tan\alpha\tan\beta}{\tan\alpha\tan\beta}}{\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}} = \frac{1+\tan\alpha\tan\beta}{\cancel{\tan\alpha\tan\beta}} \times \frac{\cancel{\tan\alpha\tan\beta}}{\tan\alpha-\tan\beta}$$

$$= \frac{1+\tan\alpha\tan\beta}{\tan\alpha - \tan\beta} = \frac{1}{\frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}} = \frac{1}{\tan(\alpha-\beta)} = \cot(\alpha-\beta) = \text{L.H.S}$$

iii. $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta} = \dfrac{Sin(\alpha+\beta)}{Sin(\alpha-\beta)}$

Sol. L.H.S $= \dfrac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta} = \dfrac{\frac{Sin\alpha}{Cos\alpha} + \frac{Sin\beta}{Cos\beta}}{\frac{Sin\alpha}{Cos\alpha} - \frac{Sin\beta}{Cos\beta}} = \dfrac{\frac{Sin\alpha Cos\beta + Cos\alpha Sin\beta}{Cos\alpha Cos\beta}}{\frac{Sin\alpha Cos\beta - Cos\alpha Sin\beta}{Cos\alpha Cos\beta}}$

$$= \frac{Sin(\alpha+\beta)}{\cancel{Cos\alpha Cos\beta}} \times \frac{\cancel{Cos\alpha Cos\beta}}{Sin(\alpha-\beta)} = \frac{Sin(\alpha+\beta)}{Sin(\alpha-\beta)} = \text{R.H.S}$$

8. If $\text{Sin}\,\alpha = \dfrac{4}{5}$, $\text{Cos}\,\beta = \dfrac{40}{41}$ $0<\alpha<\dfrac{\pi}{2}$, $0<\beta<\dfrac{\pi}{2}$, Show that $\sin(\alpha-\beta)=133/205$

Sol. $\text{Cos}^2\alpha = 1-\text{Sin}^2\alpha = 1-\left(\dfrac{4}{5}\right)^2 = 1-\dfrac{16}{25} = \dfrac{25-16}{25}$ (Its mean α & β are in I quad)

$\text{Cos}^2\alpha = \dfrac{9}{25} \Rightarrow \text{Cos}\,\alpha = \pm\dfrac{3}{5} \Rightarrow \text{Cos}\,\alpha = \dfrac{3}{5}$ (Because α is in I quad)

$\text{Sin}^2\beta = 1-\text{Cos}^2\beta = 1-\left(\dfrac{40}{41}\right)^2 = 1-\dfrac{1600}{1681} = \dfrac{1681-1600}{1681} = \dfrac{81}{1681}$

$\text{Sin}\,\beta = \pm\dfrac{9}{41} \Rightarrow \text{Sin}\,\beta = \dfrac{9}{41}$ (Because β is in I quad)

Now $\text{Sin}(\alpha-\beta) = \sin\beta\,\text{Cos}\,\beta - \text{Cos}\,\alpha\,\text{Sin}\,\beta$

$$= \left(\frac{4}{5}\right)\left(\frac{40}{41}\right) - \left(\frac{3}{5}\right)\left(\frac{9}{41}\right) = \frac{160}{205} - \frac{27}{205} = \frac{160-27}{205} = \frac{133}{205}$$

9. If $\text{Sin}\,\alpha = \dfrac{4}{5}$, $\text{Sin}\,\beta = \dfrac{12}{13}$ & $\dfrac{\pi}{2}<\alpha<\pi$ $(\alpha\ in\ II)$, $\dfrac{\pi}{2}<\beta<\pi$ (β in II) then find

i. $\text{Sin}(\alpha+\beta)$ ii. $\text{Cos}(\alpha+\beta)$ iii. $\tan(\alpha+\beta)$

iv. $\text{Sin}(\alpha-\beta)$ v. $\text{Cos}(\alpha-\beta)$ vi. $\tan(\alpha-\beta)$

Sol. $\cos^2\alpha = 1-\sin^2\alpha = 1-\left(\frac{4}{5}\right)^2 = 1-\frac{16}{25} = \frac{9}{25}$ **Sargodha 2009**

$\cos\alpha = \pm\frac{3}{5} \Rightarrow \cos\alpha = -\frac{3}{5}$ (Because α is in II)

$\cos^2\beta = 1-\sin^2\beta = 1-\left(\frac{12}{13}\right)^2$

$= 1-\frac{144}{169} = \frac{169-144}{169} = \frac{25}{169} \Rightarrow \cos\beta = \pm\frac{5}{13}$

$\cos\beta = -\frac{5}{13}$ (Because β is in II quad)

$\tan\alpha = \frac{Sin\alpha}{Cos\alpha} = \frac{4/\cancel{5}}{-3/\cancel{5}}$

$\tan\alpha = -4/3$

$\tan\beta = \frac{Sin\beta}{Cos\beta} = \frac{12/\cancel{13}}{-5/\cancel{13}}$

$\tan\beta = -12/5$

(i) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Sol. $= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right)$

$= -\frac{20}{65} - \frac{36}{65} = \frac{-20-36}{65} = \frac{-56}{65}$

ii. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Sol. $= \left(\frac{-3}{5}\right)\left(\frac{-5}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) = \frac{15}{65} - \frac{48}{65} = \frac{15-48}{65} = \frac{-33}{65}$

iii. $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

Sol. $= \frac{\left(\frac{-4}{3}\right)+\left(\frac{-12}{5}\right)}{1-\left(\frac{-4}{3}\right)\left(\frac{-12}{5}\right)} = \frac{\frac{-4}{3}-\frac{12}{5}}{1-\frac{48}{15}} = \frac{\frac{-20-36}{15}}{\frac{15-48}{15}} = \frac{-56}{15}\times\frac{15}{-33} = \frac{56}{33}$

iv. $\text{Sin}(\alpha-\beta)=\text{Sin}\,\alpha\ \text{Cos}\,\beta\ -\text{Cos}\,\alpha\ \text{Sin}\,\beta$

Sol. $=(\frac{4}{5})(-\frac{5}{13})-(-\frac{3}{5})(\frac{12}{13})=\frac{-20}{65}+\frac{36}{65}=\frac{16}{65}$

v. $\text{Cos}(\alpha-\beta)=\text{Cos}\,\alpha\ \text{Cos}\,\beta\ +\text{Sin}\,\alpha\ \text{Sin}\,\beta\ =(\frac{-3}{5})(\frac{-5}{13})-(\frac{4}{5})(\frac{12}{13})$

Sol. $=\frac{15}{65}+\frac{48}{65}=\frac{63}{65}$

vi. $\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\frac{\left(\frac{-4}{3}\right)-\left(\frac{-12}{5}\right)}{1+\left(\frac{-4}{3}\right)\left(\frac{-12}{5}\right)}=\frac{\frac{-4}{3}+\frac{12}{5}}{1+\frac{48}{15}}$

Sol. $\frac{\frac{-20+36}{15}}{\frac{15+48}{15}}=\frac{16}{\cancel{15}}\times\frac{\cancel{15}}{63}=\frac{16}{63}$

$\boxed{\alpha+\beta\ \text{in III quad } and\ \alpha-\beta\ \text{in I quad}}$

10. Find sin $(\alpha+\beta)$ and cos$(\alpha+\beta)$ given that

(1). $\tan\alpha=\frac{3}{4}$, $\text{Cos}\,\beta=\frac{5}{13}$, α in III Quad. β in IV Quad.

Sol. $1+\tan^2\alpha=\text{Sec}^2\alpha\Rightarrow 1+\frac{9}{16}=\text{Sec}^2\alpha=\frac{25}{16}\Rightarrow \text{Sec}\,\alpha=\pm\frac{5}{4}$

$\Rightarrow \text{Cos}\,\alpha=\pm\frac{4}{5}\Rightarrow \text{Cos}\,\alpha=\frac{-4}{5}$ (Because α in III)

$\text{Sin}^2\alpha=1-\text{Cos}^2\alpha=1-\left(\frac{-4}{5}\right)^2=1-\frac{16}{25}=\frac{9}{25}\Rightarrow \text{Sin}\,\alpha=\pm\frac{3}{5}$

$\text{Sin}\,\alpha=-\frac{3}{5}$ (Because α is in III)

$\text{Sin}^2\beta=1-\text{Cos}^2\beta=1-\left(\frac{5}{13}\right)^2=1-\frac{25}{169}=\frac{169-25}{169}=\frac{144}{169}=\ \text{Sin}\,\beta=\pm\frac{12}{13}$

$\text{Sin}\,\beta=\frac{-12}{13}$ (Because β is in IV)

i. **$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$**

Sol. $= \left(\frac{-3}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{-4}{5}\right)\left(\frac{-12}{13}\right) = \frac{-15}{65} + \frac{48}{65} = \frac{-15+48}{65} = \frac{33}{65}$

ii. **$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$**

Sol. $\left(\frac{-4}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{-3}{5}\right)\left(\frac{-12}{13}\right) = \frac{-20}{65} - \frac{36}{65} = \frac{-20-36}{65} = \frac{-56}{65}$

10 (2) $\tan\alpha = -\frac{15}{8}, \sin\beta = -\frac{7}{25}$ $(\alpha\ in\ II, \beta\ in\ III)$

Sol. $1+\tan^2\alpha = \sec^2\alpha \Rightarrow 1 + \left(\frac{-15}{8}\right)^2 = \sec^2\alpha$ $\boxed{\begin{array}{l}\textit{Given } \alpha \textit{ not in IV and } \tan\alpha = -ve \\ \textit{so } \alpha \textit{ in II Similarly } \beta \textit{ in III}\end{array}}$

$1+\frac{225}{16} = Sec^2\alpha \Rightarrow \frac{289}{64} = Sec^2\alpha \Rightarrow \sec\alpha = \pm\frac{17}{8} \Rightarrow Cos\alpha = \pm\frac{8}{17}$

$\boxed{\cos\alpha = -8/17}$ (Because α is in II)

$\sin^2\alpha = 1-\cos^2\alpha = 1-\left(\frac{-8}{176}\right)^2 = 1-\frac{64}{289} = \frac{225}{289} \Rightarrow Sin\alpha = \pm\frac{15}{17}$

$\boxed{\sin\alpha = \frac{15}{17}}$ (Because α is in II)

$\cos^2\beta = 1-\sin^2\beta = 1-\left(\frac{-7}{25}\right)^2 = 1-\frac{49}{625} = \frac{625-64}{625} = \frac{576}{625}$

$\cos\beta = \pm\frac{24}{25} \Rightarrow \boxed{\cos\beta = -\frac{24}{25}}$ (Because β is in III)

Now

i. **$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$**

Sol. $= \left(\frac{15}{17}\right)\left(\frac{-24}{25}\right) + \left(\frac{-8}{17}\right)\left(\frac{-7}{25}\right) = \frac{-360}{425} + \frac{56}{425}$

$= \frac{-360+56}{425} = \frac{-304}{425}$

ii. $\text{Cos}(\alpha+\beta) = \text{Cos}\,\alpha\ \text{Cos}\,\beta - \text{Sin}\,\alpha\ \text{Sin}\,\beta = \left(\frac{-8}{17}\right)\left(\frac{-24}{25}\right) - \left(\frac{15}{17}\right)\left(\frac{-7}{25}\right)$

Sol. $= \frac{192}{425} + \frac{105}{425} = \frac{192+105}{425} = \frac{297}{425}$

11. $\frac{Cos8^o - Sin8^o}{Cos8^o + Sin8^o} = \tan 37^o$ **Multan 2008, Sargodha 2008, Lahore 2009**

Sol. R.H.S = $\tan 37^\circ = \tan(45^\circ - 8^\circ) = \frac{\tan 45^o - \tan 8^o}{1+\tan 45^o \tan 8^o} = \frac{1-\tan 8^o}{1+1.\tan 8^o}$

$$= \frac{1-\tan 8^o}{1+\tan 8^o} = \frac{1-\frac{Sin8^o}{Cos8^o}}{1+\frac{Sin8^o}{Cos8^o}} = \frac{\frac{Cos8^o - Sin8^o}{Cos8^o}}{\frac{Cos8^o + Sin8^o}{Cos8^o}}$$

$$= \frac{Cos8^o - Sin8^o}{\cancel{Cos8^o}} \times \frac{\cancel{Cos8^o}}{Cos8^o + Sin8^o} = \frac{Cos8^o - Sin8^o}{Cos8^o + Sin8^o} = \text{L.H.S}$$

12. $\text{Cot}\,\alpha/2 + \text{Cot}\,\beta/2 + \text{Cot}\,\gamma/2 = \text{Cot}\,\alpha/2\,\text{Cot}\,\beta/2\,\text{Cot}\,\gamma/2$ **Federal**

Sol. We know that $\alpha + \beta + \gamma = 180^\circ \Rightarrow \alpha + \beta = 180^\circ - \gamma$

Divide both side by '2' $\frac{\alpha+\beta}{2} = \frac{180^o - \gamma}{2} \Rightarrow \frac{\alpha+\beta}{2} = 90^o - \frac{\gamma}{2} \Rightarrow \frac{\alpha}{2} + \frac{\beta}{2} = 90^o - \frac{\gamma}{2}$

$\tan(\alpha/2 + \beta/2) = \tan(90^\circ - \gamma/2) \Rightarrow \frac{\tan\alpha/2 + \tan\beta/2}{1-\tan\alpha/2\tan\beta/2} = \text{Cot}\,\gamma/2$

$$\frac{\frac{1}{Cot\alpha/2} + \frac{1}{Cot\beta/2}}{1 - \frac{1}{Cot\alpha/2\,Cot\beta/2}} = \text{Cot}\,\gamma/2 \Rightarrow \frac{\frac{Cot\frac{\alpha}{2} + Cot\frac{\beta}{2}}{Cot\frac{\alpha}{2}\,Cot\frac{\beta}{2}}}{\frac{Cot\frac{\alpha}{2}\,Cot\frac{\beta}{2} - 1}{Cot\frac{\alpha}{2}\,Cot\frac{\beta}{2}}} = \text{Cot}\,\frac{\gamma}{2}$$

$$\frac{Cot\alpha/2 + Cot\beta/2}{\cancel{Cot\alpha/2Cot\beta/2}} \times \frac{\cancel{Cot\alpha/2Cot\beta/2}}{Cot\alpha/2Cot\beta/2-1} = \text{Cot}\gamma/2$$

Cot α /2 + Cot β /2 = Cot γ /2 (Cot α /2Cot β /2 – 1)

Cot α /2 + Cot β /2 = Cot α /2Cot β /2Cot γ /2 – Cot γ /2

Cot α /2 + Cot β /2 + Cot γ /2 = Cot α /2Cot β /2Cot γ /2

13. **Cot α Cot β + Cot β Cot γ + Cot γ Cot γ = 1** **Faisalabad 2007, 08 Sargodha 2006,10**

Sol. α , β , γ are angle of triangle then

$\alpha + \beta + \gamma = 180^\circ \Rightarrow \alpha + \beta = 180^\circ - \gamma \Rightarrow \tan(\alpha + \beta) = \tan(180^\circ - \gamma)$

$$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = -\tan\gamma \Rightarrow \frac{\frac{1}{Cot\alpha} + \frac{1}{Cot\beta}}{1 - \frac{1}{Cot\alpha Cot\beta}} = -\frac{1}{Cot\gamma}$$

$$\frac{\frac{Cot\alpha + Cot\beta}{Cot\alpha Cot\beta}}{\frac{Cot\alpha Cot\beta - 1}{Cot\alpha Cot\beta}} = -\frac{1}{Cot\gamma} \Rightarrow \frac{Cot\alpha + Cot\beta}{\cancel{Cot\alpha Cot\beta}} \times \frac{\cancel{Cot\alpha Cot\beta}}{Cot\alpha\, Cot\beta - 1} = -\frac{1}{Cot\gamma}$$

$(Cot\alpha + Cot\beta)(Cot\gamma) = -(Cot\alpha\, Cot\beta - 1)$

$Cot\alpha\, Cot\gamma + Cot\beta Cot\gamma = -Cot\alpha Cot\beta + 1$

$Cot\alpha\, Cot\beta + Cot\beta\, Cot\gamma + Cot\gamma\, Cot\alpha = 1$

14. **Express the following in the form of r sin ($\theta + \varphi$)**

i. **12Sin θ + 5Cos θ**

Sol. Put 12 = rCos φ & 5 = rSin φ then

12Sin θ + 5Cos θ = rCos φ Sin θ + rSin φ Cos θ

= r (Sin θ Cos φ + Cos θ Sin φ) = 13 Sin ($\theta + \varphi$)

Where r = 13 and tan θ = $\frac{5}{12}$

As $r^2Cos^2\varphi + r^2 Sin^2\varphi = (12)^2 + (5)^2$

$r^2(Cos^2\varphi + Sin^2\varphi) = 144 + 25$

or $r^2 = 169 \Rightarrow r = 13$

& $\frac{\cancel{r}Sin\varphi}{\cancel{r}Cos\varphi} = \frac{5}{12} \Rightarrow Tan\varphi = \frac{5}{12}$

ii. **$3Sin\theta - 4Co\theta = 3Sin\theta + (-4\,Cos\theta)$** **Sargodha 2011**

Sol. Put $3 = r\,Cos\varphi$ & $-4 = r\,Sin\varphi$

Then $3Sin\theta - 4Cos\theta = rCos\varphi\ Sin\theta + rSin\varphi\ Cos\theta$

$r\,[Sin\theta\ Cos\varphi + Cos\theta\ Sin\varphi] = rSin\,(\theta + \varphi) = 5Sin(\theta + \varphi)$

Where $r^2\,(Cos^2\theta + Sin^2\theta) = (3)^2 + (-4)^2 \Rightarrow r^2 = 9 = 16 \Rightarrow r^2 = 25 \Rightarrow r=5$

and $\frac{rSin\theta}{rCos\theta} = \frac{-4}{3} \Rightarrow tan\theta = \frac{-4}{3}$

iii. **$Sin\theta - Cos\theta = (1)\,Sin\theta + (-1)\,Cos\theta$**

Sol. Put $1 = r\,Cos\varphi$ & $-1 = rSin\varphi$

$= r\,Cos\varphi\,Sin\theta + rSin\varphi\,Cos\theta$

$= r(Sin\theta\ Cos\varphi + Cos\theta\ Sin\varphi)$

$= rSin\,(\theta + \varphi) = \sqrt{2}\ Sin\,(\theta + \varphi)$

Where $r = \sqrt{2}$ and $tan\varphi = -1$

As $r^2\,(Cos^2\theta + Sin^2\theta) = (1)^2 + (-1)^2$

$r^2 = 2 \Rightarrow r = \sqrt{2}$

$\frac{rSin\theta}{rCos\theta} = \frac{-1}{1} \Rightarrow tan\theta = -1$

iv. **$5Sin\theta - 4Cos\theta = 5Sin\theta + (-4)\,Cos\theta$**

Sol. Put $5 = rCos\varphi$ & $-4 = rSin\varphi$ then

$= rCos\varphi\,Sin\theta + rSin\phi\,Cos\theta = r(sin\theta\,Cos\varphi + cos\theta\,sin\varphi) = rSin(\theta + \varphi)$

$= \sqrt{41}\ Sin\,(\theta + \varphi)$

$$r^2(\cos^2\theta + \sin^2\theta) = (5)^2 + (-4)^2$$

$$r^2 = 25 + 16 \Rightarrow r^2 = 41 \Rightarrow r = \sqrt{41}$$

$$\frac{r\sin\theta}{r\cos\theta} = \frac{-4}{5} \Rightarrow \tan\theta = -\frac{4}{5}$$

v. $\sin\theta + \cos\theta = (1)\sin\theta + (1)\cos\theta$

Sol. Put $r\cos\varphi = 1$, $r\sin\varphi = 1$

$= r\cos\varphi\sin\theta + r\sin\varphi\cos\theta$

$= r\sin(\theta + \varphi) = \sqrt{2}\ \sin(\theta + \theta)$

$$r^2(\cos^2\theta + \sin^2\theta) = (1)^2 + (1)^2 = 2$$

$$\Rightarrow r = \sqrt{2}$$

$$and\ \frac{r\sin\varphi}{r\cos\varphi} = \frac{1}{1} \Rightarrow \tan\varphi = 1$$

vi. $3\sin\theta - 5\cos\theta = 3\sin\theta + (-5)\cos\theta$

Sol. Put $3 = r\cos\varphi$ & $-5 = r\sin\varphi$

$= r\cos\varphi\ \sin\theta + r\sin\varphi\ \cos\theta = \gamma\ \sin(\theta + \varphi)$

Where $r = \sqrt{34}$ and $\tan\theta = -5/3$

As $r^2(\cos^2\theta + \sin^2\theta) = (3)^2 + (-5)^2$

$$r^2 = 9 + 25 = 34 \Rightarrow r = \sqrt{34}$$

$$\frac{r\sin\theta}{r\cos\theta} = \frac{-5}{3} \Rightarrow \tan\theta = \frac{-5}{3}$$

Double Angle identities.

Therom i. **Prove that $\sin 2\alpha = 2\sin\alpha\cos\alpha$**

Sol. $\sin 2\alpha = \sin(\alpha + \alpha)$

$= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$

$= \sin\alpha\cos\alpha + \sin\alpha\cos\alpha$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$ Hence proved

Similarly $\sin\alpha = 2\sin\alpha/2\ \cos\alpha/2$

Therom ii. **Prove that $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$**

Sol. $\cos 2\alpha = \cos(\alpha + \alpha)$

$= \text{Cos}\,\alpha\ \text{Cos}\,\alpha - \text{Sin}\,\alpha\ \text{Sin}\,\alpha$

$= \text{Cos}^2\alpha - \text{Sin}^2\alpha$

$\text{Cos}^2\alpha = \text{Cos}^2\alpha - \text{Sin}^2\alpha$ Hence Proved

Similarly $\text{Cos}\,\alpha = \text{Cos}^2\alpha/2 - \text{Sin}^2\alpha/2$

Therom iii. **Prove that** $\text{Tan}2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

Sol. $\tan 2\alpha = \tan(\alpha + \alpha)$

$$= \frac{\tan\alpha + \tan\alpha}{1-\tan\alpha\tan\alpha} = \frac{2\tan\alpha}{1-\tan^2\alpha}$$ Hence Proved

Similarly $\tan\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha/2}$

Example 3. **Gujranwala 2009, Rawalpindi 2009, Federal**

Reduce $\text{Cos}^4\theta$ to an expression involving only function of multiples of θ raised to the first power.

Sol. $\text{Cos}^4\theta = (\text{Cos}^2\theta)^2$

$$= \left(\frac{1+Cos2\theta}{2}\right)^2 \qquad \because \text{Cos}^2\theta = \frac{1+Cos2\theta}{2}$$

$$= \frac{1}{4}(1+\text{Cos}2\theta)^2$$

$$= \frac{1}{4}[1+2\text{Cos}2\theta + \text{Cos}^2 2\theta]$$

$$= \frac{1}{4}[1+2\text{Cos}2\theta + \frac{1+Cos4\theta}{2}] \quad \because \text{Cos}^2 2\theta = \frac{1+Cos4\theta}{2}$$

$$= \frac{1}{4}[\frac{2+4Cos2\theta+1+Cos4\theta}{2}]$$

$$= \frac{1}{8}[3+4\text{Cos}2\theta + \text{Cos}4\theta]$$

Double Angle Formulas **Double Angle Formulas**

i) $\sin 2\theta = 2\sin\theta\cos\theta \Rightarrow \sin\theta = 2\sin\theta/2\cos\theta/2$

ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta \Rightarrow \cos\theta = \cos^2\theta/2 - \sin^2\theta/2$

$\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow \cos\theta = 2\cos^2\theta/2 - 1$

$\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow \cos\theta = 1 - 2\sin^2\theta/2$

iii) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} \Rightarrow \tan\theta = \dfrac{2\tan\theta/2}{1-\tan^2\theta/2}$

iv) $1 - \cos 2\theta = 2\sin^2\theta \Rightarrow 1 - \cos\theta = 2\sin^2\theta/2$

v) $1 + \cos 2\theta = 2\cos^2\theta \Rightarrow 1 + \cos\theta = 2\cos^2\theta/2$

EXERCISE. 10.3

1. i $\sin\alpha = \dfrac{12}{13}, 0 < \alpha < \dfrac{\pi}{2}$ **Sargodha 2010**

Sol. $\cos^2\alpha = 1 - \sin^2\alpha$

$$= 1 - \left(\frac{12}{13}\right)^2 = 1 - \frac{144}{169}$$

$$= \frac{169-144}{169} = \frac{25}{169} \Rightarrow \cos\alpha = \pm\frac{5}{13}$$

$\cos\alpha = \dfrac{5}{13}$ (Because α is in I quad)

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{12/13}{5/13} = \frac{12}{\cancel{13}} \times \frac{\cancel{13}}{5} = \frac{12}{5}$$

Now $\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(\dfrac{12}{13}\right)\left(\dfrac{5}{13}\right) = \dfrac{120}{169}$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ **Multan 2008**

$$= \left(\frac{5}{13}\right)^2 - \left(\frac{12}{13}\right)^2 = \frac{25}{169} - \frac{144}{169} = \frac{25-144}{169} = \frac{-119}{169}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(\frac{12}{5}\right)}{1-\left(\frac{12}{5}\right)^2}$$

$$= \frac{24/5}{1-\frac{144}{25}} = \frac{24/5}{\frac{25-144}{25}} = \frac{24}{5} \times \frac{25}{-119} = \frac{-120}{119}$$

ii. $\mathbf{Cos\alpha = \frac{3}{5}, 0 < \alpha \ \pi/2}$

Sol. $Sin^2\alpha = 1 - Cos^2\alpha = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{25-9}{25} = \frac{16}{25}$

$Sin\alpha = \pm\frac{4}{5} \Rightarrow Sin\alpha = \frac{4}{5}$ (Because α is in I)

$$\tan\alpha = \frac{Sin\alpha}{Cos\alpha} = \frac{4/5}{3/5} = \frac{4}{\cancel{5}} \times \frac{\cancel{5}}{3} = \frac{4}{3}$$

i) $Sin2\alpha = 2Sin\alpha\, Cos\alpha = 2(4/5)(3/5) = \frac{24}{25}$

ii) $Cos2\alpha = Cos^2\alpha - Sin\alpha = \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = \frac{9-16}{25} = \frac{-7}{25}$

iii) $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2} = \frac{8/3}{1-\frac{16}{9}} = \frac{8/3}{\frac{9-16}{9}} = \frac{8/3}{-7/9} = \frac{8}{3} \times \frac{9}{-7} = \frac{-24}{7}$

2. $\mathbf{Cot\alpha - \tan\alpha = 2Cot2\alpha}$ **Faisalabad 2007, Multan 2009**

Sol. L.H.S = $Cot\alpha - \tan\alpha = \frac{Cos\alpha}{Sin\alpha} - \frac{Sin\alpha}{Cos\alpha} = \frac{Cos^2\alpha - Sin^2\alpha}{Sin\alpha Cos\alpha}$

$\frac{2(Cos^2\alpha - Sin^2\alpha)}{2Sin\alpha Cos\alpha}$ ('X' & ÷ by 2) = $\frac{2Cos2\alpha}{Sin2\alpha}$ = 2 $Cot2\alpha$ = R.H.S

3. $\mathbf{\frac{Sin2\alpha}{1+Cos2\alpha} = \tan\alpha}$ **Multan 2007,09 Gujranwala 2009, Sargodha 2008**

Sol. L.H.S = $\frac{Sin2\alpha}{1+Cos2\alpha} = \frac{2Sin\alpha Cos\alpha}{2Cos^2\alpha} = \frac{Sin\alpha}{Cos\alpha} = \tan\alpha$ = R.H.S

4. $\dfrac{1 - Cos\alpha}{Sin\alpha} = \tan\dfrac{\alpha}{2}$ **Sargodha 2008, 09**

Sol. $$\text{L.H.S} = \frac{1-Cos\alpha}{Sin\alpha} = \frac{2Sin^2\frac{\alpha}{2}}{2Sin\frac{\alpha}{2}\,Cos\frac{\alpha}{2}} = \frac{Sin\frac{\alpha}{2}}{Cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2} = \text{R.H.S}$$

5. $\dfrac{Cos\alpha - Sin\alpha}{Cos\alpha + Sin\alpha} = \text{Sec}2\alpha - \tan 2\alpha$

Sol. $$\text{L.H.S} = \frac{Cos\alpha - Sin\alpha}{Cos\alpha + Sin\alpha} = \frac{Cos\alpha - Sin\alpha}{Cos\alpha + Sin\alpha} \times \frac{Cos\alpha - Sin\alpha}{Cos\alpha - Sin\alpha}$$

$$= \frac{(Cos\alpha - Sin\alpha)^2}{Cos^2\alpha - Sin^2\alpha} = \frac{Cos^2\alpha + Sin^2\alpha - 2Sin\alpha Cos\alpha}{Cos2\alpha} = \frac{1-Sin2\alpha}{Cos2\alpha}$$

$$= \frac{1}{Cos2\alpha} - \frac{Sin2\alpha}{Cos2\alpha} = Sec2\alpha - \tan 2\alpha = R.H.S$$

6. $\sqrt{\dfrac{1+Sin\alpha}{1-Sin\alpha}} = \dfrac{Sin\alpha/2 + Cos\alpha/2}{Sin\alpha/2 - Cos\alpha/2}$

Sol. $$\text{L.H.S} = \sqrt{\frac{1+Sin\alpha}{1-Sin\alpha}} = \sqrt{\frac{Sin^2\alpha/2 + Cos^2\alpha/2 + 2Sin\alpha/2Cos\alpha/2}{Sin^2\alpha/2 + Cos^2\alpha/2 - 2Sin\alpha/2Cos\alpha/2}}$$

$$= \sqrt{\frac{(Sin\alpha/2 + Cos\alpha/2)^2}{(Sin\alpha/2 - Cos\alpha/2)^2}} = \frac{Sin\alpha/2 + Cos\alpha/2}{Sin\alpha/2 - Cos\alpha/2} = \text{R.H.S}$$

7. $\dfrac{Cos\theta + 2Co\sec 2\theta}{Sec\theta} = \text{Cos}\,\theta/2$

Sol. $$\text{L.H.S} = \frac{Cos\theta + 2Co\sec 2\theta}{Sec\theta} = \frac{\frac{1}{Sin\theta} + \frac{2}{Sin2\theta}}{\frac{1}{Cos\theta}} = \frac{\frac{1}{Sin\theta} + \frac{\cancel{2}}{\cancel{2}SinCos\theta}}{\frac{1}{Cos\theta}}$$

$$= \left(\frac{Cos\theta + 1}{Sin\theta\,\cancel{Cos\theta}}\right)\frac{\cancel{Cos\theta}}{1} = \frac{\cancel{2}Cos^2\theta/2}{\cancel{2}Sin\theta/2Cos\theta/2} = \frac{Cos\theta/2}{Sin\theta/2} = \text{Cot}\,\theta/2 = \text{R.H.S}$$

8. $1 + \tan\alpha \tan 2\alpha = \text{Sec} 2\alpha$

Sol. L.H.S $= 1 + \tan\alpha \tan 2\alpha = 1 + \dfrac{Sin\alpha}{Cos\alpha} \cdot \dfrac{Sin2\alpha}{Cos2\alpha} = \dfrac{Cos\alpha Cos2\alpha + Sin\alpha Sin2\alpha}{Cos\alpha Cos2\alpha}$

$$= \frac{Cos\alpha(1-2Sin^2\alpha) + Sin\alpha.2Sin\alpha Cos\alpha}{Cos\alpha Cos2\alpha} = \frac{\cancel{Cos\alpha}[1 - \cancel{2Sin^2\alpha} + \cancel{2Sin^2\alpha}]}{\cancel{Cos\alpha} Cos2\alpha}$$

$$= \frac{1}{Cos2\alpha} = \text{Sec}2\alpha = \text{R.H.S}$$

9. $\dfrac{2Sin\theta\, Sin2\theta}{Cos\theta + Cos3\theta} = \tan 2\theta \tan\theta$

Sol. L.H.S $= \dfrac{2Sin\theta\, Sin2\theta}{Cos\theta + Cos3\theta}$ $(\text{Cos}3\theta = 4\text{Cos}^3\theta - 3\,\text{Cos}\theta)$

$$= \frac{2Sin\theta\, Sin2\theta}{Cos\theta + 4Cos^3\theta - 3Cos\theta} = \frac{2Sin\theta\, Sin2\theta}{4Cos^3\theta - 2Cos\theta} = \frac{\cancel{2}Sin\theta\, Sin2\theta}{\cancel{2}Cos\theta(2Cos^2\theta - 1)}$$

$$= \frac{Sin\theta.Sin2\theta}{Cos\theta.Cos2\theta} = \tan\theta \tan 2\theta = \tan 2\theta \,.\, \tan\theta = \text{R.H.S}$$

10. $\dfrac{Sin3\theta}{Sin\theta} - \dfrac{Cos3\theta}{Cos\theta} = 2$ **Faisalabad 2009**

Sol. L.H.S $= \dfrac{Sin3\theta}{Sin\theta} - \dfrac{Cos3\theta}{Cos\theta} = \dfrac{Sin3\theta Cos\theta - Cos3\theta Sin\theta}{Sin\theta Cos\theta} = \dfrac{Sin(3\theta - \theta)}{Sin\theta Cos\theta}$

$$= \frac{Sin2\theta}{Sin\theta Cos\theta} = \frac{2\cancel{Sin\theta Cos\theta}}{\cancel{Sin\theta Cos\theta}} = 2 = \text{R.H.S}$$

11. $\dfrac{Cos3\theta}{Cos\theta} + \dfrac{Cos3\theta}{Sin\theta} = 4\,\text{Cos}2\theta$ **Federal**

Sol. L.H.S $= \dfrac{Cos3\theta}{Cos\theta} + \dfrac{Cos3\theta}{Sin\theta} = \dfrac{Sin\theta Cos3\theta + Cos\theta Sin3\theta}{Cos\theta Sin\theta}$

$$= \frac{Sin(\theta + 3\theta)}{Sin\theta Cos\theta} = \frac{2Sin4\theta}{2Sin\theta Cos\theta} = \frac{2Sin2(2\theta)}{Sin2\theta}$$

$$= \frac{2.2\cancel{Sin2\theta}Cos2\theta}{\cancel{Sin2\theta}} = 4\,\text{Cos}2\theta = \text{R.H.S}$$

12. $$\frac{\tan\theta/2 + Cot\theta/2}{Cot\theta/2 - \tan\theta/2} = Sec\,\theta$$

Sol. L.H.S = $$\frac{\tan\theta/2 + Cot\theta/2}{Cot\theta/2 - \tan\theta/2} = \frac{\frac{Sin\theta/2}{Cos\theta/2} + \frac{Cos\theta/2}{Sin\theta/2}}{\frac{Cos\theta/2}{Sin\theta/2} - \frac{Sin\theta/2}{Cos\theta/2}}$$

$$= \frac{\frac{Sin^2\theta/2 + Cos^2\theta/2}{Sin\theta/2\,Cos\theta/2}}{\frac{Cos^2\theta/2 - Sin^2\theta/2}{Sin\theta/2\,Cos\theta/2}} = \frac{1}{\cancel{Cos\theta/2\,Sin\theta/2}} \times \frac{\cancel{Sin\theta/2\,Cos\theta/2}}{Cos^2\theta/2 - Sin^2\theta/2}$$

$$= \frac{1}{Cos\theta} = Sec\,\theta = R.H.S$$

13. $$\frac{Sin3\theta}{Cos\theta} + \frac{Cos3\theta}{Sin\theta} = 2Cot2\,\theta$$

Sol. L.H.S = $$\frac{Sin3\theta}{Cos\theta} + \frac{Cos3\theta}{Sin\theta} = \frac{Sin3\theta\,Sin\theta + Cos3\theta\,Cos\theta}{Sin\theta\,Cos\theta}$$

$$= \frac{Cos3\theta\,Cos\theta + Sin3\theta\,Sin\theta}{Sin\theta\,Cos\theta} = \frac{2Cos(3\theta - \theta)}{2Sin\theta\,Cos\theta} = \frac{2Cos2\theta}{Sin2\theta} = 2\,Cot\,2\,\theta$$

14. $Sin^4\theta = (Sin^2\theta)^2 = \left(\frac{1 - Cos2\theta}{2}\right)^2$ **Faisalabad 2009, Federal**

Sol. $$\frac{1 - 2Cos2\theta + Cos^2 2\theta}{4} = \frac{1}{4}\left[1 - 2Cos2\theta + \frac{1 + Cos4\theta}{2}\right]$$ $\boxed{Sin^2\theta = \frac{1 - \cos 2\theta}{2}}$

$$= \frac{1}{4}\left[\frac{2 - 4Cos2\theta + 1 + Cos4\theta}{2}\right] = \frac{1}{8}\,[3 - 4\,Cos2\,\theta + Cos4\,\theta]$$ $\boxed{Cos^2 2\theta = \frac{1 + \cos 4\theta}{2}}$

$$= \frac{3 - 4Cos2\theta + Cos4\theta}{8}$$

15. When $\theta = 18°$ **Multiply by 5**

Sol. then $5\theta = 90° \Rightarrow 2\theta + 3\theta = 90° \Rightarrow 2\theta = 90° - 3\theta$

$Sin(2\theta) = Sin(90° - 3\theta) \Rightarrow 2\,Sin\theta\,Cos\theta = Cos3\theta$

$2\sin\theta\ \cancel{\cos\theta} = 4\cos^3\theta - 3\cos\theta = \cancel{\cos\theta}\ (4\cos^2\theta - 3)$

$2\sin\theta = 4(1-\sin^2\theta) - 3 \Rightarrow 2\sin\theta = 4 - 4\sin^2\theta - 3$

$4\sin^2\theta + 2\sin\theta - 1 = 0$ $\qquad a = 4,\ b = 2,\ c = -1$

$$\sin\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{(2)^2 - 4(4)(-1)}}{2(4)}$$

$$\sin\theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-2 \pm \sqrt{20}}{8} = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{\cancel{2}(-1 \pm \sqrt{5})}{\cancel{8}_4}$$

$\sin\theta = \dfrac{-1 \pm \sqrt{5}}{4}$ Put $\theta = 18^\circ$ then $\sin 18^\circ = \dfrac{-1+\sqrt{5}}{4}$ Because 18° is in I quadrant

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{-1+\sqrt{5}}{4}\right)^2 = 1 - \left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= 1 - \left(\frac{5+1-2\sqrt{5}}{16}\right) = \frac{16-6+2\sqrt{5}}{16} = \frac{10+2\sqrt{5}}{16}$$

$$\cos^2\theta = \frac{10+2\sqrt{5}}{16} \Rightarrow \cos\theta = \pm\frac{\sqrt{10+2\sqrt{5}}}{4} \Rightarrow \cos 18^\circ = \pm\frac{\sqrt{10+2\sqrt{5}}}{4}$$

$\cos 18^\circ = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$ (Because 18° in I quad)

ii. **When $\theta = 36^\circ$**

Sol. then $\cos 2\theta = 2\cos^2\theta - 1$

Put $\theta = 18^\circ \Rightarrow \cos 2(18^\circ) = 2\cos^2(18^\circ) - 1$

$$\cos 36^\circ = 2\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2 - 1 = 2\left(\frac{10+2\sqrt{5}}{16}\right) - 1$$

$$= \frac{10+2\sqrt{5}-8}{8} = \frac{2+2\sqrt{5}}{8} = \frac{2(1+\sqrt{5})}{8}$$

$\cos 36^\circ = \dfrac{1+\sqrt{5}}{4}$, $\sin^2\theta = 1 - \cos^2\theta$

$$\text{Sin}^2 36^\circ = 1 - \text{Cos}^2 36^\circ$$

$$\text{Sin}^2 36^\circ = 1 - \left(\frac{1+\sqrt{5}}{4}\right)^2 = 1 - \left(\frac{1+5+2\sqrt{5}}{16}\right) = \frac{16-6-2\sqrt{5}}{16}$$

$$= \frac{10-2\sqrt{5}}{16} \Rightarrow Sin36^o = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

iii. **When θ = 54°**

Sol. $\text{Cos}54^\circ = \text{Sin}(90^\circ - 54^\circ) = \text{Sin}36^\circ = \dfrac{10-2\sqrt{5}}{4}$

$\text{Sin}54^\circ = \text{Cos}(90^\circ - 54^\circ) = \text{Cos}36^\circ = \dfrac{1+\sqrt{5}}{4}$

iv. **When θ = 72°**

Sol. $\text{Sin}72^\circ = \text{Sin}(90^\circ - 18^\circ) = \text{Cos}18^\circ = \dfrac{\sqrt{10+2\sqrt{2}}}{4}$

$\text{Sin}72^\circ = \text{Cos}\,72^\circ = \text{Cos}(90^\circ - 18^\circ) = \text{Sin}\,18^\circ = \dfrac{-1+\sqrt{5}}{4}$

19. **$\text{Cos}36^\circ\,\text{Cos}72^\circ\,\text{Cos}108^\circ\,\text{Cos}144^\circ = \dfrac{1}{16}$**

Sol. L.H.S = $\text{Cos}36^\circ\,\text{Cos}72^\circ\,\text{Cos}108^\circ\,\text{Cos}144^\circ$

$$= \text{Cos}36^\circ\,\text{Cos}72^\circ\,\text{Cos}(180^\circ - 72^\circ)\text{Cos}(180^\circ - 36^\circ)$$

$$= \text{Cos}36^\circ\,\text{Cos}72^\circ\,(-\text{Cos}72^\circ)\,(-\text{Cos}36^\circ)$$

$$= \text{Cos}^2 36^\circ\,\text{Cos}^2 72^\circ$$

$$= \left(\frac{1+\sqrt{5}}{4}\right)^2 \left(\frac{\sqrt{5}-1}{4}\right)^2 = \left(\frac{1+5+2\sqrt{5}}{16}\right)\left(\frac{5+1-2\sqrt{5}}{16}\right)$$

$$= \frac{(6+2\sqrt{5})(6-2\sqrt{5})}{16 \times 16} = \frac{(6)^2-(2\sqrt{5})^2}{16 \times 16}$$

$$= \frac{36-(4\times 5)}{16x16} = \frac{36-20}{16x16} = \frac{16}{16x16} = \frac{1}{16} = \text{R.H.S}$$

EXERCISE. 10.4

Formulas : Product To Sum

i) $2\text{Sin}\,\alpha\ \text{Cos}\,\beta = \text{Sin}\,(\alpha+\beta) + \text{Sin}\,(\alpha-\beta)$

ii) $2\text{Cos}\,\alpha\ \text{Sin}\,\beta = \text{Sin}\,(\alpha+\beta) - \text{Sin}\,(\alpha-\beta)$

iii) $2\text{Cos}\,\alpha\ \text{Cos}\,\beta = \text{Cos}(\alpha+\beta) + \text{Cos}(\alpha-\beta)$

iv) $-2\ \text{Sin}\,\alpha\ \text{Sin}\,\beta = \text{Cos}(\alpha+\beta) - \text{Cos}(\alpha-\beta)$

Sum To Product

v) $\text{SinP} + \text{Sin}\,Q = 2\text{Sin}\,\dfrac{P+Q}{2}\ \text{Cos}\,\dfrac{P-Q}{2}$

vi) $\text{SinP} - \text{Sin}\,Q = 2\text{Cos}\,\dfrac{P+Q}{2}\ \text{Sin}\,\dfrac{P-Q}{2}$

vii) $\text{CosP} + \text{Cos}\,Q = 2\text{Cos}\,\dfrac{P+Q}{2}\ \text{Cos}\,\dfrac{P-Q}{2}$

viii) $\text{CosP} - \text{Cos}\,Q = -2\text{Sin}\,\dfrac{P+Q}{2}\ \text{Sin}\,\dfrac{P-Q}{2}$

Take $\alpha+\beta = P\ \&\ \alpha-\beta = Q$

then $\alpha+\cancel{\beta}+\alpha-\cancel{\beta} = P+Q$

$\Rightarrow 2\alpha = P+Q \Rightarrow \alpha = \dfrac{P+Q}{2}$

Similarly $\beta = \dfrac{P-Q}{2}$

Example – 1. Express 2 Sin 7θ Cos3θ as a sum or difference.

Sol. $2\text{Sin}7\theta\,\text{Cos}3\theta = \text{Sin}(7\theta+3\theta) + \text{Sin}(7\theta-3\theta)$

$= \text{Sin}10\theta + \text{Sin}4\theta$

Example – 2. Prove Sin19°Cos11° + Sin71° Sin11° = 1/2

Sol. L.H.S = $\text{Sin}19^\circ\text{Cos}11^\circ + \text{Sin}71^\circ\text{Sin}11^\circ$

$= \dfrac{1}{2}\,[2\text{Sin}19^\circ\text{Cos}11^\circ + 2\text{Sin}71^\circ\text{Sin}11^\circ]$

$= \dfrac{1}{2}\,[2\text{Sin}19^\circ\text{Cos}11^\circ - (-2\text{Sin}71^\circ\text{Sin}11^\circ)]$

$= \dfrac{1}{2}\,[\{\text{Sin}(19^\circ+11^\circ\} + \text{Sin}(19^\circ-11^\circ) - \{\text{Cos}(71^\circ+11^\circ) - \text{Cos}(71^\circ-11^\circ)\}]$

$= \dfrac{1}{2}\,[\text{Sin}30^\circ + \text{Sin}8^\circ - \text{Cos}82^\circ + \text{Cos}60^\circ]$

$$= \frac{1}{2}\left[\frac{1}{2} + Sin8^\circ - Cos(90^\circ - 8^\circ) + \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} + \cancel{Sin8^\circ} - \cancel{Sin8^\circ} + \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} + \frac{1}{2}\right] = \frac{1}{2}(1) = \frac{1}{2} = \text{R.H.S}$$

Example – 3. Express sin5x + sin7x as a product **Sargodha 2008,09**

Sol. $$Sin5x + Sin7x = 2\sin\frac{5x+7x}{2}\cos\frac{5x-7x}{2} = 2\sin 6x\cos x$$

Example – 4. Express CosA + Cos3A + CosA + Cos7A as product.

Sol. CosA + Cos3A + Cos5A + Cos7A

= [Cos7A + CosA] + [Cos5A + Cos3A]

$$= \left[2\cos\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right)\right] + \left[2\cos\left(\frac{5A+A}{2}\right)\cos\left(\frac{5A-A}{2}\right)\right]$$

= 2Cos4A Cos3A + 2Cos 4A CosA

= 2Cos 4A [Cos3A + CosA]

$$= 2\text{Cos }4A\left[2Cos\left(\frac{3A+A}{2}\right)Cos\left(\frac{3A-A}{2}\right)\right]$$

= 2Cos 4A [2Cos 2A CosA] = 4cos4ACos 2A CosA

Example – 5. Show that Cos20° Cos40° Cos80° = $\frac{1}{8}$ **Multan 2007**

Sol. L.H.S = Cos20° Cos40° Cos80°

$$= \frac{1}{2}\text{ Cos}20^\circ\,[2\text{Cos}80^\circ\text{Cos}40^\circ]$$

$$= \frac{1}{2}\text{ Cos}20^\circ\,[\text{Cos}(80^\circ + 40^\circ) + \text{Cos}(80^\circ - 40^\circ)]$$

$$= \frac{1}{2}\text{ Cos}20^\circ\,[\text{Cos}120^\circ + \text{Cos}40^\circ] = \frac{1}{2}\text{ Cos}20^\circ\left[-\frac{1}{2} + Cos40^\circ\right]$$

$$= \frac{1}{2}\ \text{Cos}20^\circ\ [\frac{-1+2Cos40^\circ}{2}] = \frac{1}{4}\ \text{Cos}20^\circ\ [-1 + 2\text{Cos}40^\circ]$$

$$= \frac{1}{4}\ [-\text{Cos}20^\circ + 2\text{Cos}40^\circ\text{Cos}20^\circ]$$

$$= \frac{1}{4}\ [-\text{Cos}20^\circ + \text{Cos}\ (40^\circ + 20^\circ) + \text{Cos}\ (40^\circ - 20^\circ)]$$

$$= \frac{1}{4}\ [-\ \cancel{\text{Cos}20^\circ} + \text{Cos}60^\circ) + \cancel{\text{Cos}20^\circ}\ ] = \frac{1}{4}\ [\text{Cos}60^\circ]$$

$$= \frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{8} = \text{R.H.S}$$

1. i. 2Sin3 θ Cos θ

Sol. $= \text{Sin}(3\theta + \theta) + \text{Sin}(3\theta - \theta)$

$= \text{Sin}4\theta + \text{Sin}2\theta$

iii. Sin5 θ Cos θ

Sol. $= \frac{1}{2}(2\text{Sin}5\theta\ \text{Cos}2\theta)$

$= \frac{1}{2}[\text{Sin}(5\theta+2\theta)+\text{Sin}(5\theta-2\theta)]$

$= \frac{1}{2}(\text{Sin}7\theta+\text{Sin}3\theta)$

ii. 2Cos5 θ Sin3 θ Faisalabad 2008

Sol. $= \text{Sin}(5\theta + 3\theta) - \text{Sin}(5\theta - 3\theta)$

$= \text{Sin}8\theta - \text{Sin}2\theta$

iv. 2Sin7 θ Sin2 θ Multan 2007,Sgd 2011

Sol. $= -(-2\text{Sin}7\theta\,\text{Sin}2\theta)$

$= -[\text{Cos}(7\theta + 2\theta) - \text{Cos}(7\theta - 2\theta)]$

$= -(\text{Cos}9\theta - \text{Cos}5\theta)$

$= \text{Cos}5\theta - \text{Cos}9\theta$

v. **Cos(x + y) Sin (x – y) Rawalpindi 2009**

Sol. $= \frac{1}{2}\left(2Cos\left(x+y\right)Sin\left(x-y\right)\right)$

$= \frac{1}{2}\left(Sin\left(\left(x+y\right)+\left(x-y\right)\right)-Sin\left(x+y\right)-\left(x-y\right)\right)$

$= \frac{1}{2}\left(Sin\left(x+\cancel{y}+x-\cancel{y}\right)-Sin\left(\cancel{x}+y-\cancel{x}+y\right)\right)$

$= \frac{1}{2}\left(Sin2x - Sin2y\right)$

vi. **Cos (2x + 30°) Cos (2x − 30°)**

Sol. $= \frac{1}{2}\left(2Cos\left(2x+30^o\right)Cos\left(2x-30^o\right)\right)$

$= \frac{1}{2}\left[Cos\left(2x+30^o+2x-30^o\right)+Cos\left(2x+30^o\right)-\left(2x-30^o\right)\right]$

$= \frac{1}{2}\left[Cos\left(2x+\cancel{30^o}+2x-\cancel{30^o}\right)+Cos\left(\cancel{2x}+30^o-\cancel{2x}+30^o\right)\right]$

$= \frac{1}{2}\left(Cos4x+Cos60^o\right)$

vii. **Sin12° Sin46°**

Sol. $= \frac{-1}{2}\,(-2Sin12^\circ Sin46^\circ) = -\frac{1}{2}\,(Cos\,(12^\circ+46^\circ)-Cos(12^\circ-46^\circ))$

$= -\frac{1}{2}\,(Cos58^\circ-Cos(-34^\circ)) = -\frac{1}{2}\,(Cos58^\circ-Cos34^\circ)$

viii. **Sin (x + 45°) Sin (x − 45°)** **Multan 2008**

Sol. $= -\frac{1}{2}\left(-2\,Sin\left(x+45^o\right)Sin\left(x-45^o\right)\right)$

$= -\frac{1}{2}\left[Cos\left[\left(x+45^o\right)+\left(x-45^o\right)\right]-Cos\left[\left(X+45^o\right)-\left(x-45^o\right)\right]\right]$

$= -\frac{1}{2}\left[Cos\left(x+\cancel{45^o}+x-\cancel{45^o}\right)-Cos\left(\cancel{x}+45^o-\cancel{x}+45^o\right)\right]$

$= -\frac{1}{2}\left[Cos2x-Cos90^o\right] = \frac{1}{2}\left(-Cos2x+Cos90^o\right)$

$= \frac{1}{2}\left(Cos90^o-Cos2x\right)$

2.i **Sin5θ + Sin3θ** **Faisalabad 2007**

Sol. $= 2Sin\frac{5\theta+3\theta}{2}Cos\frac{5\theta-3\theta}{2} = 2Sin\,4\theta\,Cos\theta$

ii. **Sin8θ − Sin4θ** **Sargodha 2008**

Sol. $= 2Cos\,\frac{8\theta+4\theta}{2}+Sin\frac{8\theta-4\theta}{2} = 2Cos6\theta\,Sin2\theta$

iii. **$Cos6\theta + Cos3\theta$**

Sol. $= 2Cos\ \frac{6\theta + 3\theta}{2}\ Cos\frac{6\theta - 3\theta}{2} = 2\ Cos\ \frac{9\theta}{2}\ Cos\frac{3\theta}{2}$

iv. **$Cos7\theta - Cos\theta$** **Multan 2008, Lahore 2009, Sargodha 2011**

Sol. $= -2Sin\frac{7\theta + \theta}{2}\ Sin\frac{7\theta - \theta}{2} = -2\ Sin\ 4\theta\ Sin3\theta$

v. **Cos12° + Cos48°**

Sol. $= 2Cos\frac{12^{o} + 48^{o}}{2}\ Cos\frac{12^{o} - 48^{o}}{2}$

$= 2Cos\ \frac{60^{o}}{2}\ Cos\ \frac{(-36^{o})}{2}$ $\boxed{Cos(-\theta) = Cos\theta}$

$= 2Cos30^{o}Cos(18^{o})$

vi. **Sin (x + 30°) + Sin (x – 30°)** **Multan 2008**

Sol. $= 2Sin\left(\frac{x + \cancel{30^{o}} + x - \cancel{30^{o}}}{2}\right) Cos\ \frac{(x+30^{o})-(x-30^{o})}{2}$

$= 2\ Sin\left(\frac{\cancel{2}x}{\cancel{2}}\right) Cos\left(\frac{\cancel{x} + 30^{o} - \cancel{x} + 30^{o}}{2}\right) = 2SinxCos30^{o}$

3.i $\boldsymbol{\frac{Sin3x - Sinx}{Cosx - Cos3x}}$ **= Cot2x** **Lahore 2009**

Sol. L.H.S $= \frac{Sin3x - Sinx}{Cosx - Cos3x}$

$$= \frac{2Cos\frac{3x+x}{2}Sin\frac{3x-x}{2}}{-2Sin\frac{3x+x}{2}Sin\frac{x-3x}{2}} = \frac{Cos2xSinx}{-Sin2xSin(-x)} = \frac{Cos2xSinx}{-Sin2x(-Sinx)} = \frac{Cos2x\ \cancel{Sinx}}{Sin2x\ \cancel{Sinx}}$$

$= Cot2x$ = R.H.S

ii. $\boldsymbol{\frac{Sin8x + Sin2x}{Cos8x + Cos2x}}$ **= tan5x**

Sol. L.H.S = $\dfrac{Sin8x + Sin2x}{Cos8x + Cos2x} = \dfrac{2Sin\dfrac{8x+2x}{2}Cos\dfrac{8x-2x}{2}}{2Cos\dfrac{8x+2x}{2}Cos\dfrac{8x-2x}{2}}$

$= \dfrac{Sin5x\ \cancel{Cos3x}}{Cos5x\ \cancel{Cos3x}}$ = tan5x = R.H.S

iii. $\dfrac{Sin\alpha - Sin\beta}{Sin\alpha + Sin\beta} = \tan\dfrac{\alpha-\beta}{2}\ \text{Cot}\ \dfrac{\alpha+\beta}{2}$ **Sargodha 2010**

Sol. L.H.S = $\dfrac{Sin\alpha - Sin\beta}{Sin\alpha + Sin\beta}$

$= \dfrac{\cancel{2}Cos(\frac{\alpha+\beta}{2})Sin(\frac{\alpha-\beta}{2})}{\cancel{2}Sin(\frac{\alpha+\beta}{2})Cos(\frac{\alpha-\beta}{2})} = \dfrac{Cos\left(\frac{\alpha+\beta}{2}\right)}{Sin\left(\frac{\alpha+\beta}{2}\right)} \cdot \dfrac{Sin\left(\frac{\alpha-\beta}{2}\right)}{Cos\left(\frac{\alpha-\beta}{2}\right)}$

$= \text{Cot}\left(\dfrac{\alpha+\beta}{2}\right)\tan\left(\dfrac{\alpha-\beta}{2}\right) = \tan\left(\dfrac{\alpha-\beta}{2}\right)\text{Cot}\left(\dfrac{\alpha+\beta}{2}\right)$

4.i **Cos20° + Cos100° + Cos140° = 0**

Sol. L.H.S = Cos20° + Cos100° + Cos140°

$= 2\text{Cos}\left(\dfrac{20^o + 100^o}{2}\right)\text{Cos}\left(\dfrac{20^o - 100^o}{2}\right) + \text{Cos}\,(180° - 40°)$

$= 2\text{Cos}\,60°\,\text{Cos}(-40°) - \text{Cos}40°$

$= \cancel{2}\left(\dfrac{1}{\cancel{2}}\right)\text{Cos}40° - \text{Cos}40°$

$= \cancel{Cos40^o} - \cancel{Cos40^o} = 0 = R.H.S$

ii. $\text{Sin}\left(\dfrac{\pi}{4} - \theta\right)\text{Sin}\left(\dfrac{\pi}{4} + \theta\right) = \dfrac{1}{2}\,\text{Cos}2\theta$

Sol. L.H.S = $\text{Sin}\left(\dfrac{\pi}{4} - \theta\right)\text{Sin}\left(\dfrac{\pi}{4} + \theta\right)$

$$= -\frac{1}{2}\left(-2Sin\left(\frac{\pi}{4}-\theta\right) Sin\left(\frac{\pi}{4}+\theta\right)\right)$$

$$= -\frac{1}{2}\left[Cos\left(\frac{\pi}{4}-\cancel{\theta}+\frac{\pi}{4}+\cancel{\theta}\right) - Cos\left[\left(\frac{\pi}{4}-\theta\right)-\left(\frac{\pi}{4}+\theta\right)\right]\right]$$

$$= -\frac{1}{2}\left[Cos\left(2\times\frac{\pi}{4}\right) - Cos\left(\cancel{\frac{\pi}{4}}-\theta-\cancel{\frac{\pi}{4}}-\theta\right)\right]$$

$$= -\frac{1}{2}\left[Cos\frac{\pi}{2} - Cos(-2\theta)\right]$$

$$= -\frac{1}{2}[0 - Cos2\theta] = \frac{1}{2}Cos2\theta = R.H.S$$

iii. $$\frac{Sin\theta+Sin3\theta+Sin5\theta+Sin7\theta}{Cos\theta+Cos3\theta+Cos5\theta+Cos7\theta} = \tan 4\theta$$ **Sargodha 2011**

Sol. $$L.H.S = \frac{Sin\theta+Sin3\theta+Sin5\theta+Sin7\theta}{Cos\theta+Cos3\theta+Cos5\theta+Cos7\theta}$$

$$= \frac{2Sin\frac{\theta+3\theta}{2}Cos\frac{\theta-3\theta}{2}+2Sin\frac{5\theta+7\theta}{2}Cos\frac{5\theta-7\theta}{2}}{2Cos\frac{\theta+3\theta}{2}Cos\frac{\theta-3\theta}{2}+2Cos\frac{5\theta+7\theta}{2}Cos\frac{5\theta-7\theta}{2}}$$

$$= \frac{2Sin2\theta\, Cos(-\theta)+2Sin6\theta\, Co(-\theta)}{2Cos2\theta\, Cos(-\theta)+2Cos6\theta\, Co(-\theta)}$$

$$= \frac{\cancel{2}\ \cancel{s(-\theta)}\ (Sin2\theta+Sin6\theta)}{\cancel{2Cos(-\theta)}\ (Cos2\theta+Cos6\theta)}$$

$$= \frac{2Sin\frac{2\theta+6\theta}{2}Cos\frac{2\theta-6\theta}{2}}{2Cos\frac{2\theta+6\theta}{2}Cos\frac{2\theta-6\theta}{2}} = \frac{\cancel{2}Sin4\theta\,\cancel{Cos(-2\theta)}}{\cancel{2}Cos4\theta\,\cancel{Cos(-2\theta)}} = \tan 4\theta = R.H.S$$

5.i $$Cos20^\circ\ Cos40^\circ\ Cos60^\circ\ Cos80^\circ = \frac{1}{16}$$ **Faisalabad 2008, Sargodha 2008**

Sol. $L.H.S = Cos20^\circ\ Cos40^\circ\ Cos60^\circ\ Cos80^\circ$

$$= \text{Cos}20^\circ \ \text{Cos}40^\circ \left(\frac{1}{2}\right) \text{Cos}80^\circ$$

$$= \frac{1}{2} \ (\text{Cos}20^\circ \ \text{Cos}40^\circ) \ \text{Cos}80^\circ$$

$$= \frac{1}{4} \ (2\text{Cos}20^\circ \ \text{Cos}40^\circ) \ \text{Cos}80^\circ \quad (\text{'x' \& '} \div \text{' by 2})$$

$$= \frac{1}{4} \ (\text{Cos}\ (20^\circ + 40^\circ) + \text{Cos}(20^\circ - 80^\circ))] \ \text{Cos}80^\circ$$

$$= \frac{1}{4} \ [\ (\text{Cos}\ (20^\circ + 40^\circ)\ ]\ \text{Cos}80^\circ + (\text{Cos}\ (20^\circ - 40^\circ))\ \text{Cos}80^\circ$$

$$= \frac{1}{4} \ (\text{Cos}60^\circ \ \text{Cos}80^\circ + \text{Cos}20^\circ \text{Cos}80^\circ \qquad (\text{Cos}(-20^\circ) = \text{Cos}20^\circ)$$

$$= \frac{1}{4} \ \left(\frac{1}{2} Cos80^\circ + \frac{2Cos20^\circ Cos80^\circ}{2} \right)$$

$$= \frac{1}{8} (\text{Cos}80^\circ + \text{Cos}\ (20^\circ + 80^\circ) + \text{Cos}\ (20^\circ - 80^\circ))$$

$$= \frac{1}{8} (\text{Cos}(180^\circ - 100^\circ) + \text{Cos}\ (100^\circ) + \text{Cos}\ (-60^\circ))]$$

$$= \frac{1}{8}\left[-\cancel{\text{Cos}100^\circ} + \cancel{\text{Cos}100^\circ} + \text{Cos}60^\circ \right]$$

$$= \frac{1}{8} \left(\frac{1}{2}\right) = \frac{1}{16} = R.H.S$$

ii. $$\boldsymbol{Sin\frac{\pi}{9}\, Sin\frac{2\pi}{2}\, Sin\frac{\pi}{3}\, Sin\frac{4\pi}{9} = \frac{3}{16}}$$ **Multan 2009**

Sol. L.H.S = $Sin\frac{\pi}{9}\, Sin\frac{2\pi}{2}\, Sin\frac{\pi}{3}\, Sin\frac{4\pi}{9}$

$$= \text{Sin}20^\circ \text{Sin}40^\circ \text{Sin}60^\circ \text{Sin}80^\circ$$

$$= \text{Sin}20^\circ\text{Sin}40^\circ \frac{\sqrt{3}}{2} \text{Sin}80^\circ$$

$$= \frac{\sqrt{3}}{2} (\text{Sin}20^\circ\text{Sin}40^\circ)\text{Sin}80^\circ$$

$$= -\frac{\sqrt{3}}{4} (-2\,\text{Sin}20^\circ\text{Sin}40^\circ)\text{Sin}80^\circ$$

$$= -\frac{\sqrt{3}}{4} [\text{Cos}\,(20^\circ + 40^\circ) - \text{Cos}\,(20^\circ - 40^\circ)]\ \text{Sin}\,80^\circ$$

$$= \frac{-\sqrt{3}}{4} [\text{Cos}\,60^\circ - \text{Cos}\,(-20^\circ)]\ \text{Sin}80^\circ = \frac{-\sqrt{3}}{4}\left(\frac{1}{2} - Cos20^O\right)Sin\,80^O$$

$$= -\frac{\sqrt{3}}{4} [\frac{1}{2}\ \text{Sin}80^\circ - \text{Cos}20^\circ\text{Sin}80^\circ]$$

$$= -\frac{\sqrt{3}}{4} [\frac{Sin80^o - 2Cos20^o\ Sin80^o}{2}]$$

$$= -\frac{\sqrt{3}}{8} \{\text{Sin}\,(180^\circ - 100^\circ) - [\text{Sin}\,(20^\circ + 80^\circ) - \text{Sin}\,(20^\circ - 80^\circ)]\}$$

$$= -\frac{\sqrt{3}}{8}\left[\cancel{\text{Sin}\,100^\circ} - \cancel{\text{Sin}100^\circ} + \text{Sin}\,(-60^\circ)\right]$$

$$= -\frac{\sqrt{3}}{8} (-\text{Sin}60^\circ) = -\frac{\sqrt{3}}{8}\left(-\frac{\sqrt{3}}{2}\right) = \frac{3}{16} = R.H.S$$

iii. **$\text{Sin}10^\circ\text{Sin}30^\circ\text{Sin}50^\circ\text{Sin}70^\circ = \frac{1}{16}$** **Multan 2007, Faisalabad 2008, Sargodha 2009**

Sol. L.H.S = $\text{Sin}10^\circ\text{Sin}30^\circ\text{Sin}50^\circ\text{Sin}70^\circ$

$$= \text{Sin}10^\circ \left(\frac{1}{2}\right) \text{Sin}50^\circ\text{Sin}70^\circ$$

$$= \frac{1}{2} (\text{Sin}10^\circ\ \text{Sin}50^\circ)\text{Sin}70^\circ$$

$$= -\frac{1}{4}(-2\,\text{Sin}10^\circ\,\text{Sin}50^\circ)\text{Sin}70^\circ$$

$$= \frac{1}{4}\,[\text{Cos}\,(10^\circ + 50^\circ) - \text{Cos}\,(10^\circ - 50^\circ)]\,\text{Sin}70^\circ\}$$

$$= -\frac{1}{4}\,[\text{Cos}60^\circ - \text{Cos}(-40^\circ)]\,\text{Sin}70^\circ$$

$$= -\frac{1}{4}\,[\text{Cos}60^\circ\,\text{Sin}70^\circ - \text{Cos}40^\circ\text{Sin}70^\circ]$$

$$= -\frac{1}{4}\,[\frac{1}{2}\text{Cos}70^\circ - \text{Cos}40^\circ\text{Sin}70^\circ]$$

$$= -\frac{1}{4}\left(\frac{Sin70^\circ - 2Cos40^\circ\,Sin70^\circ}{2}\right)$$

$$= -\frac{1}{8}\,[\text{Sin}\,(70^\circ) - \{(\text{Sin}\,(40^\circ + 70^\circ) - \text{Sin}\,(40^\circ - 70^\circ)\}]$$

$$= -\frac{1}{8}\,[\text{Sin}\,(180^\circ - 110^\circ) - \text{Sin}110^\circ + \text{Sin}\,(-30^\circ)]$$

$$= -\frac{1}{8}\left[\cancel{\text{Sin}110^\circ} - \cancel{\text{Sin}110^\circ} - \text{Sin}30^\circ\right]$$

$$= -\frac{1}{8}\left[-\frac{1}{2}\right] = \frac{1}{16} = R.H.S$$

TEST YOUR SKILLS Marks: 50

Q # 1. Select the Correct Option **(10)**

i. $Sin294° =$

a) $Sin24°$ b) $Cos24°$

c) $-Sin24°$ d) $-Cos24°$

ii. $Sin2\theta =$

a) $\dfrac{2Tan\theta}{1-Tan^2\theta}$ b) $\dfrac{2Tan\theta}{1+Tan^2\theta}$

c) $\dfrac{1-Tan^2\theta}{1+Tan^2\theta}$ d) $\dfrac{1+Tan^2\theta}{1-Tan^2\theta}$

iii. $Cos\left(\dfrac{\pi}{2}+\beta\right)=$

a) $Cos\beta$ b) $-Cos\beta$

c) $Sin\beta$ d) $-Sin\beta$

iv. $Cos\,\theta/2$ is equal to:

a) $\pm\sqrt{\dfrac{1-Cos2\theta}{2}}$ b) $\pm\sqrt{\dfrac{1-Cos\theta}{2}}$

c) $\pm\sqrt{\dfrac{1+Cos2\theta}{2}}$ d) $\pm\sqrt{\dfrac{1+Cos\theta}{2}}$

v. $Sin\,\theta/2$ is equal to:

a) $\pm\sqrt{\dfrac{1+Sin\alpha}{2}}$ b) $\pm\sqrt{\dfrac{1-Cos\alpha}{2}}$

c) $\pm\sqrt{\dfrac{1+Cos\alpha}{2}}$ d) $\pm\sqrt{\dfrac{1-Sin\alpha}{2}}$

vi. $Cos^2\,\theta$ is equal to:

a) $1+Cos2\theta$ b) $1-Cos2\theta$

c) $\dfrac{1+Cos2\theta}{2}$ d) $\dfrac{1-Cos2\theta}{2}$

vii. $Tan(\pi-\alpha)$ equals:

a) $Tan\,\alpha$ b) $Cot\,\alpha$

c) $-Tan\,\alpha$ d) $-Cot\,\alpha$

viii. $3Sin\,\alpha-4Sin^3\,\alpha$ is equal to:

a) $Cos3\alpha$ b) $Sin3\alpha$

c) $Cos2\alpha$ d) $Sin\alpha$

ix. If $\alpha+\beta+\gamma=180^{\circ}$ then $Cos(\alpha+\beta)=$

a) $Sin\gamma$ b) $Cos\gamma$

c) $-Cos\gamma$ d) $-Sin\gamma$

x. $2Sin12^{\circ}Sin46^{\circ}=$

a) $Cos34^{\circ}+Cos58^{\circ}$ b) $Sin34^{\circ}-Cos58^{\circ}$

c) $Sin34^{\circ}+Sin58^{\circ}$ d) $Sin34^{\circ}-Cos58^{\circ}$

Q # 2. Short Questions: (10 X 2 = 20)

i. State the Distance Formula:

ii. Express $Cos7\theta+Cos\theta$ as product

iii. Prove that $Cos330^{\circ}Sin600^{\circ}+Cos120^{\circ}Sin150^{\circ}=-1$

iv. Find the value of $Sin2\alpha$ and $Cos2\alpha$ when $Cos\alpha=3/5$ where $0<\alpha<\pi/2$

v. Prove that $\dfrac{Sin\alpha-Sin\beta}{Sin\alpha+Sin\beta}=\tan\left(\dfrac{\alpha-\beta}{2}\right)Cot\left(\dfrac{\alpha+\beta}{2}\right)$

vi. Prove that $Sin(90^{\circ}-\alpha)Sin(180^{\circ}+\alpha)=-Sin\alpha Cos\alpha$

vii. Show that $Cos(\alpha+\beta).Cos(\alpha-\beta)=Cos^2\beta-Sin^2\alpha$

viii. Prove that $\dfrac{1-Cos\alpha}{Sin\alpha}=Tan\,{}^{\alpha}\!/_{2}$

ix. Without using table/calculator find values of $Sin75^{\circ}$ and $Tan75^{\circ}$

x. Prove that $\dfrac{Cos11^{\circ}+Sin11^{\circ}}{Cos11^{\circ}-Sin11^{\circ}}=Tan56^{\circ}$

Long Questions: (2 X 10 = 20)

Q # 3. (a) Express $3Sin\theta+4Cos\theta$ in the form of $rSin(\theta+\varphi)$

(b) If $\alpha+\beta+\gamma=180^{\circ}$ show that $Cot\alpha Cot\beta+Cot\beta Cot\gamma+Cot\gamma Cot\alpha=1$

Q # 4. (a) If $Sin\alpha=4/5$ and $Sin\beta=12/13$ Find value of $Cos(\alpha-\beta)$

Where $\pi/2<\alpha<\pi$ and $\pi/2<\beta<\pi$

(b) Prove that $Sin10^{\circ}Sin30^{\circ}Sin50^{\circ}Sin70^{\circ}=1/16$

TRIGONOMETRIC FUNCTIONS AND THEIR GRAPHS

11

Domain & Range of Trigonometric Functions:

Function	Domain	Range
y = Sinx	$-\infty < x < +\infty$ **Mult 2007,09 Fsd 2008 Guj 2009 Sgd 2009**	$-1 \leq y \leq 1$
y = Cosx	$-\infty < x < +\infty$ **Lhr 2009**	$-1 \leq y \leq 1$
y = tanx	$-\infty < x < +\infty,\ x \neq \left(\frac{2n+1}{2}\right)\pi, n \in z$	$-\infty < y < \infty$ **Fsd 2009, Sgd 2010, Rawal 2009**
y = Cotx	$-\infty < x < \infty,\ x \neq n\pi,\ n \in z$ **Mult 2009**	$-\infty < y < \infty$ **Sgd 2006, 011**
y = Secx	$-\infty < x < \infty,\ x \neq \left(\frac{2n+1}{2}\right)\pi, n \in z$ **Sgd 2011**	$y \geq 1$ or $y \leq -1$ **Fsd 2009**
y = Cosex	$-\infty < x < +\infty,\ x \neq n\pi\ \ n \in z$	$y \geq 1$ or $y \leq -1$

Period of Trigonometric function:

The smallest + ve number which when added to the original circular measure of the angle gives same value of function is called period.

Theorem:

Sine is a periodic function and its periods is 2π.

Proof:

Suppose p is period of sine function such that **Federal**

$\text{Sin}(\theta + p) = \text{Sin}\,\theta,\ \forall\ \theta \in R$ ——— I

Now Put $\theta = 0$

$\Rightarrow$ $\text{Sin}\,p = 0 \Rightarrow$ $p = \text{Sin}^{-1} 0$

$\Rightarrow$ $p = 0 \pm \pi, \pm 2\pi, \pm 3\pi, \ldots\ldots$

If $P = \pi$ then form I

$\text{Sin}(\pi + \theta) = \text{Sin}\ \theta$ False because $\boxed{\sin(\pi + \theta) = -\sin\theta}$

if $p = 2\pi$ then from (I)

$\text{Sin}(2\pi + \theta) = \text{Sin}\ \theta$ (Which true) As 2π is smallest + ve number for which

$\text{Sin}(2\pi + \theta) = \text{Sin}\ \theta$

$\therefore 2\pi$ is period of Sin θ

Similarly we can prove that

(i). 2π is period of $\text{Cos}\,\theta$

(ii). 2π is period of $\text{Cosec}\,\theta$

(iii). 2π is period of $\text{Sec}\,\theta$

Theorem: Tangent is periodic function and its period is π : Federal

Proof: Suppose p is period of tangent function such that

$\text{Tan}(\theta + p) = \text{Tan}\ \theta,\ \forall\ \theta \in R$ ——— I

Now put $\theta = 0$

$\text{Tan}(0 + p) = \text{Tan}\ 0 \Rightarrow p = \text{Tan}^{-1}(0)$

$p = 0, \pm\pi, \pm 2\pi \pm 3\pi$

If $p = \pi$ then from I

$\text{Tan}(\theta + \pi) = \text{Tan}(\theta)$ Which is true

As π is smallest +ve number for which $\text{Tan}(\theta + \pi) = \text{Tan}\,\theta$

$\therefore \pi$ is period if Tan θ

Example. Find the period.

(i). Sin 2x (ii). $\text{Tan}\left(\dfrac{x}{3}\right)$

Sol. (i). We known that the period of Sine is 2π

$\therefore \text{Sin}(2x + 2\pi) = \text{Sin}\ 2x$ **Sargodha 2007, Multan 2008, Faisalabad 2009**

$\Rightarrow \text{Sin}\ 2(x + \pi) = \text{Sin}\ 2x$

Hence π is period of Sin 2x

(ii) We know that period of tagent is π

$\therefore$ Tan $\left(\frac{x}{3}+\pi\right)$ = Tan $\left(\frac{x}{3}\right)$

$\Rightarrow$ Tan $\frac{1}{3}(x+3\pi) = Tan\left(\frac{x}{3}\right)$

Hence period of Tan $\frac{x}{3}$ is 3 π

EXERCISE 11.1

Find the periods of the following functions:

Note:

1. Period of Sin x, Cos x Sec x, Cosec x is 2 π

2. Period of tan x and Cot x is π

1. **Sin 3x**

Sol. We know that period of sine is 2 π

$\therefore$ Sin (3x + 2 π) = Sin 3x $\Rightarrow$ Sin 3 $\left(x+\frac{2\pi}{3}\right)$ = Sin 3x

Hence $\frac{2\pi}{3}$ is period of Sin 3x

2. **Cos 2x** **Multan 2009, Rawalpindi 2009**

Sol. We know that period of Cos is 2 π

$\therefore$ Cos (2x + 2 π) = Cos 2x $\Rightarrow$ Cos 2 (x + π) = Cox 2x

Hence period of Cos 2x is π

3. **Tan 4 π** **Gujranwala 2009**

Sol. We know that period of Tan is π

$\therefore$ Tan (4x + π) = tan 4x $\Rightarrow$ Tan 4 (x + $\frac{\pi}{4}$) = Tan 4x

Hence period of Tan 4x is $\frac{\pi}{4}$

4. **Cot $\frac{x}{2}$**

Sol. We know that period of Cot is π

$\therefore \text{Cot}\left(\frac{x}{2} + \pi\right) = \text{Cot}\frac{x}{2} \Rightarrow \text{Cot}\left(\frac{x+2\pi}{2}\right) = \text{Cot}\ \frac{x}{2}$

$\Rightarrow \text{Cot}\ \frac{1}{2}\ (x + 2\pi) = \text{Cot}\frac{x}{2}$

Period of Cot $\frac{x}{2}$ is 2π

5. Sin $\frac{\pi}{3}$ **Sargodha 2008, 2011**

Sol. We know that period of Sine is 2π

$\therefore \text{Sin}\left(\frac{x}{3} + 2\pi\right) = \text{Sin}\ \frac{x}{3}$

$\Rightarrow \text{Sin}\left(\frac{x+6\pi}{3}\right) = \text{Sin}\ \frac{x}{3} \Rightarrow \text{Sin}\ \frac{1}{3}\ (x + 6\pi) = \text{Sin}\ \frac{x}{3}$

Hence 6π is period of Sin $\frac{x}{3}$

6. Cosec $\frac{x}{4}$

Sol. We know that period of Cosec is 2π

$\therefore \text{Cosec}\left(\frac{x}{4} + 2\pi\right) = \text{Cosec}\ \frac{x}{4}$

$\Rightarrow \text{Cosec}\left(\frac{x+8\pi}{4}\right) = \text{Cosec}\ \frac{x}{4} \Rightarrow \text{Cosec}\ \frac{1}{4}.(x + 8\pi) = \text{Cosec}\ \frac{x}{4}$

Hence 8π is period of Cosec $\frac{x}{4}$

7. Sin $\frac{x}{5}$ **Faisalabad 2007, Multan 2008, Sargodha 2009**

Sol. We know that period of Sine is 2π

$\therefore \text{Sin}\left(\frac{x}{5} + 2\pi\right) = \text{Sin}\ \frac{x}{5} \Rightarrow \text{Sin}\left(\frac{x+10\pi}{5}\right) = \text{Sin}\frac{x}{5}$

$\Rightarrow$ Sin $\frac{1}{5}$ (x + 10x) = Sin $\frac{x}{5}$

Hence 10π is period of Sin $\frac{x}{5}$

8. **Cos $\frac{x}{6}$** **Multan 2007, Faisalabad 2008, Sargodha 2008**

Sol. We know that period of Cos is 2π

$\therefore$ Cos $\left(\frac{x}{6} + 2\pi\right)$ = Cos $\frac{x}{6}$ $\Rightarrow$ Cos $\left(\frac{x + 12\pi}{6}\right)$ = Cos $\frac{x}{6}$

$\Rightarrow$ Cos $\frac{1}{6}$ $(x + 12\pi)$ = Cos $\frac{x}{6}$

Hence 12π is period of Cos $\frac{x}{6}$

9. **Tan $\frac{x}{7}$** **Multan 2007, Faisalabad 2008, Sargodha 2009**

Sol. We know that period of Tan is π

$\therefore$ Tan $(\frac{x}{7} + \pi)$ = tan $\frac{x}{7}$ $\Rightarrow$ Tan $\left(\frac{x + 7\pi}{7}\right)$ = Tan $\frac{x}{7}$

$\Rightarrow$ Tan $\frac{1}{7}$ $(x + 7\pi)$ = Tan $\frac{x}{7}$; Hence 7π is period.

10. **Cot 8x** **Faisalabad 2007, Sargodha 2010**

Sol. We know that period of Cot is π

$\therefore$ Cot $(8x + \pi)$ = Cot 8x $\Rightarrow$ Cot 8$\left(x + \frac{\pi}{8}\right)$ = Cot 8x

Hence $\frac{\pi}{8}$ is period of Cot 8π

11. **Sec 9x**

Sol. We know that period of Sec is 2π

$\therefore$ Sec $(9x + 2\pi)$ = Sec 9 x $\Rightarrow$ Sec $9\left(x + \frac{2\pi}{9}\right)$ = Sec 9x

Hence $\frac{2\pi}{9}$ is period of Sec 9x

12. **Cosec 10 x**

Sol. We know that period of Cosec is 2π

$\therefore$ Cosec $(10x + 2\pi)$ = Cosec 10x

$\Rightarrow$ Cosec $10\left(x + \frac{2\pi}{10}\right)$ = Cosec 10x

$\Rightarrow$ Cosec $10\left(x + \frac{\pi}{5}\right)$ = Cosec 10x

Hence $\frac{\pi}{5}$ is period of Cosec 10x

13. **3 Sin x** **Faisalabad 2009**

Sol. We know that period of Sine is 2π

$\therefore$ 3 Sin $(x + 2\pi)$ = 3 Sin x

Hence 2π is period of 3 Sin x

14. **2 Cos x**

Sol. We know that period of Cos is 2π

$\therefore$ 2 Cos $(x + 2\pi)$ = 2 Cos x

Hence 2π is period of 2 Cos x

15. **3 Cos $\frac{x}{5}$** **Sargodha 2011**

Sol. What know that period of Cos is 2π

$\therefore$ 3Cos $\left(\frac{x}{5} + 2\pi\right)$ = 3 Cos $\frac{x}{5}$

$\Rightarrow$ 3Cos $\left(\frac{x + 10\pi}{5}\right)$ =3 Cos $\frac{x}{5}$ = 3 Cos $\frac{1}{5}$ $(x + 10\pi)$ = 3 Cos $\frac{x}{5}$

Hence 10π is period of 3 Cos $\frac{x}{5}$

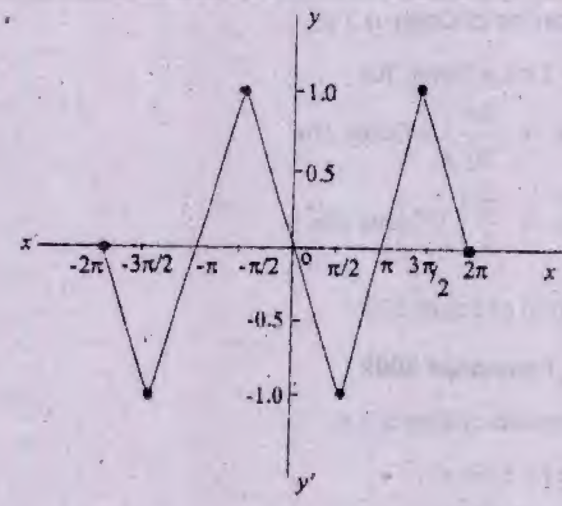
EXERCISE 11.2

1. Draw the graph of each of the following function for the intervals mentioned against each

i. $y = \text{Sin}x$ $x \in [-2\pi, 2\pi]$

Sol.

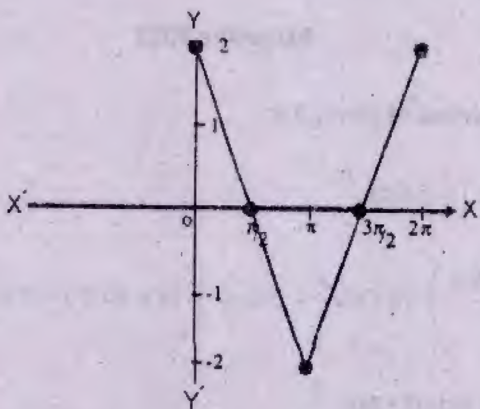
X	-2π	$-3\pi/2$	$-\pi$	$-\pi/2$	0	$\pi/2$	π	$3\pi/2$	2π
Y	0	-1	0	1	0	-1	0	1	0



ii. $y = 2\,\text{Cos}\,x$ $x \in [0, 2\pi]$

Sol. $y = 2\,\text{Cos}\,x$ $[0, 2\pi]$

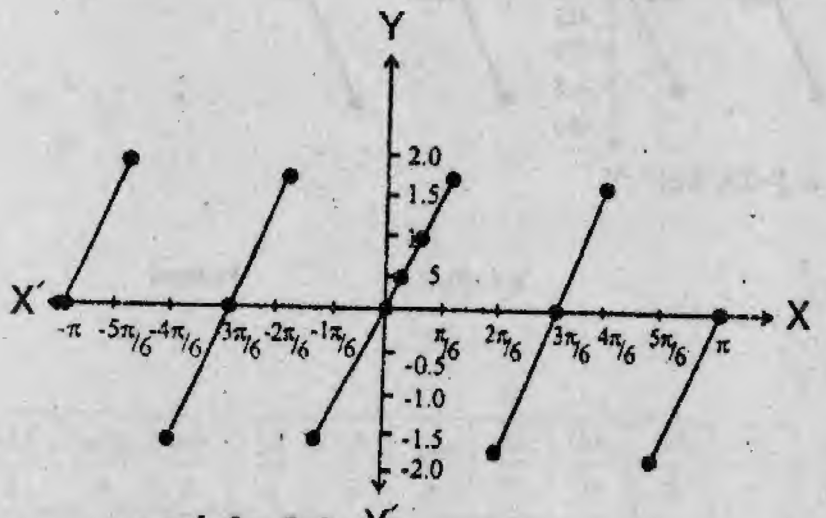
X	0	$\pi/2$	π	$3\pi/2$	2π
Y	2	0	-2	0	2



iii. y = Tan 2x $x \in [-\pi, \pi]$

Sol.

x	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-4\pi}{6}$	$\frac{-3\pi}{6}$	$\frac{-2\pi}{6}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π
y	0	1.7	−1.7	0	1.7	−1.7	0	1.7	−1.7	0	1.7	−1.7	0

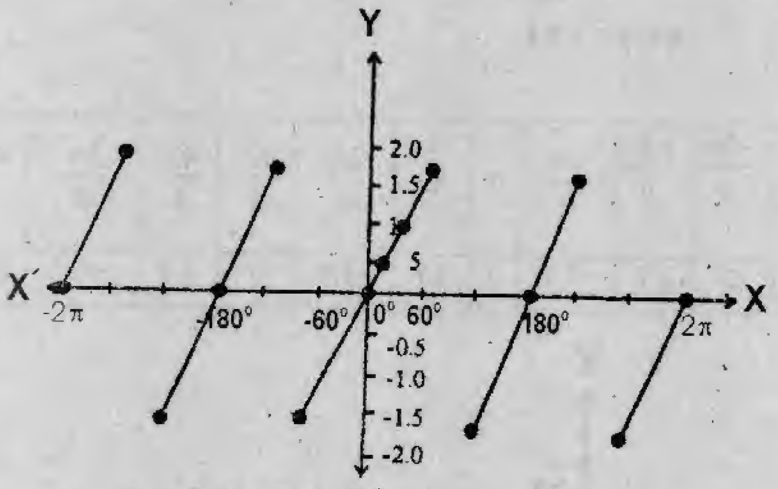


$x \in [-2\pi, 2\pi]$

iv. y=tanx $x \in [-2\pi, 2\pi]$

x	$-2\pi = -360^{O}$	-300^{O}	-240^{O}	-180^{O}	-120^{O}	-60^{O}
y	0	1.7	−1.7	0	1.7	−1.7

0	60^{O}	120^{O}	180^{O}	240^{O}	300^{O}	$2\pi = 360^{O}$
0	1.7	−1.7	0	1.7	−1.7	0

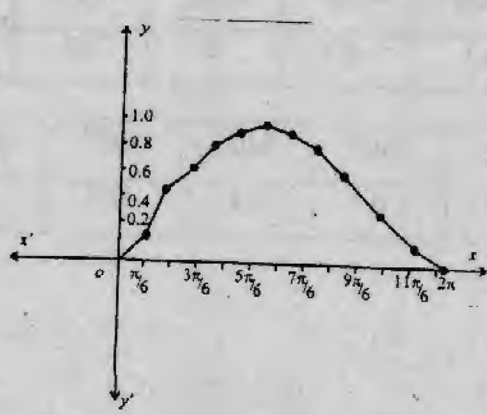


$x \in [-2\pi, 2\pi]$

v. Y = Sin $\frac{x}{2}$ x ∈ [0, 2π] **Federal**

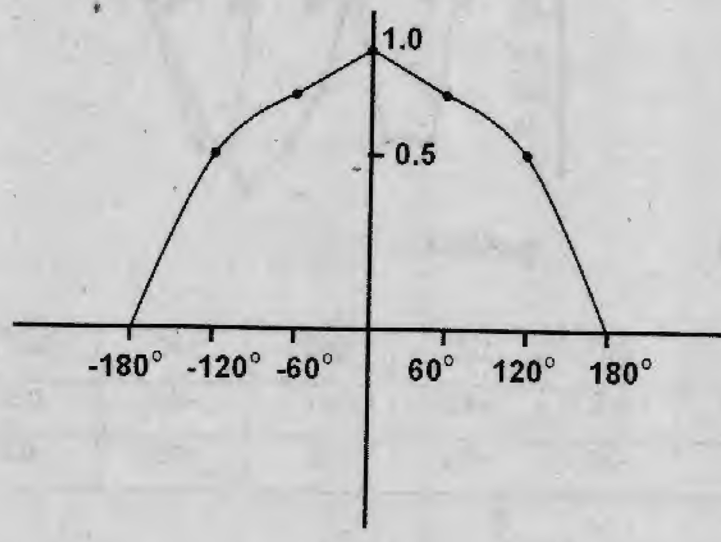
Sol. Y = Sin $\frac{x}{2}$

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$	$\frac{7\pi}{6}$	$\frac{8\pi}{6}$	$\frac{9\pi}{6}$	$\frac{10\pi}{6}$	$\frac{11\pi}{6}$	2π
y	0	0.26	0.50	0.71	0.87	0.97	1	0.97	0.87	0.71	0.5	0.26	0



vi. $y=\cos\dfrac{x}{2}$ $x \in [-\pi\ ,\ \pi\,]$

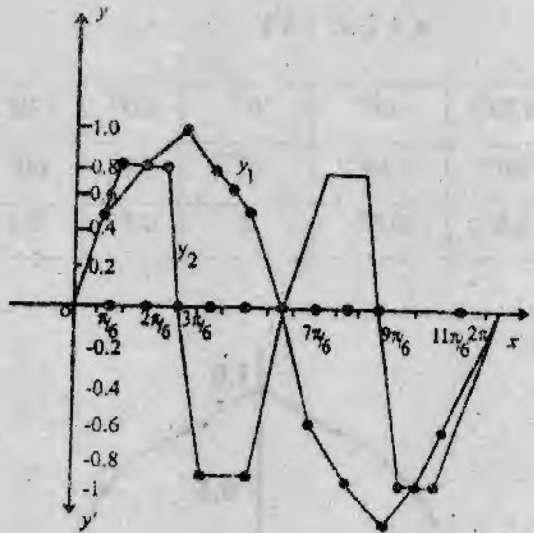
X	-180^{o}	-120^{o}	-60^{o}	0^{o}	60^{o}	120^{o}	180^{o}
X/2	-90^{o}	-60^{o}	-30^{o}	0^{o}	30^{o}	60^{o}	90^{o}
Y	0	0.5	0.87	1	0.87	0.5	0



2 (i). $y_1 = \sin x$, $y_2 = \sin 2x$ **Federal Board**

Sol.

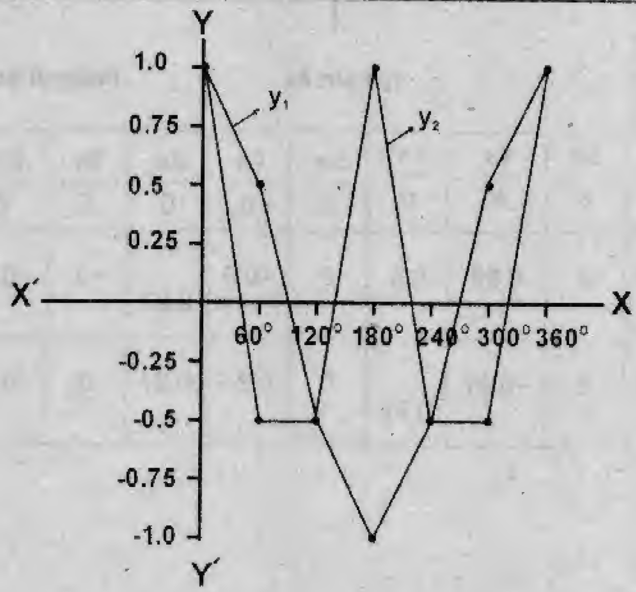
x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$	$\frac{7\pi}{6}$	$\frac{8\pi}{6}$	$\frac{9\pi}{6}$	$\frac{10\pi}{6}$	$\frac{11\pi}{6}$	2π
y_1	0	0.5	0.87	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0
y_2	0	0.87	0.87	0	−0.87	−0.87	0	0.87	0.87	0	−0.87	−0.87	0



2 (ii). $Y_1=\cos x$ $\quad$ $y_2=\cos 2x$

Sol.

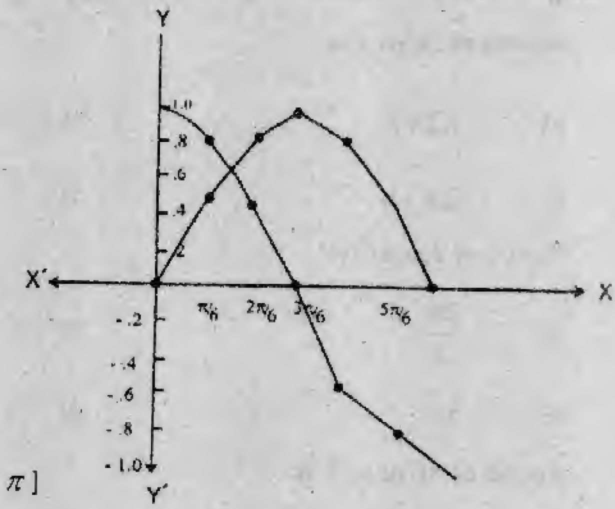
X	0	60^0	120^0	180^0	240^0	300^0	360^0
Y_1	1	0.5	–0.5	–1	–0.5	0.5	1
Y_2	1	–0.5	–0.5	1	–0.5	–0.5	1



3 (i). **Sin x = Cos x , $x \in [0, \pi]$**

Sol. Take y_1 = Sin x y_2 = Cos x

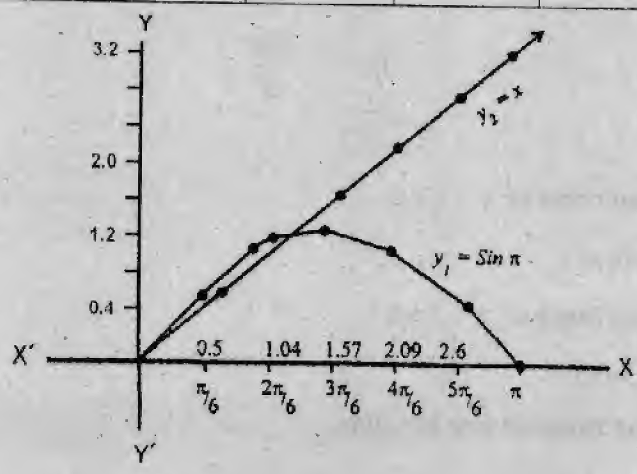
X	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π
Y_1	0	0.5	0.87	1	0.87	0.5	0
Y_2	1	0.866	0.5	0	–0.5	–0.866	–1



(ii). Y_1 = Sin x y_2 = x , $x \in [0, \pi]$

Sol.

X	0	$\frac{\pi}{6}$=0.52	$\frac{2\pi}{6}$=1.04	$\frac{3\pi}{6}$=1.57	$\frac{4\pi}{6}$=2.09	$\frac{5\pi}{6}$=2.31	π=3.14
Y_1	0	0.5	0.87	1	0.87	0.5	0
Y_2	0	0.52	1.04	1.57	2.09	2.61	3.1416



TEST YOUR SKILLS Marks: 15

Q # 1. Select the Correct Option

i. Period of $Cos\, x/6$ is

a) 12π b) 6π

c) 2π d) π

ii. Period of $3Cos\, x$ is

a) $3(2\pi)$ b) $\frac{2\pi}{3}$

c) $2\pi+3$ d) 2π

iii. Period of $Tan\, x/3$ is

a) $\frac{2\pi}{3}$ b) 6π

c) 3π d) $x/3$

iv. Period of $3Cos\, x/5$ is:

a) 4π b) 10π

c) 5π d) 2π

v. Period of $Co\sec 10x$ is:

a) $\frac{\pi}{10}$ b) $\frac{2\pi}{5}$

c) $\frac{\pi}{5}$ d) $\frac{4\pi}{5}$

Q # 2. Short Questions:

i. Write the domain and range of $y = Cotx$

ii. Find the period of $Sin\, x/3$

iii. Write the domain and range of $y = Tanx$

iv. Find the period of $Cot8x$

v. State the Domain and Range of Sine Function

Application of Trigonometry

12

EXERICISE 12.1

1. Find the values of:

i. Sin 53° 40′

Sol. Sin 53° 40° = 0.8056

ii. Cos 36° 20′

Sol. Cos 36° 20′ = 0.8056

iii. Tan 19° 30′

Sol. Tan 19° 30′ = 0.3541

iv. Cot 33° 50′

Sol. Cos 33° 50′ = 1.4920

v. Cos 42° 38′

Sol. Cos42° 38′ = 0.7357

vi. Tan 25° 34′

Sol. Tan 25° 34′ = 0.4784

vii. Sin 18° 31′

Sol. Sin 18° 31′ = 0.3176

viii. Cos 52°13′

Sol. Cos 52° 13′ = 0.6127

ix. Cot 89° 9′

Sol. Cot 89° 9′ = 0.0149

2. Find θ, if:

i. Sin θ = 0.5791

Sol. $\theta = \text{Sin}^{-1}(0.5791)$

$= 35° 23'$

ii. Cos θ = 0.9316

Sol. $\theta = \text{Cos}^{-1}(0.9316)$

$= 21° 91'$

iii. Cos θ = 0.5257

$\theta = \text{Cos}^{-1}(0.5257)$

$= 58° 17'$

iv. Tan θ = 1.705

Sol. $\theta = \text{Tan}^{-1}(1.705)$

$= 59° 36'$

v. Tan θ = 21.943

Sol. $\theta = \text{Tan}^{-1}(21.943)$

$= 87^0 23'$

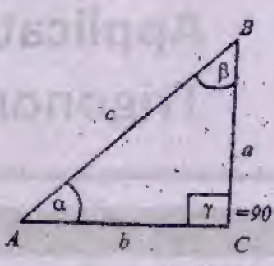
vi. Sin θ = 0.5186

Sol. $\theta = \text{Sin}^{-1}(0.5186)$

$= 31° 14'$

EXERCISE 12.2

Important Formulas in right angle triangle



1). $\alpha + \beta + \gamma = 180^\circ$

$\alpha + \beta + 90^\circ = 180^\circ$

$\alpha + \beta = 90^\circ$

2). $a^2 + b^2 = c^2$

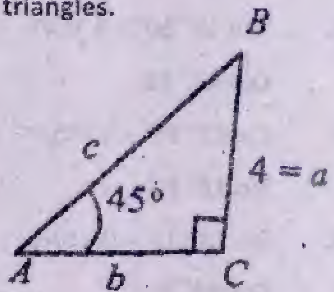
3). $\text{Sin}\,\alpha = \dfrac{a}{c} \Rightarrow \alpha = Sin^{-1}\left(\dfrac{a}{c}\right)$

4). $\text{Cos}\,\alpha = \dfrac{b}{c} \Rightarrow \alpha = Cos^{-1}\left(\dfrac{b}{c}\right)$

5). $\text{Tan}\,\alpha = \dfrac{a}{b} \Rightarrow \alpha = Tan^{-1}\left(\dfrac{a}{b}\right)$

1. Find the unknown angles and sides of the following triangles.

i.



Sol. $a = 4,\ b = ?,\ c = ?\ ,\ \alpha = 45^\circ,\ \gamma = 90^\circ,\ \beta = ?$

$\beta = 90^\circ - \alpha$

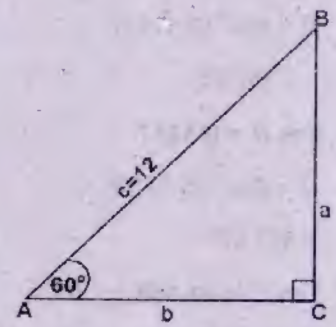
$\beta = 90^\circ - 45^\circ = 45^\circ \Rightarrow \boxed{\beta = 45^\circ}$

$\text{Sin}\,\alpha = \dfrac{a}{c} \Rightarrow Sin45^\circ = \dfrac{4}{c} \Rightarrow c = \dfrac{4}{0.7071} \Rightarrow \boxed{c = 5.656}$

$\tan\alpha = \dfrac{a}{b}$

$\tan 45^0 = \dfrac{4}{b} \Rightarrow 1 = \dfrac{4}{b} \Rightarrow \boxed{b = 4}$

ii.



Sol. $\beta = ?,\ \alpha = 60^\circ,\ \gamma = 90^\circ,\ c = 12,\ b = ?,\ a = ?$

$\beta = 90^\circ - \alpha$

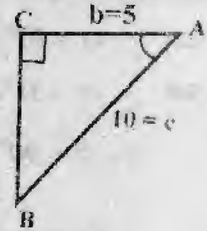
$\beta = 90^\circ - 60^\circ = 30^\circ \Rightarrow \boxed{\beta = 30^\circ}$

$\text{Sin}\ \alpha = \dfrac{a}{b} \Rightarrow \text{Sin}\ 60^\circ = \dfrac{a}{12} \Rightarrow a = 12\,(0.866) \Rightarrow \boxed{a = 10.39}$

Now $a^2 + b^2 = c^2 \Rightarrow (10.39)^2 + b^2 = (12)^2 \Rightarrow b^2 = 144 - 108 \Rightarrow b^2 = 36 \Rightarrow \boxed{b = 6}$

iii.

Sol. $b = 5, c = 10, a = ?\quad , \quad \gamma = 90^\circ,\ \alpha = ?,\ \beta = ?$



$$a^2 + b^2 = c^2 \Rightarrow a^2 = c^2 - b^2 \Rightarrow a^2 = 10^2 - 5^2 = 100 - 25 = 75 \Rightarrow a = 8.66$$

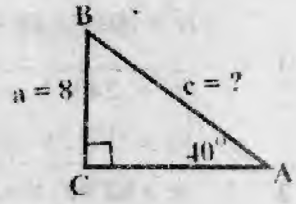
$$\text{Now } \cos\alpha = \frac{b}{c} = \frac{5}{10} = \frac{1}{2} \Rightarrow \alpha = \cos^{-1}\left(\frac{1}{2}\right)$$

$$= \cos^{-1}(0.5) \Rightarrow \boxed{\alpha = 60^\circ}$$

$$\beta = 90^\circ - \alpha = 90^\circ - 60^\circ = 30^\circ \Rightarrow \boxed{\beta = 30^\circ}$$

iv.

Sol. $a = 8, b = ?\ c = ?\quad , \quad \alpha = 40^\circ,\ \beta = ?,\ \gamma = 90^\circ$



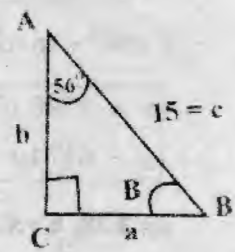
$$\beta = 90^\circ - \alpha = 90^\circ - 40^\circ \Rightarrow \boxed{\beta = 50^\circ}$$

$$\sin\alpha = \frac{a}{c} \Rightarrow \sin 40^\circ = \frac{8}{c} \Rightarrow c = \frac{8}{\sin 40^\circ} = \frac{8}{0.6427} \Rightarrow \boxed{c = 12.44}$$

$$a^2 + b^2 = c^2 \Rightarrow (8)^2 + b^2 = (12.44)^2 \Rightarrow b^2 = 154.89 - 64 = 90.89 \Rightarrow \boxed{b = 9.53}$$

v.

Sol. $a = ?, b = ?, c = 15,\ \alpha = 56^\circ,\ \beta = ?,\ \gamma = 90^\circ$



$$\beta = 90^\circ - \alpha = 90^\circ - 56^\circ \Rightarrow \boxed{\beta = 34^\circ}$$

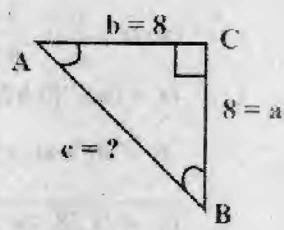
$$\sin\alpha = \frac{a}{c} = \frac{a}{15}$$

$$\sin 56^\circ = \frac{a}{15} \Rightarrow a = 15\,(0.890) \Rightarrow \boxed{a = 12.43}$$

$$a^2 + b^2 = c^2 \Rightarrow (12.43)^2 + b^2 = (15)^2 \Rightarrow b^2 = 225 - 154.50 = 70.49 \Rightarrow \boxed{b = 8.39}$$

vi.

Sol. $a = 8, b = 8, c = ?\quad , \quad \alpha = ?,\ \beta = ?,\ \gamma = 90^\circ$



$$a^2 + b^2 = c^2$$

$$8^2 + 8^2 = c^2 \Rightarrow 64 + 64 = c^2 \Rightarrow c^2 = 128 \Rightarrow c = 11.31$$

$$\tan\alpha = \frac{a}{b} = \frac{\cancel{8}}{\cancel{8}} = 1 \Rightarrow \alpha = \tan^{-1}(1) \Rightarrow \boxed{\alpha = 45^\circ}$$

$$\beta = 90^\circ - \alpha = 90^\circ - 45^\circ \Rightarrow \boxed{\beta = 45^\circ}$$

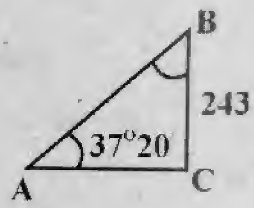
Solve the right triangle ABC, in which $\gamma = 90°$

2. $\alpha = 37° 20'$ **, a = 243** **Faisalabad 2007, Multan 2008, Sgd 2009,10, 11**

Sol. $\alpha = 37°23'$, $\beta = ?$, $\gamma = 90°$, a = 243, b = ?, c = ?

$\beta = 90° - \alpha = 90° - 37° 20' \Rightarrow \boxed{\beta = 52° 40'}$

$\text{Sin}\,\alpha = \frac{a}{c} \Rightarrow \text{Sin } 37°20' = \frac{243}{c} \Rightarrow c = \frac{243}{0.6064} \Rightarrow \boxed{c = 400.69}$



$a^2 + b^2 = c^2 \Rightarrow (243)^2 + b^2 = (400.69)^2$

$b^2 = 160552.48 - 59049 \Rightarrow \boxed{b = 318.59}$

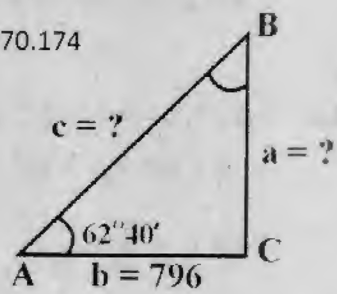
$$\boxed{\begin{array}{l} a = 243,\ b = 318.59,\ c = 400.49 \\ \alpha = 37°25',\ \beta = 52°40', \gamma = 90° \end{array}}$$

3. $\alpha = 62°40'$**, b = 796 , a = ? , c = ?** , $\beta = ?$, $\gamma = 90^0$

$\text{Cos}\,\alpha = \frac{b}{c} \Rightarrow \text{Cos } (62^0 40') = \frac{796}{c} \Rightarrow C = \frac{796}{0.4592} \Rightarrow \boxed{c = 1733.57}$

$a^2 + b^2 = c^2 \Rightarrow a^2 = c^2 - b^2 = (1733.57)^2 - (796)^2 = 2371670.174$

$\Rightarrow \boxed{a = 1540.02}$



$\beta = 90^0 - \alpha = 90^0 - 62^0 40' \Rightarrow \boxed{\beta = 27^0 20'}$

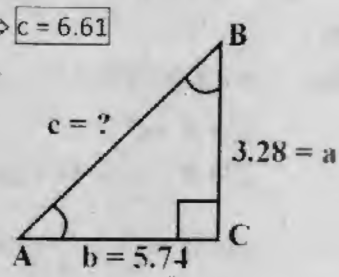
$$\boxed{\begin{array}{l} a = 1540.02\ ,\ b = 796\ ,\ c = 1733.57 \\ \alpha = 62^0 40'\ ,\ \beta = 27^0 20'\ ,\ \gamma = 90^0 \end{array}}$$

4. **a = 3.28, b = 5.74** **Sargodha 2008**

Sol. a = 3.28, b = 5.74, c = ? , $\alpha = ?$, $\beta = ?$, $\gamma = 90°$

$c^2 = a^2 + b^2 = (3.28)^2 + (5.74)^2 = 10.75 + 32.94 = 43.69 \Rightarrow \boxed{c = 6.61}$

$\tan\alpha = \frac{a}{b} = \frac{3.28}{5.74} = 0.5714$

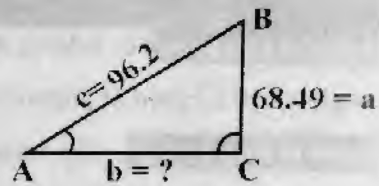


$\alpha = \tan^{-1}(0.5714) \Rightarrow \boxed{\alpha = 29°44'41''}$

$\beta = 90° - \alpha = 90° - 29°44' 41'' \Rightarrow \boxed{\beta = 60°15'18''}$

$$\boxed{\begin{array}{l} a = 3.28,\ b = 5.74,\ c = 6.61 \\ \alpha = 29°44'41'',\ \beta = 60°15'18'', \gamma = 90° \end{array}}$$

5. **b = 68.4 , c = 96.2**



Sol. $a = ?, b = 68.4, c = 96.2 \quad , \quad \alpha = ?, \ \beta = ?, \gamma = 90°$

$a^2 + b^2 = c^2 \Rightarrow a^2 + (68.4)^2 = (96.2)^2$

$a^2 = 9370.24 - 4678.56 = 6575.88 \Rightarrow a = 67.64$

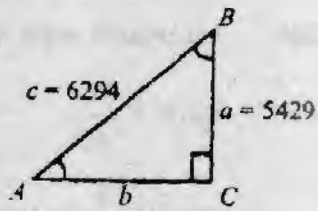
$\cos\alpha = \dfrac{b}{c} = \dfrac{67.64}{96.2}$

$\alpha = \cos^{-1}(0.7031) \Rightarrow \boxed{\alpha = 45°19'20''}$

$\beta = 90° - \alpha = 90° - 45°19'20'' = 44°40'40''$

$$\boxed{\begin{array}{l} a = 67.64,\ b = 68.4,\ c = 96.2 \\ \alpha = 45°19'20'',\ \ \beta = 44°40'40'',\ \gamma = 90° \end{array}}$$

6. **a = 5429, c = 6294**



$a = 5429, b = ?, c = 6294 \ , \ \ \alpha = ?, \ \beta = ?, \gamma = 90°$

$a^2 + b^2 = c^2 \Rightarrow (5429)^2 + b^2 = (6294)^2$

$b^2 = 39614436 - 29474041 = 10140395 \Rightarrow \boxed{b = 3184.39}$

$\sin\alpha = \dfrac{a}{c} = \dfrac{5429}{6294} = 0.862 \Rightarrow \alpha = \sin^{-1}(0.8625) \Rightarrow \boxed{\alpha = 59°36'21''}$

$\beta = 90° - \alpha = 90° - 59°36'21'' \Rightarrow \boxed{\beta = 30°23'38''}$

$$\boxed{\begin{array}{l} a = 5429,\ b = 3184.39,\ c = 6294 \\ \alpha = 59°36'21'',\ \ \beta = 30°23'38'',\ \gamma = 90° \end{array}}$$

7. **β = 50° 10' , c = 0.832**

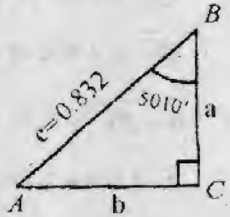
Sol. $\beta = 50°10'$, $\alpha = ?$, $\gamma = 90°$, $c = 0.832$, $b = ?$, $a = ?$

$\alpha = 90° - \beta = 90° - 50°10' \Rightarrow \alpha = 39°50'$

$\sin\alpha = \dfrac{a}{c} \Rightarrow \sin 39°50' = \dfrac{a}{0.832} \Rightarrow a = (0.832)(0.6405) \Rightarrow \boxed{a = 0.5329}$

$a^2 + b^2 = c^2 \Rightarrow (0.5329)^2 + b^2 = (0.832)^2$

$b^2 = 0.6922 - 0.2840 = 0.4082 \Rightarrow \boxed{b = 0.6389}$



$$\boxed{\begin{array}{l} a = 0.5329,\ b = 0.6389,\ c = 0.832 \\ \alpha = 39°50',\ \beta = 50°10',\ \gamma = 90° \end{array}}$$

EXERCISE 12.3

Angle of Elevation: For looking at B above the horizontal line, we have to raise our eyes then angle <AOB is called of angle of elevation. (see figureI) **Fsd 2008, Multan 2009, Sgd 2009,10**

Angle of Depression: For looking at C below the horizontal line we have to lower our eyes, then angle <AOC is called angle of depression.

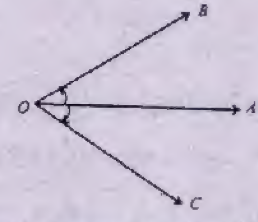
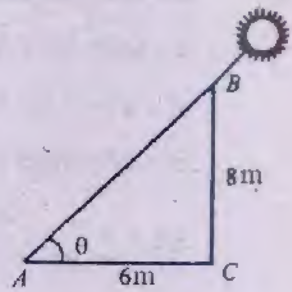


Figure: I

1. **A vertical pole is 8m high and the length of its shadow is 6m. what is the angle of elevation of the sun at that moment? Gujranwala 2009**

Sol. Let require angle is θ then



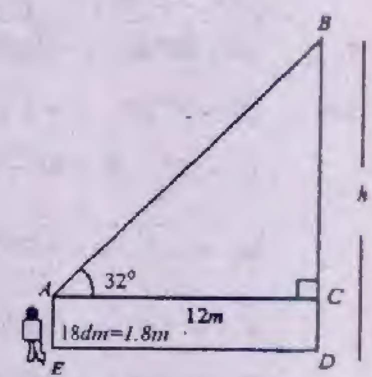
$\tan\theta = \dfrac{\overline{BC}}{\overline{AC}}$

$\tan\theta = \dfrac{8}{6} \Rightarrow \tan\theta = 1.33$

$\theta = \tan^{-1}(1.33) = \boxed{53°7'48''}$

2. **A man 18 dm tall observes that the angle of elevation of the top of a tree at a distance of 12m from him is 32°. What is the height of the tree?**

Sol. Let $\overline{AE}$ be height of man and h be the height of tree



$\overline{AE} = \overline{CD} = 18dm = 1.8m$

Then $\tan\theta = \dfrac{\overline{BC}}{\overline{AC}}$

$\text{Tan } 32° = \dfrac{\overline{BC}}{12} \Rightarrow 0.624 = \dfrac{\overline{BC}}{12}$

$\overline{BC} = 12(0.624)$

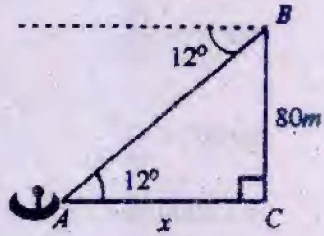
$\overline{BC} = 7.498 \text{ m} = 7.5 \text{ m}$

$h = \overline{BC} + \overline{CD}$

$= 7.5\text{m} + 1.8\text{m} \Rightarrow \boxed{h = 9.3\text{m}}$

3. At the top of a cliff 80m high, the angel of depression of a boat is 12°. How far is the boat from the cliff? Multan 2007, Faisalabad 2008

Sol. Let x be required distance



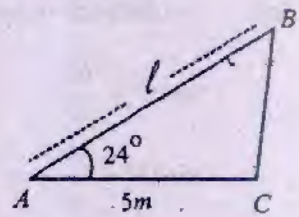
Then $\tan\theta = \dfrac{\overline{BC}}{\overline{AC}}$

$\tan 12° = \dfrac{80}{x} \Rightarrow 0.2125 = \dfrac{80}{x}$

$x = \dfrac{80}{0.2125} \Rightarrow \boxed{x = 376.37\text{m}}$

4. A ladder leaning against a vertical wall makes an angle of 24° with the wall. Its foot is 5m from the wall. Find its length.

Sol. Let ℓ be length of ladder

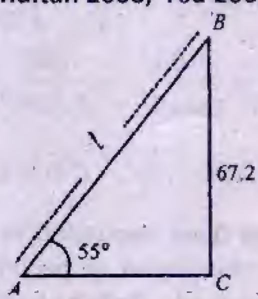


Then $\cos\theta = \dfrac{\overline{BC}}{\overline{AC}} \Rightarrow \cos 24° = \dfrac{5}{\ell}$

$\ell = \dfrac{5}{Cos24°} = \dfrac{5}{0.9135} \Rightarrow \boxed{\ell = 5.47\text{m}}$

5. A kite flying at a height of 67.2 m is attached to a fully stretched string inclined at an angle of 55° to the horizontal. Find the length of the string.

Sol. Let l be length of string then **Multan 2008, Fsd 2009**



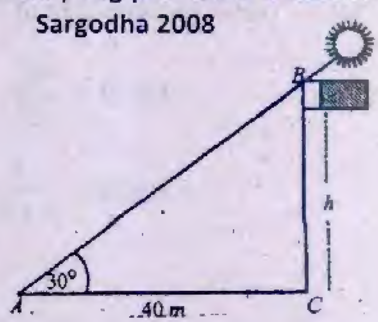
$\sin\theta = \dfrac{\overline{BC}}{\overline{AB}} \Rightarrow \sin 55° = \dfrac{6.72}{\ell}$

$\ell = \dfrac{67.2}{Sin(55^{o})}$

$\boxed{\ell = 82.03\text{m}}$

6. When the angle between the ground and the sun is 30°, flag pole casts a shadow of 40m long. Find the height of the top of the flag. Sargodha 2008

Sol. Let h be the height of plane then



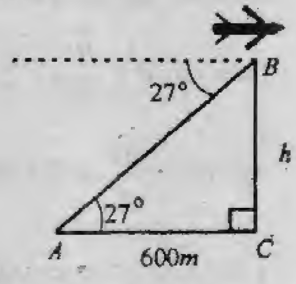
$\tan\theta = \dfrac{\overline{BC}}{\overline{AC}} \Rightarrow \tan 30° = \dfrac{h}{40}$

h = 40 (tan 30°)

h = 40 (0.577) $\Rightarrow \boxed{h = 23.1\text{m}}$

7. A plane flying directly above a post 6000 m away from an anti-aircraft gum observes the gun at an angle of depression of 27°. Find the height of the plane.

Sol. Let h be the height of plane then



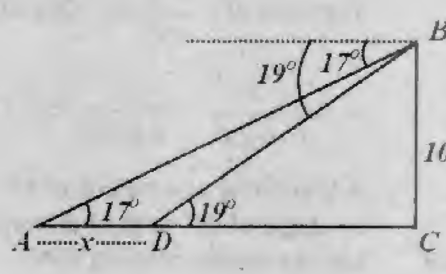
$$\tan\theta = \frac{\overline{BC}}{\overline{AC}}$$

$$\tan 27^\circ = \frac{h}{6000}$$

$h = 6000\,(\tan 27^\circ) \Rightarrow h = 600\,(0.5095) = 3057.15$

8. A man on the top of a 100 m high light-house is in line with two ships on the same side of it, whose angles of depression from the man are 17° and 19° respecting. Find the distance between the ships.

Sol. Let distance between two shop is x then



$$\tan\theta = \frac{\overline{BC}}{\overline{AC}}$$

$$\tan 17^\circ = \frac{100}{\overline{AC}} \Rightarrow \overline{AC} = \frac{100}{\tan 17^\circ}$$

$$\overline{AC} = \frac{100}{0.3057} \Rightarrow \overline{AC} = 327.08$$

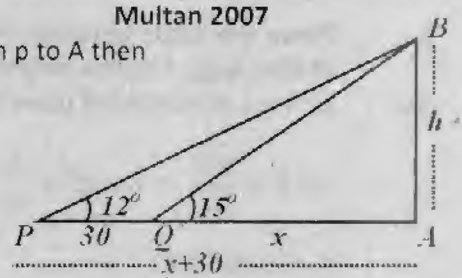
$$\text{Now } \tan 19^\circ = \frac{\overline{BC}}{\overline{CD}} = \frac{100}{\overline{CD}}$$

$$\overline{CD} = \frac{100}{\tan 19^\circ} = \frac{100}{0.3443} \Rightarrow \overline{CD} = 290.42$$

$$x = \overline{AD} = \overline{AC} - \overline{CD} = 327.08 - 290.42 \Rightarrow \boxed{x = 36.7\text{ m}}$$

9. P and Q are two points in line with a tree. If the distance between P and Q be 30m and the angles of elevation of the top of the tree at P and Q be 12° and 15° respectively, find the height of the tree. Multan 2007

Sol. Let h be the height of tree & x be distance from p to A then



$$\tan\theta = \frac{\overline{AB}}{\overline{AP}}$$

$$\tan 12^\circ = \frac{h}{x+30} \Rightarrow x+30 = \frac{h}{\tan 12^\circ}$$

$$x = \frac{h}{0.2125} - 30 \text{ ——— } I$$

Again $\tan 15^\circ = \frac{\overline{AB}}{AQ} = \frac{h}{x}$

$$x = \frac{h}{\tan 15''} \Rightarrow x = \frac{h}{0.2679}$$

$$x = \frac{h}{0.2679} \text{ ——— } II$$

Comparing I & II $\frac{h}{0.2679} = \frac{h}{0.2125} - 30$

$$\frac{h}{0.2125} - \frac{h}{0.2679} = 30 \Rightarrow h\left(\frac{1}{0.2125} - \frac{1}{0.2679}\right) = 30$$

h (4.7058 – 3.7327) = 30

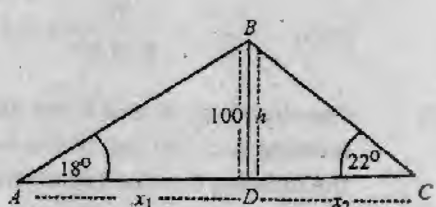
0.9730 h = 30 ⇒ h = $\frac{30}{0.9730}$ ⇒ $\boxed{h = 30.9 \text{ m}}$

10. **Two men are on the opposite sides of a 100m high tower. If the measure of the angles of elevation of the top of the tower are 18° and 22° respectively find the distance between them, (Federal Board)**

Sol. Let distance between A & is x_1 & x_2 between D & C and h is height

Tan θ = $\frac{\overline{BD}}{AD} \Rightarrow \tan 18'' = \frac{100}{x_1}$

$$\Rightarrow x_1 = \frac{100}{\tan 18''} = \frac{100}{0.3249} \Rightarrow x_1 = 307.76$$

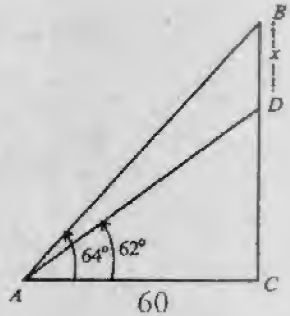


Also $\tan\theta = \frac{\overline{BD}}{DC} \Rightarrow \tan 22'' = \frac{100}{x_2}$

x_2 = $\frac{100}{\tan 22''} = \frac{100}{0.4040} \Rightarrow x_2 = 247.5$

Reluaried distance = $x_1 + x_2$ = 307.76m + 247.5m $\boxed{= 555.26\text{m}}$

11. A man standing 60m away from a tower notices that the angles of elevation of the top and the bottom of a flag staff on the top of the tower are 64° and 62° respectively. Find the length of the flag staff.



Sol. Let x be the length of flag staff then tan $\theta = \dfrac{\overline{CD}}{\overline{AC}}$

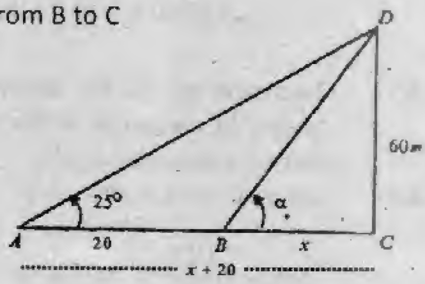
Tan 62° = $\dfrac{\overline{CD}}{60} \Rightarrow \overline{CD}$ = 60 (tan 62°) = 60(1.88)= 112.84

Now tan 64° = $\dfrac{\overline{BC}}{\overline{AC}} \Rightarrow 2.050 = \dfrac{\overline{BC}}{60}$

$\overline{BC}$ = 60 (2.0540) $\Rightarrow \overline{BC}$ = 123.01

so $x = \overline{BC} - \overline{CD}$ = 123.01 − 112.84 $\Rightarrow \boxed{x = 10.17\text{m}}$

12. The angle of elevation of the top of a 60m high tower from a point A, on the same level as the foot of the tower is 25°. Find the angle of elevation of the top of the tower from a point B, 20m nearer to A from the foot of the tower.



Sol. Let α be the required angle Let x be distance from B to C

Tan $\theta = \dfrac{\overline{CD}}{\overline{AC}} \Rightarrow$ Tan 25° = $\dfrac{60}{x+20}$

$$x + 20 = \frac{60}{\tan 25^\circ} = \frac{60}{0.4663}$$

$$\Rightarrow x+20=128.67 \Rightarrow x = 128.67 - 20 = 108.67$$

$$\tan\alpha = \frac{\overline{CD}}{\overline{BC}} = \frac{60}{108.67} = 0.5521 \Rightarrow \alpha = \tan^{-1} 0.5521 \Rightarrow \boxed{\alpha = 28^\circ 54' 16''}$$

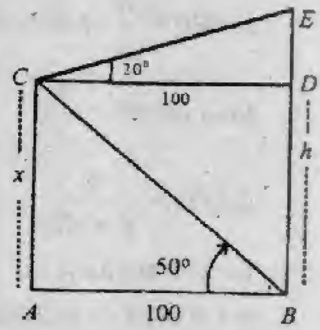
13. Two buildings A and B are 100m apart. The angle of elevation from the top of the building A to the top of the building B is 20°. the angle of elevation from the base of the building B to the top of the building A is 50°. find the height of the building B.

Sol. Let h be the height of building B and x be the height of A then

Tan $\theta = \dfrac{\overline{DE}}{\overline{CD}}$

Tan 20° = $\dfrac{\overline{DE}}{100}$

$0.3639 = \dfrac{\overline{DE}}{100} \Rightarrow \overline{DE} = 100(0.3639)$



$\overline{DE} = 36.39$

Now $\tan\theta = \dfrac{\overline{AC}}{\overline{AB}} \Rightarrow \tan 50^\circ = \dfrac{\overline{AC}}{100}$

$\overline{AC} = 100(\tan 50^\circ) = 100\ (1.1917)$

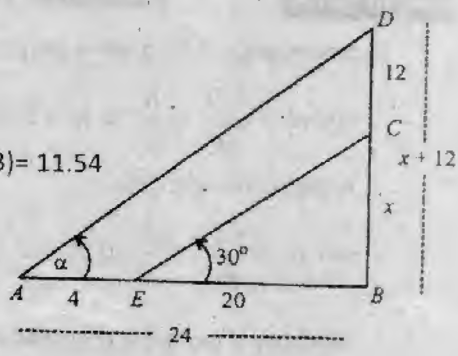
$\overline{AC} = 119.17$

So $h = \overline{BD} + \overline{DE} = \overline{AC} + \overline{DE} = 119.17 + 36.39 \Rightarrow \boxed{h = 155.56\text{ m}}$

14. A window washer is working in a hotel building. An observer at a distance of 20m from the building finds the angle of elevation of the worker to be of 30°. The worker climbs up 12m and the observer moves 4m farther away from the building. Find the new angle of elevation of the worker.

Sol. Let α be the new angle

$\text{Tan}\ \theta = \dfrac{\overline{BC}}{\overline{BE}}$



$\text{Tan } 30^\circ = \dfrac{x}{20} \Rightarrow x = 20\ \tan 30^\circ = 20\ (0.5773) = 11.54$

$\tan\alpha = \dfrac{\overline{BD}}{\overline{AB}}$

$\tan\alpha\ \dfrac{x+12}{24} = \dfrac{11.54+12}{24} = \dfrac{23.54}{24} = 0.98$

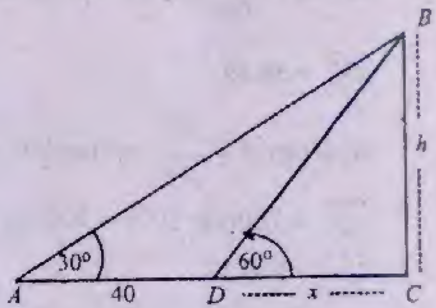
$\alpha = \tan^{-1}(0.98) \Rightarrow \boxed{\alpha = 44^\circ 25'}$

15. A man standing on the bank of canal observes that the measure of the angle of elevation of a tree on the other side of the canal, is 60°. On retreating 40 meters from the bank, he finds the measure of the angle of elevation of the tree as 30°. Find the height of the tree and the width of the canal.

Sol. Let h be required height & x be required width so

$\text{Tan}\ \theta = \dfrac{\overline{BC}}{\overline{CD}} \Rightarrow \tan 60^\circ = \dfrac{h}{x}$

$1.7320 = \dfrac{h}{x} \Rightarrow h = 1.7320\ x$ ——— I



Now $\tan 30^o = \dfrac{\overline{BC}}{\overline{AC}}$

$0.5773 = \dfrac{h}{x+40}$

$h = (x+40)(0.5773)$

$h = 0.5773\ x + 23.0940$ ——— II

Comparing I & II

$1.7320\ x = 0.5773\ x + 23.0940$

$1.1547\ x = 23.0940$

$x = \dfrac{23.0940}{1.1547} = 20m \Rightarrow$ $\boxed{\text{Width} = x = 20\text{ m}}$

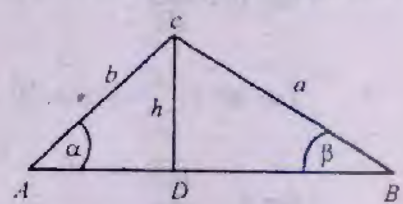
Put in I

$h = 17.320\ (20) \Rightarrow$ $\boxed{\text{Height} = h = 34.64}$

Low of Sines: **Faisalabad 2007, Rawalpindi 2009, Sgd 2006(only statement)**

In any triangle ABC draw a perpendicular form C to $\overline{AB}$ at D then. In right triangle CAD

$\text{Sin}\ \alpha = \dfrac{\overline{CD}}{\overline{AC}} = \dfrac{h}{b} \Rightarrow h = b\ Sin\ \alpha$ ——— I



In right triangle CBD

$\text{Sin}\ \beta = \dfrac{\overline{CD}}{\overline{BC}} = \dfrac{h}{a} \Rightarrow h = a\ Sin\beta$ ——— II

Similarly if we draw a perpendicular from A to $\overline{BC}$ then $\dfrac{b}{Sin\beta} = \dfrac{C}{Sin\gamma}$ ——— IV

Combining III & IV

$\boxed{\dfrac{a}{Sin\alpha} = \dfrac{b}{Sin\beta} = \dfrac{c}{Sin\gamma}}$ Hence proved

Low of cosine: **Faisalabad 2007, 08, Lhr 2009, Multan 2008, Sgd 2011**

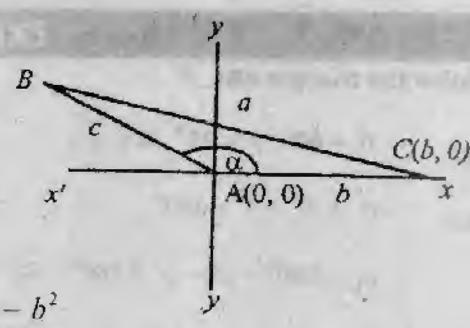
In any triangle ABC, co-ordinates of points are A(0,0), C(b,0), B(c cosα , Csinα)

Then by distance formula

$(BC)^2 = (c\ Cos\alpha - b)^2 + (c\ Sin\alpha - 0)^2$

$a^2 = c^2\ Cos^2\alpha - 2bc\ Cos\alpha + b^2 + c^2\ Sin^2\alpha$

$= c^2 (\cos^2\alpha + \sin^2\alpha) - 2bc\cos\alpha + b^2$



$\Rightarrow a^2 = c^2 + b^2 - 2bc\cos\alpha$

or $a^2 = b^2 + c^2 - 2bc\cos\alpha$

Similarly $b^2 = a^2 + c^2 - 2ac\cos\beta$

$c^2 = a^2 + b^2 - 2ab\cos\gamma$

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}, \ \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos\gamma = \frac{b^2 + c^2 - a^2}{2bc}$$

Low of Tangent: **Sargodha 2008, 2010 (only statement)**

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\frac{\alpha+\beta}{2}}$$

Proof: We have

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = D \text{ (say)}$$

then $a = D\sin\alpha$, $b = D\sin\beta$

$$\frac{a-b}{a+b} = \frac{D\sin\alpha - D\sin\beta}{D\sin\alpha + D\sin\beta} = \frac{D(\sin\alpha - \sin\beta)}{D(\sin\alpha + \sin\beta)}$$

$$\frac{a-b}{a+b} = \frac{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}} = \cot\frac{\alpha+\beta}{2}\tan\frac{\alpha-\beta}{2}$$

$$\frac{a-b}{a+b} = \frac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}}$$ **If a>b**

$$\text{Similarly}\ \frac{b-c}{b+c} = \frac{\tan\frac{\beta-\gamma}{2}}{\tan\frac{\beta+\gamma}{2}} \ \& \ \frac{a-c}{a+c} = \frac{\tan\left(\frac{\alpha-\gamma}{2}\right)}{\tan\left(\frac{\alpha+\gamma}{2}\right)}$$

EXERCISE 12.4

Solve the triangle ABC, if

1. $\beta = 60^\circ, \gamma = 15^\circ, b = \sqrt{6}$ **Faisalabad 2008, Multan 2009**

Sol. $\alpha + \beta + \gamma = 180^\circ$

$\alpha = 180^\circ - \beta - \gamma = 180^\circ - 60^\circ - 15^\circ = 105^\circ$

$$\frac{a}{Sin\alpha} = \frac{b}{Sin\beta} \Rightarrow \frac{a}{Sin105^o} = \frac{\sqrt{6}}{Sin60^o}$$

$$a = \frac{\sqrt{6}\ Sin105^o}{Sin60^o} = \frac{\sqrt{6}(0.9659)}{0.8660} = 2.7320 = \sqrt{3}+1$$

$c^2 = a^2 + b^2 - 2ab\ Cos\gamma$

$c^2 = (2.7320)^2 + (\sqrt{6})^2 - 2(2.7320)(\sqrt{6})\cos 15^\circ$

$= 7.4641 + 6 - (13.38)(40)(0.9659) = 0.5358 \Rightarrow c = 0.7320 = \sqrt{3} - 1$

$$\boxed{\begin{array}{l} a = 2.73 = \sqrt{3}-1,\ b = \sqrt{6},\ c = 0.5358 = \sqrt{3}+1 \\ \alpha = 105^o,\ \beta = 60^o,\ \gamma = 15^o \end{array}}$$

2. $\beta = 52^\circ, \gamma = 89^\circ 35', a = 89.35$

Sol. $\alpha + \beta + \gamma = 180^0 \Rightarrow \alpha = 180^\circ - \beta - \gamma$

$\alpha = 180^\circ - 50^\circ - 89^\circ 35' = 38^\circ 25'$

$$\frac{a}{Sin\alpha} = \frac{b}{Sin\beta} \Rightarrow \frac{89.35}{Sin38^o25'} = \frac{b}{Sin(52^o)}$$

$$b = b = \frac{(89.35)(Sin52^o)}{Sin38^o25'} = \frac{(89.35)(0.7880)}{0.6213} = 113.32 \Rightarrow \boxed{b = 113.32}$$

$c^2 = a^2 + b^2 - 2ab\ Cos\gamma$

$c^2 = (89.35)^2 + (113.32)^2 - 2(89.35)(113.32)\ Cos\ 89^\circ 35'$

$= 7983.42 + 12841.4 - 147.26 = 20677.56 \Rightarrow \boxed{c = 143.79}$

$$\boxed{\begin{array}{l} a = 89.35,\ b = 113.32,\ c = 143.79 \\ \alpha = 38^o 25',\ \beta = 52^o,\ \gamma = 89^o 35' \end{array}}$$

3. **b = 125, γ = 53°, α = 47°**

Sol. $\beta = 180^\circ - \alpha - \gamma = 180^\circ - 47^\circ - 53^\circ = 80^\circ \Rightarrow \boxed{\beta = 80^\circ}$

Now $\dfrac{a}{Sin\alpha} = \dfrac{b}{Sin\beta}$

$$\frac{a}{Sin47^o} = \frac{125}{Sin80^o} \Rightarrow a = \frac{125\,Sin47^o}{Sin80^o}$$

$$a = \frac{125(0.7313)}{0.9848} = 92.82 \approx 93 \Rightarrow \boxed{a = 93}$$

$$\frac{a}{Sin\alpha} = \frac{c}{Sin\gamma} \Rightarrow \frac{92.82}{Sin47^o} = \frac{c}{Sin53^o}$$

$$c = \frac{(92.82)Sin53^o}{Sin47^o} = \frac{(92.82)(0.7986)}{0.7313} = 101.36 \approx 101 \Rightarrow \boxed{c = 101}$$

$$\boxed{\begin{array}{l} a = 93, b = 125, c = 101 \\ \alpha = 47^o, \beta = 80^o, \gamma = 53^o \end{array}}$$

4. **c = 16.1, α = 42°45′, γ = 74°32′**

Sol. $\alpha + \beta + \gamma = 180^\circ$

$\beta = 180^\circ - \alpha - \gamma = 180^\circ - 42^\circ45' - 74^\circ32' \Rightarrow \boxed{\beta = 62^\circ43'}$

$$\frac{a}{Sin\alpha} = \frac{c}{Sin\gamma} \Rightarrow \frac{a}{Sin42^o45'} = \frac{16.1}{Sin74^o32'}$$

$$a = \frac{16.1\,Sin42^o45'}{Sin74^o32'} = \frac{(16.1)(0.6788)}{0.9637}$$

$$a = \frac{10.92}{0.9637} \Rightarrow a = 11.34$$

$$\frac{a}{Sin\alpha} = \frac{b}{Sin\beta}$$

$$\frac{11.34}{Sin42^o45'} = \frac{b}{Sin62^o43'} \Rightarrow b = \frac{(11.34)Sin62^o43'}{Sin42^o45'}$$

$$b = \frac{(11.34)(0.888)}{0.6788} = 14.83 \Rightarrow b = 14.83$$

$$\boxed{\begin{array}{l} a = 11.34,\ b = 14.83,\ c = 16.1 \\ \alpha = 42^\circ 45,\ \beta = 62^\circ 43',\ \gamma = 74^\circ 32' \end{array}}$$

5. **$a = 53$, $\beta = 88^\circ 36'$, $\gamma = 31^\circ 54'$**

Sol. $\alpha = 180^\circ - \beta - \gamma = 180^\circ - 88^\circ 36' - 31^\circ 54' \Rightarrow \boxed{\alpha = 59^\circ 30'}$

$$\frac{a}{Sin\alpha} = \frac{b}{Sin\beta} \Rightarrow \frac{53}{Sin\,59^\circ 30'} = \frac{b}{Sin\,88^\circ 36'}$$

$$b = \frac{53(Sin 88^\circ 36')}{Sin 59^\circ 30'} = \frac{53(0.9997)}{0.8616} \Rightarrow \boxed{b = 61.49}$$

$$\frac{a}{Sin\alpha} = \frac{c}{Sin\gamma} \Rightarrow \frac{53}{Sin 59^\circ 30'} = \frac{c}{Sin 31^\circ 54'} \Rightarrow c = \frac{(53)\,Sin 31^\circ 54'}{Sin\,59^\circ\,30'}$$

$$c = \frac{(53)(0.5284)}{0.8616} = 32.50 \Rightarrow \boxed{c = 32.50}$$

$$\boxed{\begin{array}{l} a = 53,\ b = 61.49,\ c = 32.50 \\ \alpha = 59^\circ 30',\ \beta = 88^\circ 36',\ \gamma = 31^\circ 54' \end{array}}$$

EXERCISE 12.5

Solve the triangle ABC in which:

1. **$b = 95$, $c = 34$ and $\alpha = 52^\circ$** **Faisalabad 2008, Sargodha 2009, 2011**

Sol. $a^2 = b^2 + c^2 - 2bc\,Cos\,\alpha$

$= (95)^2 + (34)^2 - 2(95)(34)\,Cos\,52^\circ$

$= 9025 + 1156 - 6460\,(0.6156)$

$= 10181 - 3977.17$

$= 6203.83 \Rightarrow \boxed{a = 78.76}$

$$Cos\,\beta = \frac{a^2 + c^2 - b^2}{2ac} = \frac{(78.76)^2 + (34)^2 - (95)^2}{2(78.76)(34)}$$

$$= \frac{6203.83 + 1156 - 9025}{5355.68} = \frac{-1665.17}{5355.68}$$

$\beta = \cos^{-1}(-0.3109) = 108°6'48''$

$\gamma = 180° - \alpha - \beta$

$= 180° - 52° - 108°6'48'' \Rightarrow \boxed{\gamma = 19°53'12''}$

$$\boxed{\begin{array}{l} a = 78.76,\ b = 95,\ c = 34 \\ \alpha = 52°,\ \beta = 108°6'48'',\ \gamma = 19°53'12'' \end{array}}$$

2. **b=12.5, c = 23, α =38°20'**

Sol. $a^2 = b^2 + c^2 - 2bc\cos\alpha$

$=(12.5)^2+(23)^2-2(12.5)(23)\cos 38°20'$

$=156.25+529-575(0.7844)=685.25-451.03=234.22 \Rightarrow \boxed{a=15.30}$

$$\cos\beta = \frac{a^2+c^2-b^2}{2ac} = \frac{(15.30)^2+(23)^2-(12.50)^2}{2(15.30)(23)}$$

$$= \frac{234.09+529-156.25}{703.8} = \frac{606.84}{703.8} = 0.862$$

$\beta = \cos^{-1}(0.862) = 30°25'$

$\gamma = 180° - \alpha - \beta = 180° - 38°20' - 30°25' = 111°15' \Rightarrow \boxed{\gamma = 111°15'}$

$$\boxed{\begin{array}{l} a = 15.30,\ b = 12.5,\ c = 23 \\ \alpha = 38°20',\ \beta = 30°25',\ \gamma = 111°15' \end{array}}$$

3. $a = \sqrt{3} - 1$, $b = \sqrt{3} + 1$ and $\gamma = 60°$

Sol. $c^2 = a^2 + b^2 - 2ab\cos\gamma$

$= (\sqrt{3}-1)^2 + (\sqrt{3}+1)^2 - 2(\sqrt{3}-1)(\sqrt{3}+1)\cos 60°$

$= 3 + 1 - 2\sqrt{3} + 3 + 1 + 2\sqrt{3} - 2(3-1)(1/2)$

$= 8 - 2(2)(0.5) = 8 - 2 = 6 \Rightarrow \boxed{c = \sqrt{6}}$

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc} = \frac{(\sqrt{3}+1)^2+(\sqrt{6})^2-(\sqrt{3}-1)^2}{2(\sqrt{3}+1)(\sqrt{6})}$$

$$= \frac{3+1+2\sqrt{3}+6-3-1+2\sqrt{3}}{13.38} = \frac{12.928}{13.38} = 0.9662$$

$\alpha = \cos^{-1}(0.9662) = 14°55'54'' \approx 15° \Rightarrow \boxed{\alpha = 15°}$

$\beta = 180^0 - \alpha - \gamma = 180° - 15° - 60° = 105^0 \Rightarrow \boxed{\beta = 105^0}$

$$\boxed{\begin{array}{l} a=\sqrt{3}-1,\ b=\sqrt{3}+1,\ \ c=\sqrt{6} \\ \alpha=15^O,\ \beta=105^O,\ \gamma=60^O \end{array}}$$

4. **a = 3, c = 6 and β = 36°26′**

Sol. $b^2 = a^2 + c^2 - 2ac\,\text{Cos}\,\beta$

$= (3)^2 + (6)^2 - 2(3)(6)\,\text{Cos}\,36°20' = 9 + 36 - 36\,(0.8055)$

$= 45 - 29.0010 = 15.99 \Rightarrow b = 3.998 \Rightarrow \boxed{b = 4}$

$$\text{Cos}\,\alpha = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(4)^2 + (6)^2 - (3)^2}{2(4)(6)} = \frac{16 + 36 - 9}{48}$$

$$\text{Cos}\,\alpha = \frac{43}{48} \Rightarrow \alpha = \text{Cos}^{-1}(0.8958) \Rightarrow \boxed{\alpha = 26°23'4''}$$

$\gamma = 180° - \alpha - \beta = 180° - 26°23'4'' - 36°20' \Rightarrow \boxed{\gamma = 117^0 16'56''}$

$$\boxed{\begin{array}{l} a=3,\ b=4,\ \ c=6 \\ \alpha=26^O 23'4'',\ \beta=36^O 26',\ \gamma=117^O 16'56'' \end{array}}$$

5. **a=7 , b = 3 , γ =38°13′**

$c^2 = a^2 + b^2 - 2ab\cos\gamma$

$c^2 = (7)^2 + (3)^2 - 2\,(7)\,(3)\cos 38^O 13' = 49 + 9 - 42\,(0.7856)$

$c^2 = 58 - 32.9984 = 25.0016 \approx 25 \Rightarrow \boxed{c = 5}$

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc} = \frac{(3)^2+(5)^2-(7)^2}{2(3)(5)} = \frac{9+25-49}{30} = \frac{-1}{2}$$

$$\alpha = \cos^{-1}\left(\frac{-1}{2}\right) = 120^O \Rightarrow \boxed{\alpha = 120^O}$$

$$\beta = 180^O - \alpha - \gamma = 180^O - 120^O - 38^O 13' = 21^O 47'$$

$$\boxed{\begin{array}{l} a=7,\ b=3,\ \ c=5 \\ \alpha=120^O,\ \beta=21^O 47',\ \gamma=38^O 13' \end{array}}$$

Solve the following triangle, using first law of tangents and then Law of sines:

6. **a = 36.21 , b = 42.09 and γ = 44°29′**

$\alpha + \beta = 180^\circ - 44^\circ 29' = 135^\circ 31'$ ——I

$$\frac{\alpha+\beta}{2} = \frac{135^\circ 31'}{2} = 67^\circ 45' 31''$$

$$\frac{b-a}{b+a} = \frac{\tan\frac{\beta-\alpha}{2}}{\tan\frac{\beta+\alpha}{2}} \Rightarrow \frac{42.09-36.21}{42.09+36.21} = \frac{\tan\left(\frac{\beta-\alpha}{2}\right)}{\tan 67^\circ 45' 30''}$$

$$\frac{5.88}{78.3} = \frac{\tan\frac{\beta-\alpha}{2}}{2.4453}$$

$$\frac{2.4453 \times 5.88}{78.3} = \tan\frac{\beta-\alpha}{2}$$

$$0.1836 = \tan\left(\frac{\beta-\alpha}{2}\right)$$

$$\Rightarrow \frac{\beta-\alpha}{2} = \tan^{-1}(0.1836) = 10.4036$$

$\beta - \alpha = 20^\circ 45' 26''$ ______ II

Solving I & II

$\beta + \cancel{\alpha} = 135^\circ 31'$

$\beta - \cancel{\alpha} = 20^\circ 48' 26''$

$2\beta = 156^\circ 19' 26'' \Rightarrow \beta = 78^\circ 9' 43''$

Put value of β in I

$\alpha + 78^\circ 9' 43'' = 135^\circ 31'$

$\alpha = 135^\circ 31' - 78^\circ 9' 43'' = 57^\circ 21' 17''$

Now $\frac{b}{Sin\beta} = \frac{c}{Sin\gamma} \Rightarrow \frac{42.09}{Sin 78^\circ 9' 43''} = \frac{c}{Sin 44^\circ 29'} \Rightarrow \frac{(42.09)(Sin 44^\circ 29')}{Sin 78^\circ 9' 43''} = c$

$c = \frac{(42.09)(0.7007)}{0.9787} = 30.13 \Rightarrow \boxed{c = 30.13}$

$$\boxed{\begin{array}{l} a = 36.21,\ b = 42.09,\ c = 30.13 \\ \alpha = 57^\circ 21' 17'',\quad \beta = 78^\circ 9' 43'',\quad \gamma = 44^\circ 29' \end{array}}$$

7. **a = 93, c = 101 and β = 80°**

Sol. $\alpha + \beta + \gamma = 180^\circ \Rightarrow \alpha + \gamma = 18^\circ - \beta$

$\alpha + \gamma = 180^\circ - 80^\circ = 100^\circ$ ——I

$$\frac{\gamma + \alpha}{2} = 50^\circ$$

$$\frac{c-a}{c+a} = \frac{\tan\frac{\gamma-\alpha}{2}}{\tan\frac{\gamma+\alpha}{2}}$$

$$\frac{101-93}{101+93} = \frac{\tan\frac{\gamma-a}{2}}{\tan 50^\circ} \Rightarrow \frac{8}{194} = \frac{\tan\frac{\gamma-\alpha}{2}}{1.1917}$$

$\tan\frac{\gamma-\alpha}{2} = 0.04914 \Rightarrow \gamma - \alpha = 5^\circ 37' 37''$ ——II

Solving I & II

$\gamma - \cancel{\alpha} = 5^\circ 37' 37''$

$\gamma + \cancel{\alpha} = 100^\circ$

$2\gamma = 105^\circ 37' 37'' \Rightarrow \gamma = 52^\circ 48' 48'' \approx 53^\circ \Rightarrow \boxed{\gamma = 53^\circ}$

Put in I

$\alpha + 52^\circ 48' 48'' = 100^\circ \Rightarrow \alpha = 100^\circ - 52^\circ 48' 48'' = 47^\circ 11' 11'' \approx 48^\circ \Rightarrow \boxed{\alpha = 48^\circ}$

Now $\dfrac{a}{Sin\alpha} = \dfrac{b}{Sin\beta} \Rightarrow \dfrac{93}{Sin 47^\circ 11' 12''} = \dfrac{b}{Sin 80^\circ}$

$b = \dfrac{93 \times Sin 80^\circ}{Sin 47^\circ 11' 11''} = \dfrac{93(0.9848)}{0.7335} \Rightarrow$ b = 124.86 ≈ 125 $\Rightarrow \boxed{b=125}$

$$\boxed{\begin{array}{l} a = 93,\ c = 101,\ \ b = 125 \\ \alpha = 48^\circ,\ \ \beta = 80^\circ,\ \ \gamma = 53^\circ \end{array}}$$

8. **b = 14.8, c = 16.1 and α = 42°45′**

Sol. $\alpha + \beta + \gamma = 180° \Rightarrow \beta + \gamma = 180° - \alpha$

$\beta + \gamma = 180° - 42°45' \Rightarrow \gamma + \beta = 137°15'$ —— I

Now $\dfrac{c-b}{c+b} = \dfrac{\tan\dfrac{\gamma-\beta}{2}}{\tan\dfrac{\gamma+\beta}{2}}$

$$\frac{16.1-14.8}{16.1+14.8} = \frac{\tan\dfrac{\gamma-\beta}{2}}{\tan 68°37'30''} \Rightarrow \frac{1.3}{30.9} = \frac{Tan\dfrac{\gamma-\beta}{2}}{2.5549}$$

$$Tan\frac{\gamma-\beta}{2} = \frac{1.3 \times 2.5549}{30.9} = 0.1074$$

$$\frac{\gamma-\beta}{2} = Tan^{-1}(0.1074) = 6.1350$$

$\gamma - \beta = 12°16'12''$ —— II

Solving I & II

$\gamma - \cancel{\beta} = 12°16'12''$

$\gamma + \cancel{\beta} = 137°15'$

$2\gamma = 149°31'21'' \Rightarrow \boxed{\gamma = 74°45'36''}$

Put value in I

$\beta + 74°45'36'' = 137°15'$

$\beta = 137°15' - 74°45'36'' = \boxed{\beta = 62°29'24''}$

Now $\dfrac{a}{Sin\alpha} = \dfrac{b}{Sin\beta}$

$$\frac{a}{Sin 42°45'} = \frac{14.8}{Sin 62°29'24''} \Rightarrow a = \frac{14.8(0.6788)}{0.8869} \Rightarrow \boxed{a = 11.32}$$

$$\boxed{\begin{array}{l} b = 14.8, \quad a = 11.32,\ c = 16.1 \\ \alpha = 42^{O}45', \quad \beta = 62^{O}29'24'', \quad \gamma = 74^{O}45'36'' \end{array}}$$

9. **a = 319, b = 168, γ = 110°22′**

Sol. $\alpha + \beta + \gamma = 180^\circ \Rightarrow \alpha + \beta = 180^\circ - \gamma$

$\alpha + \beta = 180^\circ - 110^\circ 22' \Rightarrow \alpha + \beta = 69^\circ 38'$ —— I

We know that $\dfrac{a-b}{a+b} = \dfrac{\tan\dfrac{\alpha-\beta}{2}}{\tan\dfrac{\alpha+\beta}{2}} \Rightarrow \dfrac{319-168}{319+168} = \dfrac{\tan\dfrac{\alpha-\beta}{2}}{\tan\left(\dfrac{69^{O}38'}{2}\right)}$

$$\frac{151}{487} = \frac{\tan\dfrac{\alpha-\beta}{2}}{\tan 34^{o}49'} \Rightarrow \frac{151}{487} = \frac{\tan\dfrac{\alpha-\beta}{2}}{0.6954} \Rightarrow \frac{151\times 0.6954}{487} = \tan\frac{\alpha-\beta}{2}$$

$$\tan\frac{\alpha-\beta}{2} = 0.2156 \Rightarrow \frac{\alpha-\beta}{2} = \tan^{-1}(0.2156) = 12.1684$$

$$\tan\frac{\alpha-\beta}{2} = 0.2156 \Rightarrow \frac{\alpha-\beta}{2} = \tan^{-1}(0.2156) = 12.1684$$

$\Rightarrow \alpha - \beta = 24.3369 = 24^{O}20'13''$ —— II

Solving I & II

$\alpha - \cancel{\beta} = 69^\circ 38'$

$\alpha + \cancel{\beta} = 24^\circ 20' 13''$

$2\alpha = 93^\circ 58' 13'' \Rightarrow \boxed{\alpha = 46^\circ 59'}$

Put in I $\Rightarrow 46^\circ 59' + \beta = 69^{O}38' \Rightarrow \beta = 69^{O}38' - 46^\circ 59' = 22^{O}39' \Rightarrow \boxed{\beta = 22^{O}39'}$

$$Also\ \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \Rightarrow c = \frac{b\sin\gamma}{\sin\beta}$$

$$c = \frac{(168)\sin 110^{O}22'}{\sin 22^{O}39'} = \frac{(168)(0.9374)}{0.3851} = \frac{157.4832}{0.3851} = 409 \Rightarrow \boxed{c = 409}$$

$$\boxed{\begin{array}{l} a = 319, \quad b = 168,\ c = 409 \\ \alpha = 46^{O}59', \quad \beta = 22^{O}39', \quad \gamma = 110^{O}22' \end{array}}$$

10. **b = 61, c = 32 and α = 59°30′**

Sol. $\alpha + \beta + \gamma = 180^\circ \Rightarrow \beta + \gamma = 180^\circ - \alpha$

$\beta + \gamma = 180^\circ - 59^\circ 30' \Rightarrow \gamma + \beta = 120^\circ 30'$ —— I

Now $\dfrac{b-c}{b+c} = \dfrac{\tan\dfrac{\beta-\gamma}{2}}{\tan\dfrac{\beta+\gamma}{2}}$

$$\frac{61-32}{61+32} = \frac{\tan\dfrac{\beta-\gamma}{2}}{\tan 60^{\circ}15'} \Rightarrow \frac{29}{93} = \frac{Tan\dfrac{\beta-\gamma}{2}}{1.7496}$$

$$Tan\frac{\beta-\gamma}{2} = \frac{1.7496\times 29}{93} = 0.5456$$

$$\frac{\beta-\gamma}{2} = Tan^{-1}(0.5456) = 28.6162$$

$\beta - \gamma = 57^\circ 14'$ —— II

Solving I & II

$\beta - \cancel{\gamma} = 57^\circ 14'$

$\beta + \cancel{\gamma} = 120^\circ 30'$

$2\beta = 177^\circ 44' \Rightarrow \beta \boxed{= 88^\circ 52'}$

Put value in I

$\gamma + 88^\circ 52' = 120^\circ 30'$

$\gamma = 120^\circ 30' - 88^\circ 52' = \boxed{\gamma = 31^\circ 38'}$

Now $\dfrac{a}{Sin\alpha} = \dfrac{b}{Sin\beta}$

$$\frac{a}{Sin59^{\circ}30'} = \frac{61}{Sin88^{\circ}52'} \Rightarrow a = \frac{61(0.8616)}{0.9998} \Rightarrow \boxed{a = 53}$$

$$\boxed{\begin{array}{l} b = 61,\ a = 53,\ c = 32 \\ \alpha = 59^{\circ}30',\ \beta = 88^{\circ}52',\ \gamma = 31^{\circ}38' \end{array}}$$

Q. # 11: Give b = 2, c = 3, α = 57° β = ?, γ = ? **Faisalabad 2009**

Sol. $\alpha + \beta + \gamma = 180^\circ \Rightarrow \beta + \gamma = 180^\circ - \alpha$

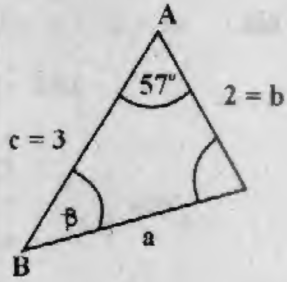
$\beta + \gamma = 180^\circ - 57^\circ = 123^\circ$ —— I

note

$c : b = 3 : 2$

$\frac{c}{b} = \frac{3}{2}$

$\frac{c-b}{c+b} = \frac{3-2}{3+2}$



$$\frac{c-b}{c+b} = \frac{\tan\frac{\gamma-\beta}{2}}{\tan\frac{\gamma+\beta}{2}}$$

$$\frac{3-2}{3+2} = \frac{\tan\frac{\gamma-\beta}{2}}{\tan 61''30'} \Rightarrow \frac{1}{5} = \frac{Tan\frac{\gamma-\beta}{2}}{1.8417}$$

$$tan\frac{\gamma-\beta}{2} = \frac{1.8417}{5} = 0.3683 \Rightarrow \frac{\gamma-\beta}{2} = Tan^{-1}(0.3683)$$

$$\frac{\gamma-\beta}{2} = 20^\circ 13' 17''$$

$\gamma - \beta = 40^\circ 26' 34''$ —— II

Solving I & II

$\gamma - \cancel{\beta} = 40^\circ 26' 34''$

$\gamma + \cancel{\beta} = 123^\circ$

$2\gamma = 163^\circ 26' 34'' \Rightarrow \boxed{\gamma = 81^\circ 43' 17''}$

Put in I

$\beta + 81^\circ 43' 17'' = 123^\circ$

$\beta = 123^\circ - 81^\circ 43' 17''$

$\boxed{\beta = 41^\circ 16' 43''}$

12. **Two forces of 40N and 30 N are represented by $\overrightarrow{AB}$ and $\overrightarrow{AB}$ which are inclined at an angle of 147° 25'. Find $\overrightarrow{AB}$, the resultant $\overrightarrow{AB}$ and $\overrightarrow{AB}$.**

Sol. a = 30, b = ?, c = 40, β = 147°25' (Federal Board)

$b^2 = a^2 + c^2 - 2ac\cos\beta$

$$\left(\overline{AC}\right)^2 = \left(\overline{BC}\right)^2 + \left(\overline{AB}\right)^2 - 2\left(\overline{AB}\right)\left(\overline{BC}\right)Cos\beta$$

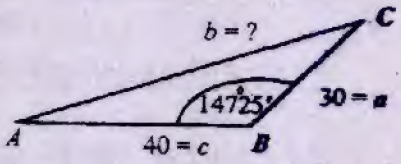
$= (30)^2 + (40)^2 - 2(40)(30) \text{ Cos } 147°25'$

$= 1600 + 900 - 2400\,(-0.8426)$

$= 2500 + 2022.26$

$\left(\overline{AC}\right)^2 = 4522.26$

$\boxed{\overline{AC} = 67.29}$



Theorem I $\sin\frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{bc}}$ **Sargodha 2008, 2010**

Prof: We known that

$2\sin^2\frac{\alpha}{2} = 1 - Cos\alpha$

$\boxed{\cos\alpha = \frac{b^2+c^2-a^2}{2bc}}$

$2\sin^2\frac{\alpha}{2} = 1 - \frac{b^2+c^2-a^2}{2bc}$

$= \frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (-2bc + b^2 + c^2)}{2bc}$

$= \frac{a^2 - (b-c)^2}{2bc} = \frac{[(a-(b-c)][(a+(b-c)]}{2bc} = \frac{(a-b+c)(a+b-c)}{2bc}$

$= \frac{(a+c+b-b-b)(a+b+c-c-c)}{2bc}$

$= \frac{(2S-2b)(2S-2c)}{2bc} = \frac{4(S-b)(S-c)}{2bc}$

$2\sin^2\frac{\alpha}{2} = \frac{4(S-b)(S-c)}{2bc} \Rightarrow \sin^2\frac{\alpha}{2} = \frac{\cancel{4}(S-b)(S-c)}{\cancel{4}bc} \Rightarrow Sin\frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{bc}}$

Similarly $\sin\frac{\beta}{2} = \sqrt{\frac{(S-a)(S-c)}{ac}}$, $Sin\frac{\gamma}{2} = \sqrt{\frac{(S-a)(S-b)}{ab}}$

Theorem II $\text{Cos}\,\dfrac{\alpha}{2}=\sqrt{\dfrac{S(S-a)}{bc}}$ **Sargodha 2008**

Proof: we know that $2\,\text{Cos}^2\dfrac{\alpha}{2}=1+Cos\alpha$

$$2\text{Cos}^2\frac{\alpha}{2}=1+\frac{b^2+c^2-a^2}{2bc}=\frac{2bc+b^2+c^2-a^2}{2bc}=\frac{(b+c)^2-a^2}{2bc}$$

$$=\frac{(b+c-a)(b+c+a)}{2bc}=\frac{(b+c+a-a-a)(a+b+c)}{2bc}$$

$$=\frac{(2S-2a)(2S)}{2bc}=\frac{4S(S-a)}{2bc}$$

$$\text{Cos}^2\frac{\alpha}{2}=\frac{\cancel{4}S(S-b)}{\cancel{4}bc}\Rightarrow Cos\frac{\alpha}{2}=\sqrt{\frac{S(S-a)}{bc}}$$

Similarly $\text{Cos}\dfrac{\beta}{2}=\sqrt{\dfrac{S(S-b)}{ac}}\ \&\ Cos\dfrac{\gamma}{2}=\sqrt{\dfrac{S(S-c)}{ab}}$

Theorem III $\tan\dfrac{\alpha}{2}=\sqrt{\dfrac{(S-b)(S-c)}{S(S-a)}}$

Proof: $\tan\dfrac{\alpha}{2}=\dfrac{Sin\dfrac{\alpha}{2}}{Cos\dfrac{\alpha}{2}}=\dfrac{\sqrt{\dfrac{(S-b)(S-c)}{bc}}}{\sqrt{\dfrac{S(S-a)}{bc}}}$

$$=\sqrt{\frac{(S-b)(S-c)}{\cancel{bc}}}\times\sqrt{\frac{\cancel{bc}}{S(S-a)}}=\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}$$

Similarly $\tan\dfrac{\beta}{2}=\sqrt{\dfrac{(S-a)(S-c)}{S(S-b)}}\ \&\ \tan\dfrac{\gamma}{2}=\sqrt{\dfrac{(S-a)(S-b)}{S(S-c)}}$

EXERCISE 12.6

Formulas for this exercise when three sides are given.

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}, \quad \cos\frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$$

$$\cos\beta = \frac{a^2+c^2-b^2}{2ac}, \quad \cos\frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}} \qquad \text{where } S = \frac{a+b+c}{2}$$

$$\cos\gamma = \frac{a^2+b^2-c^2}{2ab}, \quad \cos\frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}}$$

Solve the following triangles, in which

Note : just for practice one question is solve by half angle formula

1. **a = 7 , b = 7 , c = 9**

Sol. $S = \frac{a+b+c}{2} = \frac{7+7+9}{2} = 11.5$

$$\cos\frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$$

$$= \sqrt{\frac{(11.5)(11.5-7)}{(7)(9)}} = \sqrt{\frac{(11.5)(4.5)}{63}} = \sqrt{\frac{51.75}{63}} = \sqrt{0.8214}$$

$$\cos\frac{\alpha}{2} = 0.9063 \Rightarrow \frac{\alpha}{2} = \cos^{-1}(0.9063) = 25^0 \Rightarrow \boxed{\alpha = 50^0}$$

$$\cos\frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}}$$

$$= \sqrt{\frac{(11.5)(11.5-7)}{(7)(9)}} = \sqrt{\frac{(11.5)(4.5)}{63}} = \sqrt{\frac{51.75}{63}} = \sqrt{0.8214}$$

$$\cos\frac{\beta}{2} = 0.9063 \Rightarrow \frac{\beta}{2} = \cos^{-1}(0.9063) = 25^0 \Rightarrow \boxed{\beta = 50^0}$$

$$\gamma = 180° - \alpha - \beta = 180° - 50° - 50° = 80° \Rightarrow \boxed{\gamma = 80^0}$$

2. **a = 32, b = 40, c = 66**

Sol. $\cos\alpha = \frac{b^2+c^2-a^2}{2bc} = \frac{(40)^2+(66)^2-(32)^2}{2(40)(66)}$

$$\cos\alpha = \frac{1600+4356-1024}{5280} = 0.9340 \Rightarrow \alpha = \text{Cos}^{-1}(0.9340) \Rightarrow \boxed{\alpha = 20°56'6''}$$

$$\cos\beta = \frac{a^2+c^2-b^2}{2ac} = \frac{(32)^2+(66)^2-(40)^2}{2(32)(66)} = \frac{1024+4356-1600}{4224} = 0.8948$$

$\beta = \text{Cos}^{-1}(0.8948) \Rightarrow \beta \boxed{= 26^\circ 30' 22''}$

$\gamma = 180^\circ - \alpha - \beta = 180^\circ - 20^\circ 59' 6'' - 26^\circ 30' 22'' \Rightarrow \boxed{\gamma = 132^\circ 30' 31''}$

3. **a = 28.3, b = 31.7, c = 42.8**

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc} = \frac{(31.7)^2+(42.8)^2-(28.3)^2}{2(31.7)(42.8)}$$

$$= \frac{1004.89+1831.84-800.89}{2713.52} = \frac{2035.84}{2713.52} = 0.7502$$

$$\alpha = \cos^{-1}(0.7502) = 41^O 23'$$

$$\cos\beta = \frac{a^2+c^2-b^2}{2ac} = \frac{(28.3)^2+(42.8)^2-(31.7)^2}{2(28.3)(42.8)}$$

$$= \frac{800.89+1831.84-1004.89}{2422.48}$$

$$\beta = \frac{1627.84}{2422.48} = 0.6719$$

$$\beta = \cos^{-1}(0.6719) = 47^O 46'$$

$$\gamma = 180^O - \alpha - \beta = 180^O - 41^O 23' - 47^O 46' = 90^O 50'$$

4. **a = 28.3, b = 31.7, c = 42.8**

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc} = \frac{(56.31)^2+(40.27)^2-(31.9)^2}{2(56.31)(40.27)}$$

$$= \frac{3170.81+1621.67-1017.61}{4535.2} = \frac{3774.87}{4535.2} = 0.8323$$

$$\alpha = \cos^{-1}(0.8323) = 33^O 39'$$

$$\cos\beta = \frac{a^2+c^2-b^2}{2ac} = \frac{(31.9)^2+(40.27)^2-(56.31)^2}{2(31.9)(40.27)}$$

$$= \frac{1017.61+1621.67-3170.81}{2596.66} = \frac{-531.53}{2596.66}$$

$$\beta = -0.2046 \Rightarrow \beta = \cos^{-1}(-0.2046)$$

$$\beta = 101^O 48'$$

$$\gamma = 180^O - \alpha - \beta = 180^O - 33^O 39' - 101^O 48'$$

$$\gamma = 44^O 33'$$

5. **a = 4584, b = 5140, c = 3624**

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc} = \frac{(5140)^2+(3624)^2-(4584)^2}{2(5140)(3624)}$$

$$= \frac{26419600+13133376-21013056}{37254720} = \frac{18539920}{37254720} = 0.4976$$

$$\alpha = \cos^{-1}(0.4976) = 60^0 9'$$

$$\cos\beta = \frac{a^2+c^2-b^2}{2ac} = \frac{(4584)^2+(3624)^2-(5140)^2}{2(4584)(3624)}$$

$$= \frac{21013056+13133376-26419600}{33224832} = \frac{7726832}{33224832} = 0.2325$$

$$\beta = \cos^{-1}(0.2325) = 76^0 33'$$

$$\gamma = 180^0 - \alpha - \beta = 180^0 - 60^0 9' - 76^0 33' = 43^0 18'$$

6. **Find the smallest angle of the triangle ABC,**
when a = 37.34, b = 3.24, c = 35.06 **Federal**

Sol. a = 37.34, b = 3.24, c = 35.06. b < c < a so smallest angle is β

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos\beta = \frac{(37.34)^2+(35.06)^2-(3.24)^2}{2(37.34)(35.06)} = \frac{1394.27+1229.20-10.49}{2618.28}$$

$\cos\beta$ = 0.9979 $\Rightarrow$ β = $\cos^{-1}$ (0.9979) $\Rightarrow$ $\boxed{\beta = 3°38'46''}$

7. **Find the measure of the greatest angle, if sides of the triangle are 16, 20, 33.**

Sol. a = 16, b = 20, c = 33 **Sargodha 2007, Faisalabad 2007, Federal**
c > b > a So γ is greatest

$$\cos\gamma = \frac{a^2+b^2-c^2}{2ab} = \frac{(16)^2+(20)^2-(33)^2}{2(16)(20)} = \frac{256+400-1089}{640} = -0.6765$$

γ = $\cos^{-1}$ (-0.6765) $\Rightarrow$ $\boxed{\gamma = 132°34'}$

8. **The sides of triangle are x^2+x+1, $2x+1$ and x^2-1. Prove that the greatest angle of the triangle is 120°.** **Faisalabad 2007, Multan 2007, 2009**

Sol. $a = x^2+x+1,\ b = 2x+1,\ c = x^2-1$ *Clearly* $a > b > c$ *so* $\alpha = ?$

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(2x+1)^2 + (x^2-1)^2 - (x^2+x+1)^2}{2(2x+1)(x^2-1)}$$

$$= \frac{4x^2+4x+1+x^4-2x^2+1-(x^4+x^2+1+2x^3+2x+2x^2)}{2(2x^3-2x+x^2-1)}$$

$$=\frac{4x^2+4x+1+x^4-2x^2+1-x^4-x^2-1-2x^3-2x-2x^2}{2(2x^3-2x+x^2-1)}$$

$$=\frac{-2x^3+2x-x^2+1}{2(2x^3-2x+x^2-1)}=\frac{-(2x^3-2x+x^2-1)}{2(2x^3-2x+x^2-1)}=\frac{-1}{2}$$

$$\alpha=\text{Cos}^{-1}\left(\frac{-1}{2}\right)=120^{o}$$

9. The measure of side of a triangular plot are 413, 214 and 375 meters. Find the measures of the corner angles of the plot.

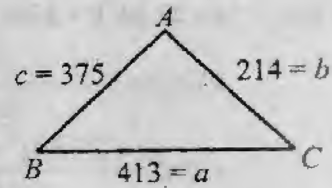
Sol. a = 413, b = 214, c = 375

$$\text{Cos}\,\alpha=\frac{b^2+c^2-a^2}{2bc}=\frac{(214)^2+(375)^2-(413)^2}{2(214)(375)}=\frac{15852}{160500}=0.0987$$

$\alpha=\text{Cos}^{-1}(0.0987)=\boxed{84^\circ 19'54''}$

$$\text{Cos}\,\beta=\frac{a^2+c^2-b^2}{2ac}=\frac{(413)^2+(375)^2-(214)^2}{2(413)(375)}$$

$$=\frac{170569+140625-45796}{309750}=\frac{265398}{309750}$$



$\beta=\text{Cos}^{-1}(0.8568)\Rightarrow\boxed{\beta=31^\circ 21'21''}$

$\gamma=180^\circ-\alpha-\beta=180^\circ-84^\circ 19'45''-31^\circ 3'21''$

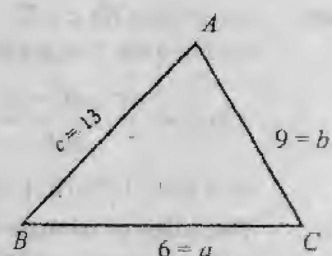
$\boxed{\gamma=64^\circ 37'45''}$

10. Three villages A, B and C are connected by straight roads 6km, 9km and 13km. What angles these roads make with each other?

Sol. a = 6, b = 9, c = 13

$$\text{Cos}\,\alpha=\frac{b^2+c^2-a^2}{2bc}=\frac{(9)^2+(13)^2-(6)^2}{2(9)(13)}$$

$$=\frac{81+169-36}{234}=\frac{214}{234}=0.1945$$



$\alpha=\text{Cos}^{-1}(0.9145)=23^\circ 51'39''\Rightarrow\boxed{\alpha=23^\circ 51'39''}$

$$\text{Cos}\,\beta=\frac{a^2+c^2-b^2}{2ac}$$

$$=\frac{(6)^2+(13)^2-(9)^2}{2(6)(13)}=\frac{36+169-81}{156}=\frac{124}{156}=0.7948$$

$\beta=\text{Cos}^{-1}(0.7948)\Rightarrow\boxed{\beta=37^\circ 21'25''}$

$\gamma=180^\circ-\alpha-\beta=180^\circ-23^\circ 15'39''-37^\circ 21'25''\Rightarrow\boxed{\gamma=118^\circ 46'56''}$

EXERCISE 12.7

Area of Triangle when one side is given.

$$\Delta = \frac{1}{2}\frac{a^2\ Sin\beta\ Sin\gamma}{Sin\alpha}$$

$$\Delta = \frac{1}{2}\frac{b^2\ Sin\alpha\ Sin\gamma}{Sin\beta}$$

$$\Delta = \frac{1}{2}\frac{c^2\ Sin\alpha\ Sin\beta}{Sin\gamma}$$

Two sides are given

$$\Delta = \frac{1}{2}ab\ Sin\gamma$$

$$\Delta = \frac{1}{2}ac\ Sin\beta$$

$$\Delta = \frac{1}{2}bc\ Sin\alpha$$

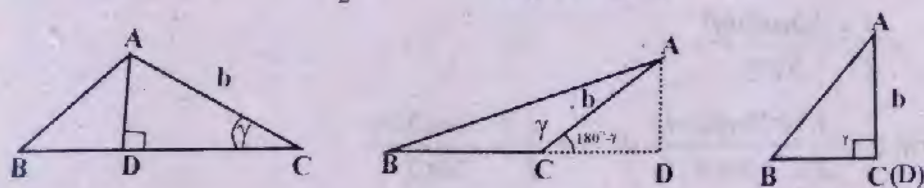
Three sides are given

$$\Delta = \sqrt{S(S-a)\,(S-b)\,(S-c)}$$ Where $S = \dfrac{a+b+c}{2}$

Two sides are given then Area of Triangle

Theorem I. Area of triangle = $\Delta = \dfrac{1}{2}ab\ Sin\gamma$ **Sargodha 2008**

Proof:



(i) (ii) (iii)

In fig (i) $\dfrac{AD}{AC} = Sin\gamma$ ——— I

In fig (ii) $\dfrac{AD}{AC} = Sin(180^\circ - \gamma) = Sin\gamma$ ——— II

In fig (iii) $\dfrac{AD}{AC} = 1 = Sin90^\circ = Sin\gamma$ ——— III

From I, II, & III it is clear that

$$\frac{AD}{AC} = Sin\gamma \Rightarrow AD = AC\ Sin\gamma = b\,Sin\gamma$$

Now Area of triangle = $\Delta = \dfrac{1}{2}$ (base) (attitude)

$$= \frac{1}{2}\text{ (BC) (AD)} = \frac{1}{2}\text{ ab Sin}\gamma$$

Similarly $\Delta = \frac{1}{2} bc\, Sin\alpha$ & $\Delta = \frac{1}{2}\, ac\; Sin\beta$

Theorem II $\Delta = \frac{1}{2}\frac{c^2 Sin\alpha Sin\beta}{Sin\gamma}$

Proof: We know that $\frac{a}{Sin\alpha} = \frac{b}{Sin\beta} = \frac{c}{Sin\gamma}$

$$\Rightarrow a = \frac{cSin\alpha}{Sin\gamma}\ \&\ b = \frac{cSin\beta}{Sin\gamma}$$

Now $\Delta = \frac{1}{2} ab\; Sin\gamma$

$$= \frac{1}{2}\frac{cSin\alpha}{Sin\gamma}\frac{cSin\beta}{Sin\gamma}\, Sin\gamma\, (Put\ values\ of\ a\ \&\ b)$$

$$= \frac{1}{2}\frac{c^2 Sin\alpha Sin\beta}{Sin\gamma}$$

Similarly $\Delta = \frac{1}{2}\frac{a^2 Sin\beta Sin\gamma}{Sin\alpha}$ & $\Delta = \frac{1}{2}\frac{b^2 Sin\alpha Sin\gamma}{Sin\beta}$

Theorem III Heroes formula $\Delta = \sqrt{S(S-a)(S-b)(S-c)}$ **Sgd2008,09, Rawalpindi 09**

Sol. We know that $\Delta = \frac{1}{2} bc\, Sin\alpha$

$$\therefore\ Since\ Sin\alpha = 2Sin\frac{\alpha}{2}Cos\frac{\alpha}{2}$$

$$\Delta = \frac{1}{2} bc\, 2\, Sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$$= bc\sqrt{\frac{(S-b)(S-c)}{bc}}\sqrt{\frac{S(S-a)}{bc}} = bc\sqrt{\frac{S(S-a)(S-b)(S-c)}{b^2c^2}}$$

$$= \cancel{bc}\frac{\sqrt{S(S-a)(S-b)(S-c)}}{\cancel{bc}}$$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$$

1. Find the area of the triangle ABC, given two sides and their included angle:

i. a = 200, b = 120, γ = 150° **Multan 2007, 2009**

Sol. $\Delta = \frac{1}{2}ab\ Sin\gamma$

$= \frac{1}{2}\ (200)\ (120)\ Sin150^{o} = \frac{1}{2}\ (24000)\ (0.5) = 6000\, sq\, unit$

ii. b = 37, c = 45, α = 30°50′

Sol. $\Delta = \frac{1}{2}bc\ Sin\alpha = \frac{1}{2}\ (37)\ (45)\ Sin(30^{o}50') = 426.69\, sq\, unit$

iii. a= 4.33 , b = 9.25, γ = 56°44′

Sol. $\Delta = \frac{1}{2}ab\ Sin\gamma = \frac{1}{2}(4.33)(9.25)Sin(56^{o}44') = (20.02)(0.8361) = 16.73 squnit$

2. Find the area of the triangle ABC, given one side and two angles:

i. b = 25.4, γ = 36°41′ , α = 45°17′

β = 180° − α − γ

= 180° − 45°17′ − 36°41′ = 98°2′

$$\Delta = \frac{1}{2}\frac{b^2\ Sin\alpha\ Sin\gamma}{Sin\beta} = \frac{1}{2}\frac{(25.4)^2\ Sin45^{o}17'\ Sin36^{o}41'}{Sin98^{o}2'}$$

$$= \frac{1}{2}\frac{(645.16)\ (0.7105)\ (0.5973)}{0.99018} = 138.25 \text{ sq unit}$$

ii. c = 32, α = 47°24′, β = 70°16′

γ = 180° − α − β

= 180° − 47°24′ − 70°16′= 62°20′

$$\Delta = \frac{1}{2}\frac{c^2\ Sin\alpha\ Sin\beta}{Sin\gamma} = \frac{1}{2}\frac{(32)^2\ Sin47^{o}24'\ Sin70^{o}16'}{Sin62^{o}20'}$$

$$= \frac{1}{2}\frac{(1024)\ (0.7360)\ (0.9412)}{0.8856} = \frac{1}{2}\frac{801.09}{2} = 400.49\, sq\ units$$

iii. a = 4.8 , α = 83°42′ , γ = 37°12′

β = 180° − α − γ

= 180° − 83°42′ − 37°12′= 59°6′

$$\Delta = \frac{1}{2}\frac{a^2\ Sin\beta\ Sin\gamma}{Sin\alpha} = \frac{1}{2}\frac{(4.8)^2\ Sin59''6'\ Sin37''12'}{Sin83^\circ 42'}$$

$$= \frac{1}{2}\frac{(23.04)\ (0.8580)\ (0.6045)}{0.9939} = 6.0116 \text{ sq unit}$$

3. Find the area of the triangle ABC, given three sides;

i. α = 18 , b = 24 , c = 30 Fsd 2007, 2008, Multan 2008, Lahore 2009

Sol. $S = \dfrac{a+b+c}{2} = \dfrac{18+24+30}{2} = 36$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)} = \sqrt{36(36-18)(36-24)(36-30)}$$

$$= \sqrt{36(18)(12)(6)} = \sqrt{46656} = 216 \text{ sq unit}$$

ii. a= 524 , b = 276 , c = 315

Sol. $S = \dfrac{524+276+315}{2} = \dfrac{1115}{2} = 557.5$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)} = \sqrt{557.5(557.5-524)(557.5-276)(557.5-315)}$$

$$= \sqrt{(557.5)(33.5)(281.5)(242.5)} = \sqrt{1274910861} = 35705.89 squint$$

iii. a= 32.65 , b = 42.81 , c = 64.92

Sol. $S = \dfrac{a+b+c}{2} = \dfrac{32.65+42.81+64.92}{2} = \dfrac{140.38}{2} = 70.19$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)} = \sqrt{70.19(70.19-32.65)(70.19-42.81)(70.19-64.92)}$$

$$= \sqrt{(70.19)(37.54)(27.38)(5.27)} = \sqrt{380201.27} = 616\ squint$$

4. The area of triangle is 2437. If a = 79, and c = 97, then find angle β.

Sol. $\Delta = 2437,\ a = 79\ c = 97,\ \beta = ?$ **Sargodha 2006, Raiwalpindi 2009**

$$\Delta = \frac{1}{2}\ ac\ Sin\beta \Rightarrow 2437 = \frac{1}{2}(79)(97)\ Sin\beta$$

$$2437 = 3831.5\ Sin\ \beta \Rightarrow sin\ \beta = \frac{2437}{3813.5} = 0.6360 \Rightarrow \boxed{\beta = 39^\circ 29' 5''}$$

5. **The area of triangle is 121.34. If $\alpha = 32°$, $\beta = 65°37'$, c = ? , γ = ?**

$\gamma = 180° - \alpha - \beta$

$\gamma = 180° - 32°23' - 65°37' \Rightarrow \boxed{\gamma = 82°}$

Now $\Delta = \dfrac{1}{2} c^2 \dfrac{Sin\alpha \ Sin\beta}{Sin\gamma}$

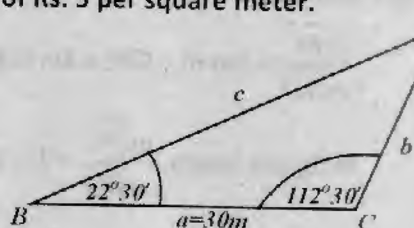
$$121.34 = \frac{1}{2} \frac{c^2 \ Sin32° \ Sin65°37'}{Sin82°23'}$$

$$121.34 = \frac{1}{2} \frac{c^2 (0.5299)(0.9108)}{0.9912}$$

$$\Rightarrow c^2 = \frac{(121.34)(0.9912) \times 2}{(0.5299)(0.9108)} = \frac{240.54}{0.4826} = 498.434 \Rightarrow \boxed{c = 22.24}$$

6. **One side of a triangle garden is 30m. If it two corner angles are 22° ½ and 112° ½, find the cost of planting the grass at the rate of Rs. 5 per square meter.**

Sol. Given a = 30, $\beta = 22°30'$ $\gamma = 112°30'$



$\alpha = 180° - \beta - \gamma$

$= 180° - 22°30' - 112°30' = 45°$

Now = $\Delta = \dfrac{1}{2} \dfrac{a^2 \ Sin\beta \ Sin\gamma}{Sin\alpha}$

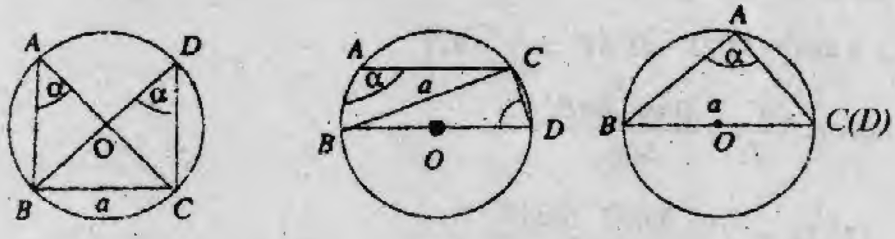
$$= \frac{1}{2} \frac{(30)^2 \ Sin22°30' \ Sin112°30'}{Sin45°}$$

Note :
$22°½ = 22° 30'$
$112°½ = 112° 30'$

$$= \frac{1}{2} \frac{(900)(0.3826)(0.9238)}{0.7071}$$ = 224.43 squint / per rupees

Total Cost = 244.43 x 5 = 1125 squint

Theorem I R = $\dfrac{a}{2Sin\alpha} = \dfrac{b}{2Sin\beta} = \dfrac{c}{2Sin\gamma}$



(i) (ii) (iii)

Sol. In fig (i) In right triangle Δ BCD

$$\frac{m\overline{BC}}{mBD} = Sin\alpha \text{ ——} I(\alpha \cong m < BDC)$$

In fig (ii) m $\angle$ BDC + m $\angle$ BAC = 180° (Sum of opposite angle of cyclical quadrilateral = 180°)

$\Rightarrow \angle BDC = 180^\circ - m\angle A$ = 180° − α

In right triangle BCD

$\dfrac{m\overline{BC}}{mBD}$ = Sin m < BDC = Sin (180° − α) = Sin α ____ II

In fig (iii) clearly $\dfrac{m\overline{BC}}{mBD}$ = 1 = Sin90° = Sin α ________ III

From I, III, III $\dfrac{m\overline{BC}}{mBD}$ = Sin α $\Rightarrow$ $\dfrac{a}{2R} = Sin\alpha$ *when* $m\overline{BC} = a$ & m BD = 2R

$\Rightarrow$ 2R Sin α = a $\Rightarrow$ R = $\dfrac{a}{2Sin\alpha}$

Similarly R = $\dfrac{b}{2Sin\beta}$ & $R\dfrac{c}{2Sin\gamma}$

Theorem II R = $\dfrac{abc}{4\Delta}$ **Fsd 2007, Multan 2007, Sgd 2010, Federal Board**

Sol. we know that R = $\dfrac{a}{2Sin\alpha}$ $\left(Since\ \ Sin\alpha = 2Sin\dfrac{\alpha}{2}Cos\dfrac{\alpha}{2}\right)$

$$R = \frac{a}{2.2Sin\dfrac{\alpha}{2}Cos\dfrac{\alpha}{2}}$$

$$= \frac{a}{4\sqrt{\frac{(S-b)(S-c)}{bc}}\sqrt{\frac{S(S-a)}{bc}}} = \frac{a}{4\sqrt{\frac{S(S-a)(S-b)(S-c)}{b^2c^2}}} = \frac{a}{4\frac{\Delta}{bc}} = \frac{abc}{4\Delta}$$

Theorem III $r = \dfrac{\Delta}{S}$ **Faisalabad 2007, 08, Federal**

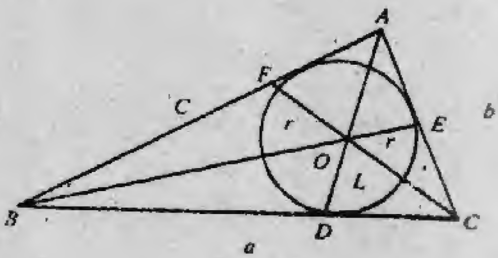
Proof In triangle ABC OD, OE, OF are perpendicular to BC, AC and AB respectively then Area Δ ABC = Area of Δ OBC + area of Δ OCA + area of Δ OAB.

$$\Delta = \frac{1}{2}BC \times OD + \frac{1}{2}CA \times OE + \frac{1}{2}AB \times OF$$

$$= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$$

$$\Delta = \frac{1}{2}r(a+b+c) = \frac{1}{2}r(2S)$$

$$\Delta = rS \Rightarrow r = \frac{\Delta}{S}$$



Theorem IV $r_1 = \dfrac{\Delta}{S-a}, r_2 = \dfrac{\Delta}{S-b}, b_3 = \dfrac{\Delta}{S-c}$ **Faisalabad 2008**

Proof Let "o" be the centre of escribed circle Draw ⊥ar, D, E, F then

ΔABC= Δ OAB + Δ OAC − ΔOBC

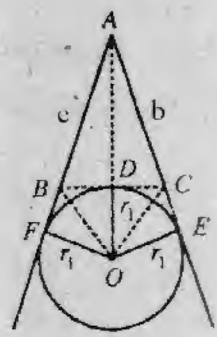
$$= \frac{1}{2}(AB)(OF) + \frac{1}{2}(AC)(OE) - \frac{1}{2}(BC)(OD)$$

$$\Delta = \frac{1}{2}cr_1 + \frac{1}{2}br_1 - \frac{1}{2}ar_1$$

$$= \frac{1}{2}r_1(c+b-a) = \frac{1}{2}r_1(a+b+a-a-a)$$

$$\Delta = \frac{1}{2}r_1(2S-2a) = \frac{1}{2}2r_1(S-a)$$

$$\Delta = r_1(S-a) \Rightarrow r_1 = \frac{\Delta}{S-a}$$



Similarly $r_2 = \dfrac{\Delta}{S-b}$ & $r_3 = \dfrac{\Delta}{S-c}$ **Federal**

EXERCISE 12.8

Important formulas about 12.8

$$r = \frac{\Delta}{S}$$

$$r_1 = \frac{\Delta}{S-a}$$

$$r_2 = \frac{\Delta}{S-b}$$

$$r_3 = \frac{\Delta}{S-c}$$

$$R = \frac{abc}{4\Delta}$$

$$S = \frac{a+b+c}{2}$$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$$

$$R = \frac{a}{2Sin\alpha} = \frac{b}{2Sin\beta} = \frac{c}{2Sin\gamma}$$

$$Cos\frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$$

$$Cos\frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}}$$

$$Cos\frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}}$$

$$Sin\frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{bc}}$$

$$Sin\frac{\beta}{2} = \sqrt{\frac{(S-a)(S-c)}{ac}}$$

$$Sin\frac{\gamma}{2} = \sqrt{\frac{(S-a)(S-b)}{ab}}$$

$$\tan\frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{S(S-a)}}$$

$$tan\frac{\beta}{2} = \sqrt{\frac{(S-a)(S-c)}{S(S-b)}}$$

$$\tan\frac{\gamma}{2} = \sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$$

1. **Show that:**

I. **$r = 4R \text{ Sin } \frac{\alpha}{2} \text{ Sin } \frac{\beta}{2} \text{ Sin } \frac{\gamma}{2}$** **Sargodha 2010**

$$= 4R \sqrt{\frac{(S-b)(S-c)}{bc}} \sqrt{\frac{(S-a)(S-c)}{ac}} \sqrt{\frac{(S-a)(S-b)}{ab}}$$

$$= 4\frac{abc}{4\Delta} \sqrt{\frac{(S-a)^2 (S-b)^2 (S-c)^2}{a^2b^2c^2}}$$

$$= \frac{abc}{\Delta} \sqrt{\frac{S^2(S-a)^2 (S-b)^2 (S-c)^2}{S^2a^2b^2c^2}}$$

$$= \frac{abc}{\Delta} \frac{S(S-a)(S-b)(S-c)}{S(abc)}$$

$$= \frac{1}{\Delta}\frac{\Delta^2}{S} = \frac{\Delta}{S} = r = L.H.S$$

ii. $S = 4R \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$

$$= \frac{4abc}{4\Delta}\sqrt{\frac{S(S-a)}{bc}}\sqrt{\frac{S(S-b)}{ac}}\sqrt{\frac{S(S-c)}{ab}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{S^2.S(S-a)\,(S-a)\,(S-a)}{a^2b^2c^2}}$$

$$= \frac{\cancel{abc}}{\Delta}\,\frac{S\sqrt{S(S-a)(S-b)(S-c)}}{\cancel{abc}}$$

$$= \frac{1}{\cancel{\Delta}}S.\cancel{\Delta} = S = R.H.S$$

2. **Show that: $r = a \sin\frac{\beta}{2} \sin\frac{\gamma}{2} \sec\frac{\alpha}{2} = b \sin\frac{\gamma}{2} \sin\frac{\alpha}{2} \sec\frac{\beta}{2} = c \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sec\frac{\gamma}{2}$**

Sol. Now Considers

$$a \sin\frac{\beta}{2} \sin\frac{\gamma}{2} \sec\frac{\alpha}{2} = a \sin\frac{\beta}{2} \sin\frac{\gamma}{2}\frac{1}{\cos\frac{\alpha}{2}}$$

$$= a\sqrt{\frac{(S-a)\,(S-c)}{ac}}\sqrt{\frac{(S-a)(S-b)}{ab}}\frac{1}{\sqrt{\frac{S(S-a)}{bc}}}$$

$$= a\sqrt{\frac{(S-a)\,(S-c)}{ac}}\sqrt{\frac{(S-a)(S-b)}{ab}}\sqrt{\frac{bc}{S(S-a)}}$$

$$= a\sqrt{\frac{(S-a)^2\,(S-b)(S-c)bc}{S(S-a)a^2bc}}$$

$$= a\sqrt{\frac{(S-a)\,(S-b)(S-c)}{Sa^2}}$$

$$= a\sqrt{\frac{S(S-a)\,(S-b)(S-c)}{S^2a^2}}\ (\text{"}\times\text{" \& "}\div\text{" by } S)$$

$$= \frac{a\sqrt{S(S-a)(S-b)(S-c)}}{aS} = \frac{\Delta}{S} = r$$

Again $b\,\text{Sin}\frac{\gamma}{2}\text{Sin}\frac{\alpha}{2}\;\text{Sec}\frac{\beta}{2} = b\,\text{Sin}\frac{\gamma}{2}\;Sin\frac{\alpha}{2}\,\frac{1}{Cos\frac{\beta}{2}}$

$$= b\sqrt{\frac{(S-a)\,(S-b)}{ab}}\;\sqrt{\frac{(S-b)(S-c)}{bc}}\,\frac{1}{\sqrt{\frac{S(S-b)}{ac}}}$$

$$= b\sqrt{\frac{(S-a)\,(S-b)}{ab}}\;\sqrt{\frac{(S-b)(S-c)}{bc}}\sqrt{\frac{ac}{S(S-b)}}$$

$$= b\sqrt{\frac{ac(S-a)\,(S-b)^{\cancel{2}}(S-c)}{S(\cancel{S-b})ab^2c}}$$

$$= b\sqrt{\frac{S(S-a)\,(S-b)(S-c)}{S^2b^2}}$$

$$= b\sqrt{\frac{S(S-a)\,(S-b)(S-c)}{S^2b^2}}$$

$$= \cancel{b}\,\frac{\Delta}{S\cancel{b}} = \frac{\Delta}{S} = r$$

Again $c\,\text{Sin}\frac{\alpha}{2}\;\text{Sin}\frac{\beta}{2}\;\text{Sec}\frac{\gamma}{2}$

$$= c\sqrt{\frac{(S-b)\,(S-c)}{bc}}\;\sqrt{\frac{(S-a)(S-c)}{ac}}\sqrt{\frac{ab}{S(S-c)}}$$

$$= c\sqrt{\frac{ab(S-a)\,(S-b)(S-c)^{\cancel{2}}}{abc^2S(\cancel{S-c})}}$$

$$= c\sqrt{\frac{S(S-a)\,(S-b)(S-c)}{c^2S^2}} = \cancel{c}\,\frac{\Delta}{\cancel{c}S} = \frac{\Delta}{S} = r$$

Hence proved that

$$r = a\,\text{Sin}\frac{\beta}{2}\;\text{Sin}\frac{\gamma}{2}\;\text{Sec}\frac{\alpha}{2} = b\,\text{Sin}\frac{\gamma}{2}\text{Sin}\frac{\alpha}{2}\;\text{Sec}\frac{\beta}{2} = c\,\text{Sin}\frac{\alpha}{2}\;\text{Sin}\frac{\beta}{2}\;\text{Sec}\frac{\gamma}{2}$$

3. Show that:

i. $r_1 = 4R\,\text{Sin}\frac{\alpha}{2}\;\text{Cos}\frac{\beta}{2}\;\text{Cos}\frac{\gamma}{2}$ **Multan 2007, Faisalabad 2007, Sargodha 2008**

$$= \cancel{4}\,\frac{abc}{\cancel{4}\Delta}\sqrt{\frac{(S-b)\,(S-c)}{bc}}\;\sqrt{\frac{S(S-b)}{ac}}\sqrt{\frac{S(S-c)}{ab}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{S^2(S-b)^2\,(S-c)^2}{a^2b^2c^2}} = \frac{abc}{\Delta}\sqrt{\frac{S^2(S-a)^2(S-b)^2(S-c)^2}{a^2b^2c^2(S-a)^2}}$$

$$= \frac{\cancel{abc}}{\Delta}\,\frac{S(S-a)(S-b)(S-c)}{\cancel{abc}(S-a)}$$

$$= \frac{\Delta^{\cancel{2}}}{\cancel{\Delta}}\,\frac{1}{S-a} = \frac{\Delta}{S-a} = r_1 = L.H.S$$

ii. $r_2 = 4R\,\text{Cos}\frac{\alpha}{2}\,\text{Sin}\frac{\beta}{2}\,\text{Cos}\frac{\gamma}{2}$

R.H.S $= 4R\,\text{Cos}\frac{\alpha}{2}\,\text{Sin}\frac{\beta}{2}\,\text{Cos}\frac{\gamma}{2} = \frac{\cancel{4}abc}{\cancel{4}\Delta}\sqrt{\frac{S(S-a)}{bc}}\sqrt{\frac{(S-a)(S-c)}{ac}}\sqrt{\frac{S(S-c)}{ab}}$

$$= \frac{abc}{\Delta}\sqrt{\frac{S^2(S-a)^2\,(S-c)^2}{a^2b^2c^2}}$$

$$= \frac{\cancel{abc}}{\Delta}\,\frac{S(S-a)(S-b)(S-c)}{\cancel{abc}(S-b)} = \frac{1}{\cancel{\Delta}}\,\frac{\Delta^{\cancel{2}}}{S-b} = \frac{\Delta}{S-b} = r_2 = R.H.S$$

iii. $r_3 = 4R\,\text{Cos}\frac{\alpha}{2}\,\text{Cos}\frac{\beta}{2}\,\text{Sin}\frac{\gamma}{2}$

Sol. $= \frac{\cancel{4}abc}{\cancel{4}\Delta}\sqrt{\frac{S(S-a)}{bc}}\sqrt{\frac{S(S-b)}{ac}}\sqrt{\frac{(S-a)(S-b)}{ab}}$

$$= \frac{abc}{\Delta}\sqrt{\frac{S^2(S-a)^2\,(S-b)^2}{a^2b^2c^2}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{S^2(S-a)^2\,(S-b)^2(S-c)^2}{a^2b^2c^2(s-c)^2}}$$

$$= \frac{\cancel{abc}}{\Delta}\,\frac{S(S-a)(S-b)(S-c)}{\cancel{abc}(S-c)}$$

$$= \frac{1}{\cancel{\Delta}}\,\frac{\Delta^{\cancel{2}}}{(S-c)} = \frac{\Delta}{S-c} = r_3$$

4. **Show that:** **Multan 2008, Sargodha 2009**

i. $r_1 = S\tan\alpha/2$

Sol. R.H.S $= S\tan\frac{\alpha}{2} = S\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}$

$$= S\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2(S-a)^2}} = \cancel{S}\frac{\Delta}{\cancel{S}(S-a)} = r_1 = L.H.S$$

ii. $r^2 = S\tan\frac{\beta}{2}$ **Sargodha 2010**

Sol. R.H.S = $S\tan\frac{\beta}{2} = S\sqrt{\frac{(S-a)(S-c)}{S(S-b)}}$

$$= S\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2(S-b)^2}} = \cancel{S}\frac{\Delta}{\cancel{S}(S-b)} = \frac{\Delta}{S-b} = r_2 = L.H.S$$

iii. $r_3 = S\tan\frac{\gamma}{2}$ **Multan 2008, Sargodha 2010**

Sol. $r_3 = S\tan\frac{\gamma}{2} = S\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$

$$= S\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2(S-c)^2}} = \cancel{S}\frac{\Delta}{\cancel{S}(S-c)} = \frac{\Delta}{S-c} = r_3 = L.H.S$$

5. Prove that:

i. $r_1r_2 + r_2r_3 + r_3r_1 = s^2$ **Lahore 2009, Sargodha 2011**

Sol. L.H.S = $r_1r_2 + r_2r_3 + r_3r_1$

$$= \frac{\Delta}{S-a}\times\frac{\Delta}{S-b} + \frac{\Delta}{S-b}\times\frac{\Delta}{S-c} + \frac{\Delta}{S-c}\times\frac{\Delta}{S-a}$$

$$= \frac{\Delta^2}{(S-a)(S-b)} + \frac{\Delta^2}{(S-b)(S-c)} + \frac{\Delta^2}{(S-c)(S-a)}$$

$$= \Delta^2\left[\frac{1}{(S-a)(S-b)} + \frac{1}{(S-b)(S-c)} + \frac{1}{(S-c)(S-a)}\right] = \Delta^2\left[\frac{S-c+S-a+S-b}{(S-a)(S-b)(S-c)}\right]$$

$$= \Delta^2 S\left[\frac{3S-(a+b+c)}{S(S-a)(S-b)(S-c)}\right] \qquad \boxed{\frac{a+b+c}{2} = S \Rightarrow a+b+c = 2S}$$

$$= \cancel{\Delta^2} S\left[\frac{3S-2S}{\cancel{\Delta^2}}\right] = S(S) = S^2 = R.S.H$$

ii. $rr_1\, r_2\, r_3 = \Delta^2$ **Multan 2007, Faisalabad 2009, Sargodha 2008, 10**

Sol. L.H.S = $rr_1\, r_2\, r_3$

$$= \frac{\Delta}{S} \cdot \frac{\Delta}{S-a} \cdot \frac{\Delta}{S-b} \cdot \frac{\Delta}{S-c}$$

$$= \frac{\Delta^4}{S(S-a)(S-b)(S-c)} = \frac{\Delta^4}{\Delta^2} = \Delta^2 = R.H.S$$

iii. $r_1 + r_2 + r_3 - r = 4R$ **Multan 2008, Faisalabad 2008, Sargodha 2008**

Sol. L.H.S $= r_1 + r_2 + r_3 - r$

$$= \frac{\Delta}{S-a} + \frac{\Delta}{S-b} - \frac{\Delta}{S-c} - \frac{\Delta}{S}$$

$$= \Delta\left[\frac{1}{S-a} + \frac{1}{S-b} + \frac{1}{S-c} - \frac{1}{S}\right]$$

$$= \Delta\left[\frac{S-b+S-a}{(S-a)(S-b)} + \frac{\cancel{S}-\cancel{S}+c}{S(S-c)}\right] = \Delta\left[\frac{2S-a-b}{(S-a)(S-b)} + \frac{c}{S(S-c)}\right]$$

$$= \Delta\left[\frac{\cancel{a}+\cancel{b}+c-\cancel{a}-\cancel{b}}{(S-a)(S-b)} + \frac{c}{S(S-c)}\right] = \Delta\left[\frac{c}{(S-a)(S-b)} + \frac{c}{S(S-c)}\right]$$

$$= c\Delta\left[\frac{1}{(S-a)(S-b)} + \frac{1}{S(S-c)}\right] = \Delta\left[\frac{S(S-c)+(S-a)(S-b)}{S(S-a)(S-b)(S-c)}\right]$$

$$= c\Delta\left[\frac{S^2-cS-S^2-bS-aS+ab}{\Delta^2}\right] = c\Delta\left[\frac{2S^2-S(a+b+c)+ab}{\Delta^2}\right]$$

$$= c\left[\frac{2S^2-S(2S)+ab}{\Delta}\right] = c\left[\frac{\cancel{2S^2}-\cancel{2S^2}+ab}{\Delta}\right] = \frac{abc}{\Delta} = 4\frac{abc}{4\Delta} = 4R = R.H.S$$

iv. $r_1 r_2 r_3 = rs^2$

Sol. L.H.S $= r_1 r_2 r_3$

$$= \frac{\Delta}{S-a} \cdot \frac{\Delta}{S-b} \cdot \frac{\Delta}{S-c}$$

$$= \frac{\Delta^3}{(S-a)(S-b)(S-c)}$$

$$\boxed{r = \frac{\Delta}{S} \Rightarrow \Delta = rs}$$

$$= \frac{S\Delta^3}{S(S-a)(S-b)(S-c)} = \frac{S\Delta^3}{\Delta^2} = S\Delta = S(rs) = rs^2 = R.H.S$$

6. **Find R, r, r_1, r_2 and r_3, if measures of the sides of triangle ABC are**

i. **a = 13 , b = 14 , c = 15**

Sol. $S = \frac{a+b+c}{2} = \frac{13+14+15}{2} = 21$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$$

$$= \sqrt{21(21-13)(21-14)(21-15)} = \sqrt{21(8)(7)(6)} = \sqrt{7056} = 84$$

Now

$$r = \frac{\Delta}{S} = \frac{84}{21} = 4$$ **Gujranwala 2009**

$$\left.\begin{aligned} r_1 &= \frac{\Delta}{S-a} = \frac{84}{21-13} = \frac{84}{8} = 10.5 \\ r_2 &= \frac{\Delta}{S-b} = \frac{84}{21-14} = \frac{84}{7} = 12 \end{aligned}\right] \text{Multan 2009}$$

$$r_3 = \frac{\Delta}{S-c} = \frac{84}{21-15} = \frac{84}{6} = 14$$

$$R = \frac{abc}{4\Delta} = \frac{(13)(14)(15)}{4(84)} = \frac{2730}{336} = 8.125$$ **Lahore 2009**

ii. **a = 34 , b = 20 , c = 42**

Sol. $$S = \frac{a+b+c}{2} = \frac{34+20+42}{2} = \frac{96}{2} = 48$$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)} = \sqrt{48(48-34)(48-20)(48-42)}$$

$$\Delta = \sqrt{(48)(14)(28)(6)} = \sqrt{112896} = 336$$

$$r = \frac{\Delta}{S} = \frac{336}{48} = 7$$

$$r_1 = \frac{\Delta}{S-a} = \frac{336}{48-34} = \frac{336}{14} = 24$$

$$r_2 = \frac{\Delta}{S-b} = \frac{336}{48-20} = \frac{336}{28} = 12$$

$$r_3 = \frac{\Delta}{S-c} = \frac{336}{48-42} = \frac{336}{6} = 56$$

$$R = \frac{abc}{4\Delta} = \frac{(34)(20)(42)}{4(336)} = \frac{28560}{1344} = 21.25$$

7. **Prove that in an equilateral triangle,** **Faisalabad 2008, 09 Sargodha 2009, 2010**

i. **$r : R : r_1 = 1 : 2 : 3$**

Sol. **In equilateral triangle a = b = c**

So

$$S = \frac{a+b+c}{2} = \frac{a+a+a}{2} = \frac{3a}{2}, \ \Delta = \sqrt{S(S-a)(S-b)(S-c)} = \sqrt{S(S-a)(S-a)(S-a)}$$

$$\Delta = \sqrt{S(S-a)^3} = \sqrt{\frac{3a}{2} \times \left(\frac{3a}{2} - a\right)^3} = \sqrt{\frac{3a}{2} \times \left(\frac{3a-2a}{2}\right)^3} = \sqrt{\frac{3a}{2} \times \left(\frac{a}{2}\right)^3}$$

$$=\sqrt{\frac{3a}{2}\times\frac{a^3}{8}}=\sqrt{\frac{3a^4}{16}}=\frac{\sqrt{3}a^2}{4}$$

$$r=\frac{\Delta}{S}=\frac{\frac{\sqrt{3}a^2}{4}}{\frac{3a}{2}}=\frac{\sqrt{3}a^{\cancel{2}}}{4_2}\times\frac{\cancel{2}}{3\cancel{a}}=\frac{\sqrt{3}a}{2}\times\frac{1}{\sqrt{3}.\sqrt{3}}=\frac{a}{2\sqrt{3}}$$

$$r_1=\frac{\Delta}{S-a}=\frac{\frac{\sqrt{3}a^2}{4}}{\frac{3a}{2}-a}=\frac{\frac{\sqrt{3}a^2}{4}}{\frac{3a-2a}{2}}=\frac{\frac{\sqrt{3}a^2}{4}}{\frac{a}{2}}=\frac{\sqrt{3}a^2}{4}\times\frac{\cancel{2}}{\cancel{a}}=\frac{\sqrt{3}a}{2}$$

$$r_1=r_2=r_3=\frac{\sqrt{3}a}{2}\quad(because\,a=b=c\,so\,r_1=r_2=r_3)$$

$$R=\frac{abc}{4\Delta}=\frac{a.a.a}{4\frac{\sqrt{3}a^2}{4}}=\frac{a}{\sqrt{3}}$$

(i) **$r : R : r_1 = 1 : 2 : 3$**

$$\text{L.H.S}=r:R:r_1=\frac{a}{2\sqrt{3}}:\frac{a}{\sqrt{3}}:\frac{\sqrt{3}a}{2}$$

Multiplying by $\frac{2\sqrt{3}}{a}$

$$=\frac{\cancel{a}}{\cancel{2\sqrt{3}}}\times\frac{\cancel{2\sqrt{3}}}{\cancel{a}}:\frac{\cancel{a}}{\cancel{\sqrt{3}}}\times\frac{2\cancel{\sqrt{3}}}{\cancel{a}}:\frac{\sqrt{3}\cancel{a}}{\cancel{2}}\times\frac{\cancel{2}\sqrt{3}}{\cancel{a}}$$

$= 1 : 2 : 3 : = \text{R.H.S}$

ii. **$r : R : r_1 : r_2 : r_3 = 1 : 2 : 3 : 3 : 3$**

Sol. **L.H.S = $r:R:r_1:r_2:r_3$**

$$=\frac{a}{2\sqrt{3}}:\frac{a}{\sqrt{3}}:\frac{\sqrt{3}a}{2}:\frac{\sqrt{3}a}{2}:\frac{\sqrt{3}a}{2}$$

Multiplying by $\frac{2\sqrt{3}}{a}$

$$\frac{\cancel{a}}{\cancel{2\sqrt{3}}}\times\frac{\cancel{2\sqrt{3}}}{\cancel{a}}:\frac{\cancel{a}}{\cancel{\sqrt{3}}}\times\frac{2\cancel{\sqrt{3}}}{\cancel{a}}:\frac{\sqrt{3}\cancel{a}}{\cancel{2}}\times\frac{\cancel{2}\sqrt{3}}{\cancel{a}}:\frac{\sqrt{3}\cancel{a}}{\cancel{2}}\times\frac{\cancel{2}\sqrt{3}}{\cancel{a}}:\frac{\sqrt{3}\cancel{a}}{\cancel{2}}\times\frac{\cancel{2}\sqrt{3}}{\cancel{a}}$$

= 1:2:3:3:3 = R.H.S

8. (i) $\Delta = r^2 \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$

$$\text{R.H.S} = r^2 \text{Cot}\frac{\alpha}{2}\ \text{Cot}\frac{\beta}{2}\ \text{Cot}\frac{\gamma}{2} = r^2 \frac{1}{\tan\frac{\alpha}{2}}\cdot\frac{1}{\tan\frac{\beta}{2}}\cdot\frac{1}{\tan\frac{\gamma}{2}}$$

$$= r^2 \frac{1}{\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}}\cdot\frac{1}{\sqrt{\frac{(S-a)(S-c)}{S(S-b)}}}\cdot\frac{1}{\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}}$$

$$= r^2 \sqrt{\frac{S(S-a)}{(S-b)(S-c)}}\cdot\sqrt{\frac{S(S-b)}{(S-a)(S-c)}}\cdot\sqrt{\frac{S(S-c)}{(S-a)(S-b)}}$$

$$= r^2 \sqrt{\frac{S^2S(S-a)(S-b)(S-c)}{(S-a)^2(S-b)^2(S-c)^2}}$$

$$= r^2 \sqrt{\frac{S^3}{(S-a)(S-b)(S-c)}}$$

$$= r^2 \sqrt{\frac{S^4}{S(S-a)(S-b)(S-c)}} \quad (\textit{Multiply and divided by S})$$

$$= \frac{r^2S^2}{\Delta} = \frac{\Delta^{\cancel{2}}}{\cancel{S^2}}\cdot\frac{\cancel{S^2}}{\cancel{\Delta}} = \Delta = L.H.S$$

8. (ii) $\mathbf{r = s \tan\frac{\alpha}{2}\ \tan\frac{\beta}{2}\ \tan\frac{\gamma}{2}}$

Sol. $\text{R.H.S.} = s\tan\frac{\alpha}{2}\ \tan\frac{\beta}{2}\ \tan\frac{\gamma}{2}$

$$= S\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}\ \sqrt{\frac{(S-a)(S-c)}{S(S-b)}}\ \sqrt{\frac{(S-a)(S-b)}{S(S-c)}} = S\sqrt{\frac{(S-a)^2(S-b)^2(S-c)^2}{S^2\ S(S-a)(S-b)(S-c)}}$$

$$= \frac{S(S-a)(S-b)(S-c)}{S\sqrt{S(S-a)(S-b)(S-c)}} = \frac{\cancel{\Delta}^2}{S\cancel{\Delta}} = \frac{\Delta}{S} = r = L.H.S$$

8(iii). $\Delta = 4RrCos\frac{\alpha}{2}\ Cos\frac{\beta}{2}\ Cos\frac{\gamma}{2}$

Sol. **R.H.S** $= 4RrCos\frac{\alpha}{2}\ Cos\frac{\beta}{2}\ Cos\frac{\gamma}{2}$

$$= 4\frac{abc}{4\Delta}r\sqrt{\frac{S(S-a)}{bc}}\ \sqrt{\frac{S(S-b)}{ac}}\sqrt{\frac{S(S-c)}{ab}}$$

$$= \frac{abc}{\Delta} r \sqrt{\frac{S(S-a)S(S-b)(S)(S-c)}{(bc)(ac)(ab)}}$$

$$= \frac{abc}{\Delta} r \sqrt{\frac{S^2.S(S-a)S(S-b)(S-c)}{a^2b^2c^2}}$$

$$= \frac{\cancel{abc}}{\Delta} \frac{rS\sqrt{S(S-a)(S-b)(S-c)}}{\cancel{abc}} = \frac{rS\cancel{\Delta}}{\cancel{\Delta}} = rS = \frac{\Delta}{\cancel{S}}.\cancel{S} = \Delta = L.H.S$$

9 (i). $\frac{1}{2rR} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$ **Gujranwala 2009**

Sol. L.H.S = $\frac{1}{2rR} = \frac{1}{\cancel{2}.\frac{\cancel{\Delta}}{S}.\frac{abc}{_2\cancel{4\Delta}}} = \frac{1}{\frac{abc}{2S}} = \frac{2S}{abc}$

$$= \frac{a+b+c}{abc} = \frac{\cancel{a}}{\cancel{a}bc} + \frac{\cancel{b}}{a\cancel{b}c} + \frac{\cancel{c}}{ab\cancel{c}} = \frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \text{R.H.S}$$

9 (ii). $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$ **Sargodha 2006**

Sol. R.H.S= $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{S-a}{\Delta} + \frac{S-b}{\Delta} + \frac{S-c}{\Delta}$

$$= \frac{1}{\Delta}[S-a+S-b+S-c]$$

$$= \frac{1}{\Delta}[3S-(a+b+c)]$$

$$= \frac{1}{\Delta}[3S-2S)] = \frac{S}{\Delta} = \frac{1}{r} \quad Since \frac{\Delta}{S} = r \Rightarrow \frac{S}{\Delta} = \frac{1}{r} = L.H.S$$

10 (i) r = $\frac{a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{Cos\frac{\alpha}{2}}$

$$= \frac{a\sqrt{\frac{(S-a)(S-c)}{ac}}\sqrt{\frac{(S-a)(S-b)}{ab}}}{\sqrt{\frac{S(S-a)}{bc}}} = a\sqrt{\frac{\frac{(S-a)(S-c).(S-a)(S-b)}{a^2\cancel{bc}}}{\frac{S(S-a)}{\cancel{bc}}}}$$

$$= a\sqrt{\frac{S(S-a)^2(S-b)(S-c)}{a^2.S(S-a)}} = \frac{\cancel{a}}{\cancel{a}}\sqrt{\frac{(S-a)(S-b)(S-b)}{S}}$$

$$= \sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2}} = \frac{\Delta}{S} = r = L.H.S$$

10 (ii). r = $\dfrac{b\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}}$

Sol. $R.H.S = \dfrac{b\,sin\frac{\alpha}{2}\,sin\frac{\gamma}{2}}{cos\frac{\beta}{2}}$

$$= \frac{b\sqrt{\frac{(S-b)(S-c)}{bc}}.\sqrt{\frac{(S-a)(S-b)}{ab}}}{\sqrt{\frac{S(S-b)}{ac}}} = b\sqrt{\frac{(S-b)(S-c)}{bc}}.\sqrt{\frac{(S-a)(S-b)}{ab}}.\sqrt{\frac{ac}{S(S-b)}}$$

$$= b\sqrt{\frac{ac(S-a)(S-b)^2(S-c)}{acb^2S(S-b)}} = b\sqrt{\frac{S(S-a)(S-b)(S-c)}{b^2S^2}} = \cancel{b}\frac{\Delta}{\cancel{b}S} = \frac{\Delta}{S} = r = L.H.S$$

10 (iii) r = $\dfrac{c\,Sin\frac{\alpha}{2}\,Sin\frac{\beta}{2}}{Cos\frac{\gamma}{2}}$

Sol. $R.H.S = \dfrac{c\,Sin\frac{\alpha}{2}\,Sin\frac{\beta}{2}}{Cos\frac{\gamma}{2}}$

$$= \frac{c\sqrt{\frac{(S-b)(S-c)}{bc}}.\sqrt{\frac{(S-a)(S-c)}{ac}}}{\sqrt{\frac{S(S-c)}{ab}}} = c\sqrt{\frac{(S-b)(S-c)}{bc}}.\sqrt{\frac{(S-a)(S-c)}{ac}}.\sqrt{\frac{ab}{S(S-c)}}$$

$$= c\sqrt{\frac{ab(S-a)(S-b)(S-c)^2}{abc^2S(S-c)}} = c\sqrt{\frac{S(S-a)(S-b)(S-c)}{c^2S^2}}$$

$$= \cancel{c}\frac{\Delta}{\cancel{c}S} = \frac{\Delta}{S} = r = L.H.S$$

Hence Proved

11. Prove that: abc (Sinα + Sinβ + Sinγ) = 4ΔS

Sol. L.H.S = abc (Sinα + Sinβ + Sinγ)

$$= abc\left(\frac{a}{2R}+\frac{b}{2R}+\frac{c}{2R}\right) = abc\left(\frac{a+b+c}{2R}\right)$$

$$= \cancel{abc}\left(\frac{\cancel{2}S}{\cancel{2}\frac{\cancel{abc}}{4\Delta}}\right) = \frac{S}{\frac{1}{4\Delta}} = 4\Delta S = R.H.S$$

$$\boxed{\begin{aligned} &As\ R = \frac{a}{2Sin\alpha} = \frac{b}{2Sin\beta} = \frac{c}{2Sin\gamma} \\ &\Rightarrow sin\alpha = \frac{a}{2R}, sin\beta = \frac{b}{2R}, sin\gamma = \frac{c}{2R}\end{aligned}}$$

12. Prove that:

i. **($r_1 + r_2$) tan** $\frac{\gamma}{2} = c$ **Faisalabad 2008, Multan 2009**

Sol. L.H.S = ($r_1 + r_2$) tan $\frac{\gamma}{2} = \left(\frac{\Delta}{S-a}+\frac{\Delta}{S-b}\right)\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$

$$= \Delta\left[\frac{1}{S-a}+\frac{1}{S-b}\right]\sqrt{\frac{(S-a)^2(S-b)^2}{S(S-a)(S-b)(S-c)}} \ ("\times"\ \&\ "\div"\ by (S-a)(S-b))$$

$$= \cancel{\Delta}\left(\frac{S-b+S-a}{\cancel{(S-a)}\ \cancel{(S-b)}}\right)\frac{\cancel{(S-a)}\ \cancel{(S-b)}}{\cancel{\Delta}}$$

$$= 2S - a - b = \cancel{a} + \cancel{b} + c - \cancel{a} - \cancel{b} = c = R.H.S$$

ii. **(r_3 – r) Cot**$\frac{\gamma}{2} = c$

Sol. L.H.S = (r_3 – r) Cot $\frac{\gamma}{2} = \left(\frac{\Delta}{S-c}-\frac{\Delta}{S}\right)\frac{1}{tan\frac{\gamma}{2}}$

$$= \Delta\left(\frac{1}{S-c}-\frac{1}{S}\right)\sqrt{\frac{S(S-c)}{(S-a)(S-b)}}$$

$$= \Delta\left(\frac{S-(S-c)}{S(S-c)}\right)\sqrt{\frac{S^2(S-c)^2}{S(S-a)(S-b)(S-c)}}$$

$$= \cancel{\Delta}\left[\frac{\cancel{S}-\cancel{S}+c}{\cancel{S}\cancel{(S-c)}}\right]\frac{\cancel{S}\ \cancel{(S-c)}}{\cancel{\Delta}} = c$$

TEST YOUR SKILLS Marks: 50

Q # 1. Select the Correct Option **(10)**

i. $Sin\gamma =$

a) Area of triangle b) 2(Area of triangle)

c) $\frac{1}{2}$ (Area of triangle) d) 3(Area of Triangle)

ii. $\frac{1}{2rR}$

a) $\frac{S}{2abc}$ b) $\frac{abc}{2s}$

c) $\frac{2bc}{a}$ d) $\frac{2s}{abc}$

iii. $r_2 =$

a) $\frac{2\Delta}{s-c}$ b) $\frac{2\Delta}{s-b}$

c) $\frac{\Delta}{s-b}$ d) $\frac{\Delta}{s-b}$

iv. $2s = a+b+c$ then $Sin\,\alpha/2 =$

a) $\sqrt{\frac{(s-b)(s-c)}{bc}}$ b) $\sqrt{\frac{(s-a)(s-c)}{bc}}$

c) $\sqrt{\frac{(s-b)(s-a)}{bc}}$ d) $\sqrt{\frac{s(s-a)}{bc}}$

v. $Cos\,\theta/2$ is equal to:

a) $\pm\sqrt{\frac{1+Sin\alpha}{2}}$ b) $\pm\sqrt{\frac{1-Cos\alpha}{2}}$

c) $\pm\sqrt{\frac{1+Cos\alpha}{2}}$ d) $\pm\sqrt{\frac{1-Sin\alpha}{2}}$

vi. $R =$

a) $\frac{4\Delta}{abc}$ b) $\frac{abc}{4\Delta}$

c) $\frac{\Delta}{s}$ d) $\frac{\Delta}{s-a}$

vii. $Cos\,\alpha/2 =$ equals:

a) $\sqrt{\frac{s(s-a)}{bc}}$ b) $\sqrt{\frac{s(s-b)}{ac}}$

c) $\sqrt{\dfrac{s(s-c)}{ab}}$ d) $\sqrt{\dfrac{(s-b)(s-c)}{bc}}$

viii. In any triangle $\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} =$

a) $Tan\,\alpha/2$ b) $Tan\,\beta/2$
c) $Tan\,\gamma/2$ d) $Cot\,\alpha/2$

ix. If $a = 3,\ b = 4,\ c = 5$ then $S =$

a) 9 b) 6
c) 12 d) 7

x. A triangle which is not right is called

a) Isosceles b) Equilateral
c) Oblique d) Quadrilateral

Q # 2. Short Questions: **(10 X 2 = 20)**

i. In right triangle $\alpha = 37^\circ 20',\ a = 243,\ \gamma = 90^\circ,\ c = ?$

ii. Prove that $r.r_1.r_2r_3 = \Delta^2$

iii. Prove that $R = \dfrac{abc}{4\Delta}$

iv. Write any two law of Tangents

v. Solve right triangle if $\alpha = 58^\circ 13',\ b = 125.7,\ \gamma = 90^\circ$

vi. Prove that $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

vii. Prove that $r_1 = s\tan\alpha/2$

viii. Define angle of Elevation and Depression:

ix. Find Area of Triangle if $b = 21.6,\ c = 30.2,\ \alpha = 52^\circ 40'$

x. Prove that $Cos\,\beta/2 = \sqrt{\dfrac{s(s-b)}{ac}}$

Long Questions: **(2 X 10 = 20)**

Q # 3. **(a)** Prove that $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

(b) Prove that $r = 4R\ Sin\,\alpha/2\,Sin\,\beta/2\ Sin\,\gamma/2$

Q # 4. **(a)** Prove that in equilateral triangle $r : R : r_1 = 1 : 2 : 3$

(b) Solve triangle if $b = 95,\ c = 34,\ \alpha = 52^\circ$

Inverse Trigonometric Functions

13

EXERCISE 13.1

1. Evaluate without using tables/calculator.

i. $Sin^{-1}(1)$

Sol. Let $y = Sin^{-1}(1)$ ——— I

$\Rightarrow Sin\ y = 1 \Rightarrow y = \dfrac{\pi}{2}$

I become $Sin^{-1}(1) = \dfrac{\pi}{2}$

ii. $Sin^{-1}(-1)$

Sol. Let $y = Sin^{-1}(-1)$ ——— I

$\Rightarrow Sin\ y = -1 \Rightarrow y = -\dfrac{\pi}{2}$

I become $\sin^{-1}(-1) = -\dfrac{\pi}{2}$

iii. $Cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

Sol. Let $y = \cos^{-1}\dfrac{\sqrt{3}}{2}$ ——— I

$\Rightarrow Cos\ y = \dfrac{\sqrt{3}}{2} \Rightarrow y = \dfrac{\pi}{6}$

I become $Cos^{-1}\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{6}$

iv. $Tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)$

Sol. Let $y = \tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)$ ——— I

$\Rightarrow \tan y = \left(\dfrac{-1}{\sqrt{3}}\right)$

$\Rightarrow y = \tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right) \Rightarrow y = -\dfrac{\pi}{6}$

I become $\tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right) = -\dfrac{\pi}{6}$

v. $Cos^{-1}\dfrac{1}{2}$

Sol. Let $y = Cos^{-1}\dfrac{1}{2}$ ——— I

$\Rightarrow Cos\ y = \dfrac{1}{2} \Rightarrow y = \dfrac{\pi}{6}$

I become $Cos^{-1}\dfrac{1}{2} = \dfrac{\pi}{6}$

vi. $\text{Tan}^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Sol. Let $y=\text{Tan}^{-1}\left(\frac{1}{\sqrt{3}}\right)$ —I

$\frac{1}{\sqrt{3}} = \tan y \Rightarrow y = \frac{\pi}{6}$

I becomes $\text{Tan}^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

vii. $\text{Cot}^{-1}(-1)$

Sol. Let $y=\text{Cot}^{-1}(-1)$ ——I

$\Rightarrow \text{Cot } y = -1$

$\Rightarrow \tan y = \frac{1}{-1} = -1$

$\Rightarrow y = \frac{-\pi}{4} \text{ or } y = \frac{3\pi}{4}$

$(\pi - \frac{\pi}{4} = \frac{3\pi}{4}$

because Domain of $\cot^{-1}$ *is* $[0, \pi])$

$\Rightarrow$ I become $\text{Cot}^{-1}(-1) = \frac{3\pi}{4}$

viii. $\text{Cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right)$

Sol. Let $y=\text{Cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ —I

$\Rightarrow \text{Cosec } y = \frac{-2}{\sqrt{3}} \Rightarrow \text{Sin} y = \frac{-\sqrt{3}}{2} \Rightarrow y = \frac{-\pi}{3}$

I become $\text{Cosee}^{-1}\left(\frac{-2}{\sqrt{3}}\right) = \frac{-\pi}{3}$

ix. $\text{Sin}^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

Sol. Let $y = \text{Sin}^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ —I

$\Rightarrow \text{Sin } y = \left(\frac{-1}{\sqrt{2}}\right) \Rightarrow y = \frac{-\pi}{4}$

I become $\text{Sin}^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{-\pi}{4}$

2. **Without using table/ calculator show that:**

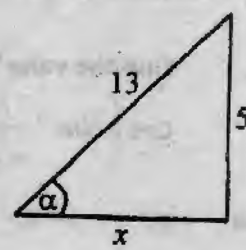
i. $\tan^{-1}\frac{5}{12} = Sin^{-1}\frac{5}{13}$

Sol. Let $\text{Sin}^{-1}\left(\frac{5}{13}\right) = \alpha$ ——I $\Rightarrow \text{Sin } \alpha = \frac{5}{13}$

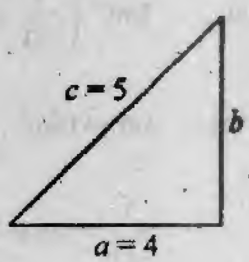
$\tan\alpha = \frac{12}{5} \Rightarrow \alpha = \tan^{-1}\frac{5}{12}$

$\text{Sin}^{-1}\frac{5}{13} = \tan^{-1}\frac{5}{12}$ (use I)

$$\boxed{\begin{aligned} &x^2 + (5)^2 = (13)^{12} \\ &x^2 = 169 - 25 = 144 \\ &x = 12 \end{aligned}}$$



ii. $2\,\mathrm{Cos}^{-1}\dfrac{4}{5} = \mathrm{Sin}^{-1}\dfrac{24}{25}$



Sol. Let $\cos^{-1}\dfrac{4}{5} = \alpha$ ——— I

By Pythagoras

$a^2 + b^2 = c^2$

$b^2 = c^2 - a^2 = (5)^2 - (4)^2$

$b^2 = 9 \Rightarrow b = 3$

$\Rightarrow \mathrm{Cos}\,\alpha = \dfrac{4}{5}$

$\mathrm{Sin}\,\alpha = \dfrac{3}{5}$

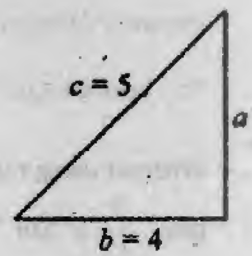
Now $\mathrm{Sin}\,2\alpha = 2\,\mathrm{Sin}\,\alpha\,\mathrm{Cos}\,\alpha$

$\mathrm{Sin}\,2\alpha = 2\left(\dfrac{3}{5}\right)\left(\dfrac{4}{5}\right) = \dfrac{24}{25}$

$2\alpha = \mathrm{Sin}^{-1}\left(\dfrac{24}{25}\right)$

$2\left(Cos^{-1}\dfrac{4}{5}\right) = \mathrm{Sin}^{-1}\left(\dfrac{24}{25}\right)$ $\boxed{\text{use I}}$

iii. $\mathrm{Cos}^{-1}\dfrac{4}{5} = \mathrm{Cot}^{-1}\dfrac{4}{3}$



Sol. Let $\mathrm{Cos}^{-1}\dfrac{4}{5} = \alpha$ —— I

By Pythagoras

$a^2 + b^2 = c^2$

$a^2 = c^2 - b^2$

$a^2 = (5)^2 - (4)^2$

$a^2 = 9 \Rightarrow a = 3$

$\Rightarrow \mathrm{Cos}\,\alpha = \dfrac{4}{5}$

$\tan\alpha = \dfrac{3}{4}$

$\mathrm{Cot}\,\alpha = \dfrac{4}{3} \Rightarrow \alpha = \mathrm{Cot}^{-1}\left(\dfrac{4}{3}\right) \Rightarrow Cos^{-1}\left(\dfrac{4}{5}\right) = Cot^{-1}\left(\dfrac{4}{3}\right)(use\ I)$

3. **Find the value of each expression:**

i. $\mathrm{Cos}\left(Sin^{-1}\dfrac{1}{\sqrt{2}}\right)$

Sol. Let $y = \mathrm{Sin}^{-1}\dfrac{1}{\sqrt{2}}$ —— I $\Rightarrow \sin y = \dfrac{1}{\sqrt{2}}$

$\Rightarrow y = \frac{\pi}{4} \Rightarrow \text{Sin}^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$ use I

Now $\text{Cos}\left(Sin^{-1}\frac{1}{\sqrt{2}}\right) = Cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

ii. $\text{Sec}\left(Cos^{-1}\frac{1}{2}\right)$

Sol. Let $y = \text{Cos}^{-1}\frac{1}{2}$ ——I $\Rightarrow \text{Cos}\, y = \frac{1}{2}$

$\Rightarrow y = \frac{\pi}{3} \Rightarrow \text{Cos}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ (use I)

Now $\text{Sec}\left(Cos^{-1}\frac{1}{2}\right) = Sec\frac{\pi}{3} = \frac{1}{Cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$

iii. $\tan\left(Cos^{-1}\frac{\sqrt{3}}{2}\right)$ **Sargodha 2008**

sol. Let $\text{Cos}^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$ ——I $\Rightarrow \text{Cos}\, y = \frac{\sqrt{3}}{2} \Rightarrow y = \frac{\pi}{6}$ (use I)

Now $\text{Tan}\left(Cos^{-1}\frac{\sqrt{3}}{2}\right) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

iv. **Cosee ($\tan^{-1}(-1)$)**

Sol. Let $y = \tan^{-1}(-1)$ ——I $\Rightarrow \tan y = -1$

$\Rightarrow y = -\frac{\pi}{4} \Rightarrow \tan^{-1}(-1) = \frac{-\pi}{4}$ (use I)

Now $\text{cosec}(\tan^{-1}(-1)) = \text{Cosec}\left(\frac{-\pi}{4}\right) = \frac{1}{Sin\left(\frac{-\pi}{4}\right)} = \frac{1}{-\frac{1}{\sqrt{2}}} = -\sqrt{2}$

v. $\text{Sec}\left(Sin^{-1}\left(-\frac{1}{2}\right)\right)$ **Multan 2007, 2008**

Sol. Let $y=\text{Sin}^{-1}\left(\frac{-1}{2}\right)$ —— $I \Rightarrow Sin\, y=\frac{-1}{2}$

$\Rightarrow y=\frac{-\pi}{6} \Rightarrow Sin^{-1}\left(\frac{-1}{2}\right)=\frac{-\pi}{6}$

Now $\text{Sec}\left(Sin^{-1}\left(\frac{-1}{2}\right)\right)=Sec\left(\frac{-\pi}{6}\right)=\frac{1}{Cos\left(\frac{-\pi}{6}\right)}=\frac{1}{Cos\frac{\pi}{6}}=\frac{1}{\frac{\sqrt{3}}{2}}=\frac{2}{\sqrt{3}}$

vi. **$\tan(\tan^{-1}(-1))$**

Sol. Let $y=\tan^{-1}(-1)$

$\Rightarrow \tan y=-1 \Rightarrow y=\frac{-\pi}{4}$

$\Rightarrow$ Now $\tan(\tan^{-1}(-1))=\tan\left(\frac{-\pi}{4}\right)=-\tan\frac{\pi}{4}=-1$

vii. $\text{Sin}\left(Sin^{-1}\left(\frac{1}{2}\right)\right)$

Sol. Let $y=\text{Sin}^{-1}\left(\frac{1}{2}\right)$ —— $I \Rightarrow \text{Sin}\, y=\frac{1}{2} \Rightarrow y=\frac{\pi}{6} \Rightarrow Sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$

Now $\text{Sin}\left(Sin^{-1}\frac{1}{2}\right)=Sin\frac{\pi}{6}=\frac{1}{2}$

viii. $\tan\left(Sin^{-1}\left(\frac{-1}{2}\right)\right)$

Let $y=\text{Sin}^{-1}\left(\frac{-1}{2}\right)$ —— I $\Rightarrow$ Sin y $=\frac{-1}{2} \Rightarrow y=\frac{-\pi}{6} \Rightarrow \text{Sin}^{-1}\left(\frac{-1}{2}\right)=\frac{-\pi}{6}$ (use I)

Now $\tan\left(Sin^{-1}\left(\frac{-1}{2}\right)\right)=\tan\left(\frac{-\pi}{6}\right)=\frac{-1}{\sqrt{3}}$

ix. **$\text{Sin}(\tan^{-1}(-1))$**

Sol. Let $y=\tan^{-1}(-1)$ —— I

$\Rightarrow \tan y=-1 \Rightarrow y=\frac{-\pi}{4} \Rightarrow \tan^{-1}(-1)=\frac{-\pi}{4}$

Now $\text{Sin}(\tan^{-1}(-1))=\text{Sin}\left(\frac{-\pi}{4}\right)=-Sin\frac{\pi}{4}=\frac{-1}{\sqrt{2}}$

EXERCISE 13. 2

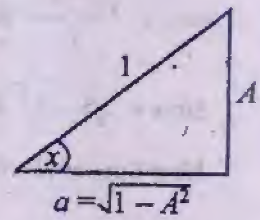
Important Note: **In whole exercise 13.2 take values of $\cos\theta$ and $\sin\theta$ positive because $\cos\theta$ is positive in domain of $\sin\theta$ and $\sin\theta$ is positive in domain of $\cos\theta$.**

Theorem I $\quad \text{Sin}^{-1}A + \text{Sin}^{-1}B = \text{Sin}^{-1}(A\sqrt{1-B^2} + B\sqrt{1-A^2})$ $\quad$ Lahore 2009

Proof: Let $x = \text{Sin}^{-1}A$ and $y = \text{Sin}^{-1}B$ —— I

$\Rightarrow \text{Sin } x = \dfrac{A}{1}$ *and* $\text{Sin } y = \dfrac{B}{1}$

$$a^2 + A^2 = 1$$
$$a^2 = 1 - A^2 \Rightarrow a = \sqrt{1-A^2}$$



$\text{Cos } x = \sqrt{1-A^2}$

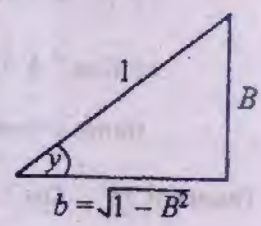
$\text{Cos } y = \sqrt{1-B^2}$

Now $\text{Sin}(x+y) = \text{Sin } x \text{ Cos } y + \text{Cos } x \text{ Sin } y$

$\text{Sin}(x+y) = A\sqrt{1-B^2} + B\sqrt{1-A^2}$

$x + y = \text{Sin}^{-1}(A\sqrt{1-B^2} + B\sqrt{1-A^2})$

$$b^2 + B^2 = 1$$
$$b^2 = 1 - B^2$$
$$b = \sqrt{1-B^2}$$



(use I) $\text{Sin}^{-1}A + \text{Sin}^{-1}B = \text{Sin}^{-1}(A\sqrt{1-B^2} + B\sqrt{1-A^2})$ Hence proved

Theorem II $\quad \text{Sin}^{-1}A - \text{Sin}^{-1}B = \text{Sin}^{-1}(A\sqrt{1-B^2} + 3\sqrt{1-A^2})$

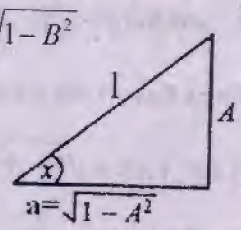
Proof: Let $\text{Sin}^{-1}A = x$ and $\text{Sin}^{-1}B = y$

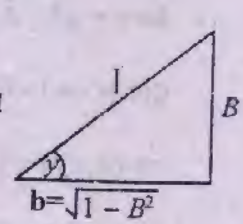
$$a^2 + A^2 = 1$$
$$a^2 = 1 - A^2 \Rightarrow a = \sqrt{1-A^2}$$

$\Rightarrow \text{Sin } x = \dfrac{A}{1}$ *and* $\text{Sin } y = \dfrac{B}{1}$

$\text{Cos } x = \dfrac{\sqrt{1-A^2}}{1} \Rightarrow \text{Cos } x = \sqrt{1-A^2}$ & $\text{Cos } y = \sqrt{1-B^2}$

Now $\text{Sin}(x-y) = \text{Sin } x \text{ Cos } y - \text{Cos } x \text{ Sin } y$





$\text{Sin}(x-y) = A\sqrt{1-B^2} - B\sqrt{1-A^2}$

$x - y = \text{Sin}^{-1}(A\sqrt{1-B^2} - B\sqrt{1-A^2})$

$$b^2 + B^2 = 1$$
$$b^2 = 1 - B^2$$
$$b = \sqrt{1-B^2}$$

$\Rightarrow Sin^{-1}A - Sin^{-1}B = Sin^{-1}(A\sqrt{1-B^2} - B\sqrt{1-A^2})$

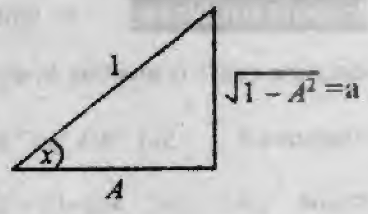
Hense proved

Theorem III $\quad \text{Cos}^{-1} A + \text{Cos}^{-1} B = \text{Cos}^{-1}(AB - \sqrt{(1-A^2)(1-B^2)})$

Sol. Let $\text{Cos}^{-1} A = x$ and $\text{Cos}^{-1} B = y$

$$\Rightarrow \text{Cos}\, x = \frac{A}{1} \text{ and } Cos\, y = \frac{B}{1}$$

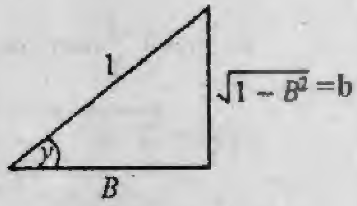
$$\boxed{\begin{aligned} a^2 + A^2 &= 1 \\ a^2 &= 1 - A^2 \\ a &= \sqrt{1-A^2} \end{aligned}}$$



$$\text{Sin}\, x = \frac{\sqrt{1-A^2}}{1} \ \& \ Sin\, y = \frac{\sqrt{1-B^2}}{1}$$

$$\text{Sin}\, x = \sqrt{1-A^2} \ \& \ \text{Sin}\, y = \sqrt{1-B^2}$$

$$\boxed{\begin{aligned} b^2 + B^2 &= 1 \\ b^2 &= 1 - B^2 \\ b &= \sqrt{1-B^2} \end{aligned}}$$



Now $\text{Cos}(x+y) = \text{Cos}\, x\, \text{Cos}\, y - \text{Sin}\, x\, \text{Sin}\, y$

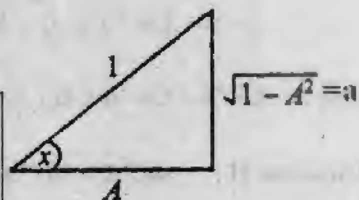
$$\text{Cos}\,(x+y) = AB - \sqrt{1-A^2}\ \sqrt{1-B^2}$$

$$\Rightarrow (x+y) = \text{Cos}^{-1}(AB - \sqrt{(1-A^2)(1-B^2)})$$

$$\Rightarrow \text{Cos}^{-1} A + \text{Cos}^{-1} B = \text{Cos}^{-1}(AB - \sqrt{(1-A^2)(1-B^2)})$$

Hense proved.

Theorem IV $\quad \text{Cos}^{-1} A - \text{Cos}^{-1} B = (AB + \sqrt{(1-A^2)(1-B^2)})$



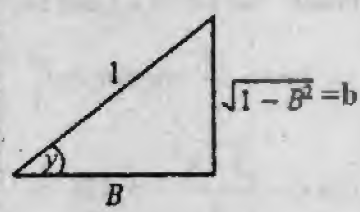
Sol. Let $\text{Cos}^{-1} A = x$ and $\text{Cos}^{-1} B = y$

$$\boxed{\begin{aligned} a^2 + A^2 &= 1 \\ a^2 &= 1 - A^2 \\ a &= \sqrt{1-A^2} \end{aligned}}$$

$$\Rightarrow \text{Cos}\, x = \frac{A}{1} \quad \Rightarrow \text{Cos}\, y = \frac{B}{1}$$

$$\text{Sin}\, x = \frac{\sqrt{1-A^2}}{1} \text{ and } \text{Sin}\, y = \frac{\sqrt{1-B^2}}{1}$$

$$\boxed{\begin{aligned} b^2 + B^2 &= 1 \\ b^2 &= 1 - B^2 \\ b &= \sqrt{1-B^2} \end{aligned}}$$



$$\text{Sin}\, x = \sqrt{1-A^2} \text{ and } \text{Sin}\, y = \sqrt{1-B^2}$$

$$\text{Cos}\,(x-y) = \text{Cos}\, x\, \text{Cos}\, y + \text{Sin}\, x\, \text{Sin}\, y = (AB + \sqrt{1-B^2}\ \sqrt{1-A^2})$$

$$\Rightarrow (x-y) = Cos^{-1}(AB + \sqrt{1-A^2}\sqrt{1-B^2})$$

$$\Rightarrow \text{Cos}^{-1} A - \text{Cos}^{-1} B = (AB + \sqrt{(1-A^2)(1-B^2)})$$

Theorem V $\tan^{-1} A + \tan^{-1} B = \tan^{-1}\left(\dfrac{A+B}{1-AB}\right)$ **Sargodha 2008, 2011**

Sol. Let $\tan^{-1} A = x$ and $\tan^{-1} B = y \Rightarrow \tan x = A$ & $\tan y = B$

Now $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y} = \dfrac{A+B}{1-AB}$

$$\Rightarrow x + y = \tan^{-1}\left(\frac{A+B}{1-AB}\right) \Rightarrow \tan^{-1} A + \tan^{-1} B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

Similarly $\tan^{-1} A - \tan^{-1} B = \tan^{-1}\left(\dfrac{A-B}{1+AB}\right)$ **Federal**

Exercise 13.2

Prove the following:

1. $\mathbf{Sin^{-1}\dfrac{5}{13} + Sin^{-1}\dfrac{7}{25} = Cos^{-1}\dfrac{253}{325}}$ **Sargodha 2009**

Sol. Let $\sin^{-1}\dfrac{5}{13} = x \Rightarrow \text{Sin } x = \dfrac{5}{13}$ and $\text{Sin}^{-1}\dfrac{7}{25} = y \Rightarrow \text{Sin } y = \dfrac{7}{25}$

By Pythagrras

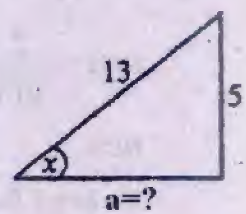
$a^2 + b^2 = c^2$

$a^2 = c^2 - b^2$

$a^2 = (13)^2 - (5)^2$

$a^2 = 144 \Rightarrow a = 12$

$\text{Cos } x = \dfrac{12}{13}$



Now $\text{Cos}(x+y) = \text{Cos } x \text{ Cos } y - \text{Sin } x \text{ Sin } y$

$\text{Cos}(x+y) = \left(\dfrac{12}{13}\right)\left(\dfrac{24}{25}\right) - \left(\dfrac{5}{13}\right)\left(\dfrac{7}{25}\right)$

$= \dfrac{288}{325} - \dfrac{35}{325} = \dfrac{288-35}{325}$

$\text{Cos}(x+y) = \dfrac{253}{325} \Rightarrow x + y = Cos^{-1}\left(\dfrac{253}{325}\right)$

$\Rightarrow \text{Sin}^{-1}\dfrac{5}{13} + \text{Sin}^{-1}\dfrac{7}{25} = \text{Cos} -1\left(\dfrac{253}{325}\right)$

$b^2 + (7)^2 = (25)^2$

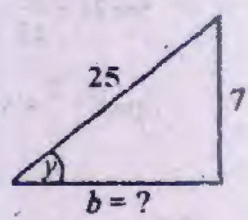
$b^2 + (25)^2 = (7)^2$

$b^2 = 625 - 49$

$b^2 = 576$

$b = 24$

$Cos\, y = \dfrac{24}{25}$



2. $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{9}{19}$ **Fsd 2008, Multan 2007,08, 09, Rawalpindi 2009**

Sol. We know that $\tan^{-1}A+\tan^{-1}B=\tan^{-1}\left(\frac{A+B}{1-AB}\right)$

Put $A=\frac{1}{4}$ and $B=\frac{1}{5}$

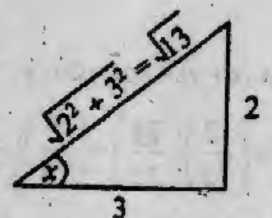
Then $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}=\tan^{-1}\left(\frac{\frac{1}{4}+\frac{1}{5}}{1-\left(\frac{1}{4}\right)\left(\frac{1}{5}\right)}\right)=\tan^{-1}\frac{\frac{5+4}{20}}{1-\frac{1}{20}}$

$\Rightarrow \tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}=\tan^{-1}\left(\frac{9}{\cancel{20}}\times\frac{\cancel{20}}{19}\right)=\tan^{-1}\frac{9}{19}$

Hence proved

3. $2\tan^{-1}\frac{2}{3}=Sin^{-1}\frac{12}{13}$ **Federal Board**

Sol. Let $\tan^{-1}\frac{2}{3}=x\Rightarrow \tan x=\frac{2}{3}$



$\text{Sin } x=\frac{2}{\sqrt{13}}\ \&\ Cos\ x=\frac{3}{\sqrt{13}}$

Now

Sin 2x = 2 Sin x Cos x

$\text{Sin } 2x=2\left(\frac{2}{\sqrt{13}}\right)\left(\frac{3}{\sqrt{13}}\right)$

$\text{Sin } 2x=\frac{12}{13}\Rightarrow 2x=\text{Sin}^{-1}\frac{12}{13}$

$2\tan^{-1}\frac{2}{3}=Sin^{-1}\frac{12}{13}\ (put\ value\ of\ x)$

4. $\tan^{-1}\left(\dfrac{120}{119}\right)=2Cos^{-1}\dfrac{12}{13}$

Sol. Take $\quad Cos^{-1}\dfrac{12}{13}=x\Rightarrow Cos\,x=\dfrac{12}{13}$

$$\text{Sin}^2 x = 1-\text{Cos}^2 x = 1-\frac{144}{169}=\frac{25}{169}$$

$\text{Sin}\,x = \dfrac{5}{13}$ (Sin is + ve in Domain of Cos x)

$$\text{Tan}\,x = \frac{Sin\,x}{Cos\,x} = \frac{\frac{5}{\cancel{13}}}{\frac{12}{\cancel{13}}} = \frac{5}{12}$$

$$\text{Now } \tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{2\left(\frac{5}{12}\right)}{1-\frac{25}{144}}$$

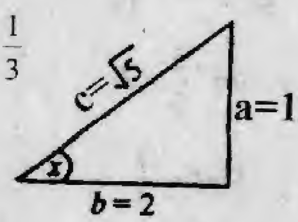
$$\tan 2x = \frac{\frac{10}{12}}{\frac{144-25}{144}} = \frac{\frac{10}{12}}{\frac{119}{144}} = \frac{10}{12}\times\frac{144}{119}$$

$$\tan 2x = \frac{120}{119} \Rightarrow 2x = \tan^{-1}\left(\frac{120}{119}\right) \Rightarrow 2Cos^{-1}\frac{12}{13} = \tan^{-1}\left(\frac{120}{119}\right)$$

5. $\text{Sin}^{-1}\dfrac{1}{\sqrt{5}}+Cot^{-1}3=\dfrac{\pi}{4}$ **Sargodha 2011**

Sol. take $\sin^{-1}\dfrac{1}{\sqrt{5}}=x\Rightarrow Sin\,x=\dfrac{1}{\sqrt{5}}\Rightarrow \tan x=\dfrac{1}{2}$

and $Cot^{-1}3 = y \Rightarrow \text{Cot y} = 3 \Rightarrow \tan y = \dfrac{1}{3}$



By Pythagoras

$a^2+b^2=c^2$

$b^2=c^2-a^2$

$a^2=\left(\sqrt{5}\right)^2-(1)^2$

$a^2=4\Rightarrow a=2$

$\tan x=\dfrac{1}{2}$

Now Tan (x + y) = $\dfrac{\tan x + \tan y}{1 - \tan x \tan y} = \dfrac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}$

Tan (x + y) = $\dfrac{\frac{5}{6}}{1-\frac{1}{6}} = \dfrac{\frac{\cancel{5}}{\cancel{6}}}{\frac{\cancel{5}}{\cancel{6}}} = 1 \Rightarrow$ x + y = $\tan^{-1}(1) = \dfrac{\pi}{4}$

Hence $\text{Sin}^{-1}\dfrac{1}{\sqrt{5}} + Cot^{-1}3 = \dfrac{\pi}{4}$ **(Put values of x & y)**

6. $\text{Sin}^{-1}\dfrac{3}{5} + Sin^{-1}\dfrac{8}{17} = Sin^{-1}\dfrac{77}{85}$ **Multan 2008, 2009 Sargodha 2011**

Sol. we know that

$\text{Sin}^{-1} A + \text{Sin}^{-1} B = \text{Sin}^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)$

Put A=3/5 and B = 8/17

$$\text{Sin}^{-1}\frac{3}{5} + Sin^{-1}\frac{8}{17} = Sin^{-1}\left(\frac{3}{5}\left(\sqrt{1-\frac{64}{289}} + \frac{8}{17}\sqrt{1-\frac{9}{25}}\right)\right)$$

$$= \text{Sin}^{-1}\left(\frac{3}{5}\sqrt{\frac{289-64}{289}} + \frac{8}{17}\sqrt{\frac{16}{25}}\right) = Sin^{-1}\left(\frac{3}{5}.\frac{15}{17} + \frac{8}{17}.\frac{4}{5}\right)$$

$$= \text{Sin}^{-1}\left(\frac{9}{17} + \frac{32}{85}\right) = Sin^{-1}\left(\frac{45+32}{85}\right) = Sin^{-1}\left(\frac{77}{85}\right)$$

Hencd $\text{Sin}^{-1}\dfrac{3}{5} + Sin^{-1}\dfrac{8}{17} = Sin^{-1}\dfrac{77}{85}$

7. $\text{Sin}^{-1}\dfrac{77}{85} - Sin^{-1}\dfrac{3}{5} = Cos^{-1}\dfrac{15}{17}$ **Faisalabad 2008,Multan 2009, Lahore 2009**

Sol. Take $\text{Sin}^{-1}\dfrac{77}{85} = \alpha$, $Sin^{-1}\dfrac{3}{5} = \beta$

$\Rightarrow Sin\alpha = \dfrac{77}{85}$, $\quad Sin\beta = \dfrac{3}{5}$

$\text{Cos}^2\alpha = 1 - \text{Sin}^2\alpha$

$\text{Cos}^2\alpha = 1 - \left(\frac{77}{85}\right)^2$, $Cos^2\beta = 1 - Sin^2\beta$

$\text{Cos}^2\alpha = 1 - \frac{5929}{7225}$, $Cos^2\beta = 1 - \left(\frac{3}{5}\right)^2$

$\text{Cos}^2\alpha = \frac{7225 - 5929}{7225}, Cos\beta = \frac{4}{5}$ (Cos in +ve be in Domain of Sin)

$\text{Cos}^2\alpha = \frac{1296}{7225} \Rightarrow \text{Cos}\,\alpha = \frac{36}{85}$ (Cos is +ve in Domain of Sine)

Now $\text{Cos}(\alpha - \beta) = \text{Cos}\,\alpha\ \text{Cos}\,\beta + \text{Sin}\,\alpha\ \text{Sin}\,\beta$

$$= \left(\frac{35}{85}\right)\left(\frac{4}{5}\right) + \left(\frac{77}{85}\right)\left(\frac{3}{5}\right) = \frac{144}{425} + \frac{231}{425} = \frac{375}{425} = \frac{15}{17}$$

$\alpha - \beta = \text{Cos}^{-1}\left(\frac{15}{17}\right) \Rightarrow \text{Sin}^{-1}\frac{77}{85} - Sin^{-1}\frac{3}{5} = Cos^{-1}\frac{15}{17}$ (*Put values of* α & β)

Hence proved.

8. $\mathbf{Cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5} = Sin^{-1}\frac{3}{5}}$ **Faisalabad 2008, Sgd 2009**

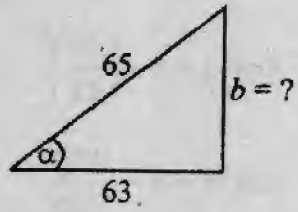
Sol. $\text{Cos}^{-1}\frac{63}{65} + \tan^{-1}\left(\frac{2.\frac{1}{5}}{1 - \left(\frac{1}{5}\right)^2}\right) = Sin^{-1}\frac{3}{5}$ $\boxed{\text{Use } 2\tan^{-1}A = \tan^{-1}\frac{2A}{1-A^2}}$

$\text{Cos}^{-1}\frac{63}{65} + \tan^{-1}\left(\frac{\frac{2}{5}}{1 - \frac{1}{25}}\right) = Sin^{-1}\frac{3}{5}$

$\text{Cos}^{-1}\frac{63}{65} + \tan^{-1}\frac{\frac{2}{5}}{\frac{24}{25}} = Sin^{-1}\frac{3}{5}$

$$\cos^{-1}\frac{63}{65}+\tan^{-1}\left(\frac{2}{5}\times\frac{25}{24}\right)=Sin^{-1}\frac{3}{5}$$

$$\cos^{-1}\frac{63}{65}+\tan^{-1}\frac{5}{12}=Sin^{-1}\frac{3}{5}$$

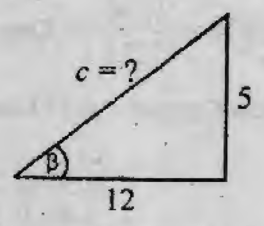


Let $\cos^{-1}\frac{63}{65}=\alpha$ & $\tan^{-1}\frac{5}{12}=\beta$

$$\boxed{\begin{aligned}&b^2+(63)^2=(65)^2\\&b=16\end{aligned}}$$

$$\cos\alpha=\frac{63}{65}\ \&\ \tan\beta=\frac{5}{12}$$

$$\sin\alpha=\frac{16}{65},\ Cos\,\beta=\frac{12}{13}$$



$$\boxed{\begin{aligned}&c^2=(12)^2+(5)^2\\&c=13\end{aligned}}$$

Now $\sin(\alpha+\beta)=\sin\alpha\ \cos\beta+\cos\alpha\ \sin\beta$

$$=\left(\frac{16}{65}\right)\left(\frac{12}{13}\right)+\left(\frac{63}{65}\right)\left(\frac{5}{13}\right)=\frac{192}{845}+\frac{315}{845}$$

$$=\frac{192+315}{845}=\frac{507}{845}=\frac{3}{5}$$

$$\alpha+\beta=\sin^{-1}\left(\frac{3}{5}\right)\Rightarrow\cos^{-1}\frac{63}{65}+\tan^{-1}\frac{5}{12}=Sin^{-1}\frac{3}{5}\ \text{(Put values of } \alpha\ \&\ \beta\text{)}$$

9. $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}=\frac{\pi}{4}$ **Multan 08, Fsd 09, Guj 09, Sgd 2010**

Sol. $$\text{L.H.S}=\tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}.\frac{3}{5}}\right)-\tan^{-1}\frac{8}{19}=\tan^{-1}\left(\frac{\frac{27}{20}}{1-\frac{9}{20}}\right)-\tan^{-1}\frac{8}{19}=\tan^{-1}\frac{\frac{27}{\cancel{20}}}{\frac{11}{\cancel{20}}}-\tan^{-1}\frac{8}{19}$$

$$=\tan^{-1}\left(\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}.\frac{8}{19}}\right)=\tan^{-1}\left(\frac{\frac{513-88}{209}}{\frac{209+216}{209}}\right)=\tan^{-1}\left(\frac{\frac{\cancel{425}}{\cancel{209}}}{\frac{\cancel{425}}{\cancel{209}}}\right)=\tan^{-1}(1)=\frac{\pi}{4}=R.H.S.$$

10. $\text{Sin}^{-1}\frac{4}{5}+Sin^{-1}\frac{5}{13}+Sin^{-1}\frac{16}{65}=\frac{\pi}{12}$ **Federal Board**

Sol. L.H.S $= \text{Sin}^{-1}\frac{4}{5}+Sin^{-1}\frac{5}{13}+Sin^{-1}\frac{16}{65}$

$$= \text{Sin}^{-1}\left(A\sqrt{1-B^2}+B\sqrt{1-A^2}\right)+Sin^{-1}\frac{16}{65}$$

$$= \text{Sin}^{-1}\left(\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}+\frac{5}{13}\sqrt{1-\frac{16}{25}}\right)+Sin^{-1}\frac{16}{65}$$

$$= \text{Sin}^{-1}\left(\frac{4}{5}\sqrt{\left(\frac{144}{169}\right)}+\frac{5}{13}\sqrt{\frac{9}{25}}\right)+Sin^{-1}\frac{16}{65}$$

$$= \text{Sin}^{-1}\left(\frac{4}{5}.\frac{12}{13}+\frac{5}{13}.\frac{3}{5}\right)+Sin^{-1}\frac{16}{65}$$

$$= \text{Sin}^{-1}\left(\frac{48}{65}+\frac{15}{65}\right)+Sin^{-1}\frac{16}{65} = \text{Sin}^{-1}\left(\frac{63}{65}\right)+Sin^{-1}\frac{16}{65}$$

$$= \text{Sin}^{-1}\left(A\sqrt{1-B^2}+B\sqrt{1-A^2}\right)$$

$$= \text{Sin}^{-1}\left(\frac{63}{65}\sqrt{1-\left(\frac{16}{65}\right)^2}+\frac{16}{65}\sqrt{1-\left(\frac{63}{65}\right)^2}\right)$$

$$= \text{Sin}^{-1}\left(\frac{63}{65}\sqrt{1-\frac{256}{4225}}+\frac{16}{65}\sqrt{1-\frac{3969}{4225}}\right)$$

$$= \text{Sin}^{-1}\left(\frac{63}{65}\sqrt{\frac{4225-256}{4225}}+\frac{16}{65}\sqrt{\frac{4225-3969}{4225}}\right)$$

$$= \text{Sin}^{-1}\left(\frac{63}{65}.\sqrt{\frac{3969}{4225}}+\frac{16}{65}\sqrt{\frac{256}{4225}}\right)$$

$$= \text{Sin}^{-1}\left(\frac{3969}{4225}+\frac{256}{4225}\right) = Sin^{-1}\left(\frac{4225}{4225}\right) = \text{Sin}^{-1}(1) = \frac{\pi}{2} = R.H.S$$

11. $\tan^{-1}\dfrac{1}{11}+\tan^{-1}\dfrac{6}{5}=\tan^{-1}\dfrac{1}{3}+\tan^{-1}\dfrac{1}{2}$ **Multan 2008, Fsd 2009, Sgd 2010**

Sol. L.H.S $=\tan^{-1}\dfrac{1}{11}+\tan^{-1}\dfrac{5}{6}$

$$=\tan^{-1}\left(\frac{A+B}{1-AB}\right)=\tan^{-1}\left(\frac{\frac{1}{11}+\frac{5}{6}}{1-\left(\frac{1}{11}\right)\left(\frac{5}{6}\right)}\right)$$

$$=\tan^{-1}\left(\frac{\frac{6+55}{66}}{1-\frac{5}{66}}\right)=\tan^{-1}\frac{\frac{61}{66}}{\frac{61}{66}}=\tan^{-1}(1)=\frac{\pi}{4}$$

Now R.H.S $=\tan^{-1}\dfrac{1}{3}+\tan^{-1}\dfrac{1}{7}$

$$=\tan^{-1}\left(\frac{A+B}{1-AB}\right)=\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}.\frac{1}{2}}\right)=\tan^{-1}\left(\frac{\frac{5}{6}}{1-\frac{1}{6}}\right)=\tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right)=\tan^{-1}(1)=\frac{\pi}{4}$$

Hence L.H.S = R.H.S

12. $2\tan^{-1}\dfrac{1}{3}+\tan^{-1}\dfrac{1}{7}=\dfrac{\pi}{7}$ **Sargodha 2008 , Faisalabad 2008**

Sol. L.H.S $=2\tan^{-1}\dfrac{1}{3}+\tan^{-1}\dfrac{1}{7}$

$\boxed{\text{Use } 2\tan^{-1}A=\tan^{-1}\dfrac{2A}{1-A^2}}$

$$=\tan^{-1}\left(\frac{2.\frac{1}{3}}{1-\frac{1}{9}}\right)+\tan^{-1}\frac{1}{7}=\tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right)+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right)+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\left(\frac{3}{4}\right)\left(\frac{1}{7}\right)}\right)=\tan^{-1}\left(\frac{\frac{21+4}{28}}{1-\frac{3}{28}}\right)=\tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right)=\tan^{-1}(1)=\frac{\pi}{4}=R.H.S$$

13. **$\cos(\sin^{-1}x) = \sqrt{1-x^2}$**

Sol. Let $\sin^{-1}x = \alpha \Rightarrow \sin\alpha = x$

$\cos^2\alpha = 1 - \sin^2\alpha = 1 - x^2$

$\cos\alpha = \sqrt{1-x^2}$ ($\cos\alpha$ is +ve in Domain of $\sin\alpha$)

$\cos(\sin^{-1}x) = \sqrt{1-x^2}$ *(Put values of α)*

14. **$\sin(2\cos^{-1}x) = 2x\sqrt{1-x^2}$**

Sol. Take $\cos^{-1}x = \alpha \Rightarrow \cos\alpha = x$

$\Rightarrow \sin^2\alpha = 1 - \cos^2\alpha \Rightarrow \sin\alpha = \sqrt{1-\cos^2\alpha}$

$\sin\alpha = \sqrt{1-x^2}$

Now $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\sin 2(\cos^{-1}x) = 2\sqrt{1-x^2}\,.x = 2x\sqrt{1-x^2}$ *(Put values of α)*

15. **$\cos(2\sin^{-1}x) = 1 - 2x^2$** **Faisalabad 2007, 09 , Federal**

Sol. Take $\sin^{-1}x = \alpha \Rightarrow \sin\alpha = x$

$\cos 2\alpha = 1 - 2\sin^2\alpha$

$\cos(2\sin^{-1}x) = 1 - 2x^2$ *(Put values of α)*

16. **$\tan^{-1}(-x) = -\tan^{-1}x$**

Sol. or $\tan^{-1}(-x) + \tan^{-1}x = 0$

$$\text{L.H.S} = \tan^{-1}\left(\frac{-x+x}{1-(-x)(x)}\right) = \tan^{-1}\left(\frac{0}{1+x^2}\right) = \tan^{-1}(0) = 0$$

$\Rightarrow \tan^{-1}(-x) + \tan^{-1}x = 0$

$\Rightarrow \tan^{-2}(-x) = -\tan^{-1}x$

17. **$\sin^{-1}(-x) = -\sin^{-1}x$** **Multan 2008, Sargodha 2008**

Sol. Let $\sin^{-1}(-x) = \alpha \Rightarrow -x = \sin\alpha$

'X' by -1 so $x = -\sin\alpha$ or $x = \sin(-\alpha)$

$\Rightarrow \sin^{-1}x = -\alpha \Rightarrow -\sin^{-1}x = \alpha$

or $\sin^{-1}(-x) = -\sin^{-1}(x)$ *(Put values of α)*

18. $\mathbf{Cos^{-1}(-x) = \pi - Cos^{-1}x}$

Sol. or $Cos^{-1}(-x) + Cos^{-1}x = \pi$

$Cos^{-1}\alpha + Cos^{-1}\beta = Cos^{-1}(\alpha\beta - \sqrt{(1-\alpha^2)(1-\beta^2)})$

Put $\alpha = -x$ & $\beta = x$

$$Cos^{-1}(-x) + Cos^{-1}x = Cos^{-1}((-x)(x) - \sqrt{(1-(-x)^2(1-x^2)})$$

$$= Cos^{-1}(-x^2 - \sqrt{(1-x^2)(1-x^2)})$$

$$= Cos^{-1}(-x^2 - \sqrt{(1-x^2)^2})$$

$$= Cos^{-1}(-x^2 - (1-x^2))$$

$$= Cos^{-1}\left(-\cancel{x^2} - 1 + \cancel{x^2}\right) = Cos^{-1}(-1)$$

$Cos^{-1}(-x) + Cos^{-1}x = \pi \Rightarrow Cos^{-1}(-x) = \pi - Cos^{-1}x$

19. $\mathbf{tan(Sin^{-1}x) = \dfrac{x}{\sqrt{1-x^2}}}$

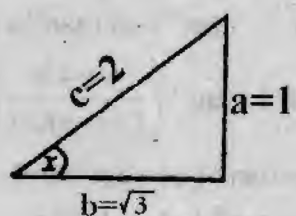
Sol. take $Sin^{-1}x = \alpha \Rightarrow Sin\,\alpha = x$

$Cos^2\alpha = 1 - Sin^2\alpha = 1 - x^2$

$Cos\,\alpha = \sqrt{1-x^2}$, $tan\,\alpha = \dfrac{Sin\alpha}{Cos\alpha}$

$Tan\,\alpha = \dfrac{x}{\sqrt{1-x^2}} \Rightarrow tan(Sin^{-1}x) = \dfrac{x}{\sqrt{1-x^2}}$

20. $\mathbf{x = Sin^{-1}\dfrac{1}{2} \Rightarrow Sin\,x = \dfrac{1}{2}}$



Sol. Now $Sin\,x = \dfrac{1}{2}$, $Cosec\,x = 2$

$Cos\,x = \dfrac{\sqrt{3}}{2}$, $Sec\,x = \dfrac{1}{\frac{\sqrt{3}}{2}} = \dfrac{2}{\sqrt{3}}$

$tan\,x = \dfrac{1}{\sqrt{3}}$, $Cot\,x = \dfrac{1}{\frac{1}{\sqrt{3}}} = \sqrt{3}$

$$\boxed{\begin{array}{l} a^2 + b^2 = c^2 \Rightarrow b^2 = c^2 - a^2 \\ b^2 = 2^2 - 1^2 = 4 - 1 = 3 \Rightarrow b = \sqrt{3} \end{array}}$$

TEST YOUR SKILLS **Marks: 30**

Q # 1. Select the Correct Option

i. For $-\pi/2 \le \theta \le \pi/2$, $Sin^{-1}(-1/2) = \theta =$ is

a) $-\pi/3$ b) $\pi/3$

c) $\pi/6$ d) $-\pi/6$

ii. Range of the function $y = Cos^{-1}x$ is

a) $0 \le y \le \pi$ b) $0 < y < \pi$

c) $-1 \le y \le 1$ d) $-1 < y < 1$

iii. $Cos^{-1}(-x) =$

a) $Cos^{-1}x$ b) $-Cos^{-1}x$

c) $\pi - Cos^{-1}x$ d) $\pi + Cos^{-1}x$

iv. $Tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) =$

a) $\pi/3$ b) $-\pi/3$

c) $\pi/6$ d) $-\pi/6$

Q # 2. Short Questions:

i. Prove that $Tan^{-1}x = \frac{\pi}{2} - Cot^{-1}x$

ii. Prove that $Tan^{-1}1/4 + Tan^{-1}1/5 = Tan^{-1}9/19$

iii. Find the Value of $Sec(Sin^{-1}(-1/2))$

Long Questions:

Q # 3. (a) Without using Calculator prove that $Sin^{-1}\frac{1}{\sqrt{5}} + Cot^{-1}3 = \frac{\pi}{4}$

(b) Prove that $\tan^{-1}\frac{1}{11} + \tan^{-1}\frac{5}{6} = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}$

Q # 4. (a) Prove that $Sin^{-1}\frac{5}{13} + Sin^{-1}\frac{7}{25} = Cos^{-1}\frac{253}{325}$

(b) Prove that $Cos^{-1}\frac{63}{65} + 2Tan^{-1}\frac{1}{5} = Sin^{-1}\frac{3}{5}$

Solution of Trigonometric Equations

14

Trigonometric equations: **Sargodha 2009, Multan 2009, Lahore 2009**

The equations, containing at least one trigonometric function, are called trigonometric equations. e.g.

Sin$x$ = $\frac{2}{5}$, Sec$x$ = tan$x$, Sin^2 – Sec$x$ + 1 = $\frac{3}{4}$

Example 1: **Solve sin$x$=1/2** **Sgd 2006,09, Multan 2008,09, Fsd 2008**

Sol. Sinx is positive in I & II quadrant

Reference angle = x = $\sin^{-1} 1/2 = \pi/6$ —— *in I quad*

$x = \pi - \pi/6 = 5\pi/6$ —— *in II quad*

$$S.S = \left\{\frac{\pi}{6} + 2n\pi\right\} \cup \left\{\frac{5\pi}{6} + 2n\pi\right\}, n \in Z$$

Example 2: **Solve 1 + cos$x$=0** **Sargodha 2009,10, Fsd 2009, Gujranwala 2009**

Sol. $1 + \text{Cos}\, x = 0 \Rightarrow \text{Cos}\, x = -1$

There is only one solution, $x = \pi$ in $[0, 2\pi]$. Since 2π is period of Cos$x$

$\therefore$ General value of x is $\pi + 2n\pi$, n $\in$ Z.

$S.S = \{\pi + 2n\pi\}, n \in Z$

Example 1(of general solution): Solve Sin x + Cos x = 0 **Sargodha 2008**

Sol. Sin$x$ + Cos$x$ = 0

$\Rightarrow \frac{Sinx}{Cosx} + \frac{Cosx}{Cosx} = 0 \Rightarrow \text{Tan}\, x + 1 = 0 \Rightarrow \tan x = -1$

$\because$ Tan x is –ve in II and IV Quadrant with the reference angle = $\frac{\pi}{4}$

$\therefore \quad x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ *in I quad*

x = $2\pi - \pi/4 = 7\pi/4$ but not in $[0, \pi]$ so it is not solution

$\therefore$ General value of x is $\frac{3\pi}{4} + n\pi$

$\therefore$ Solution set = $\left\{\frac{3\pi}{4} + n\pi\right\}$, n $\in$ Z

Example3. Solve the equation Sin2x = Cos x **Sgd 07, Multan 07, Rwl 09, Federal**

Sol. Sin2x = Cos x $\Rightarrow$ 2Sin x Cos x = Cos x

$\Rightarrow$ 2Sinx Cos x – Cos x = 0 $\Rightarrow$ Cos x (2Sinx – 1) = 0

Either Cos x = 0 or 2Sinx – 1 = 0

(i). If Cos x = 0

$\Rightarrow$ x= $\frac{\pi}{2}$ and x= $\frac{3\pi}{2}$ where x $\in$ $[0, 2\pi]$ As 2π is period of Cos x

$\therefore$ General value of x are $\frac{\pi}{2} + 2n\pi$ & $\frac{3\pi}{2} + 2n\pi$, n $\in$ z

(ii). If 2Sinx – 1 = 0 $\Rightarrow$ Sin x = $\frac{1}{2}$

Since Sinx is +ve in I and II quadrant with reference angle = $\frac{\pi}{6}$

$\therefore$ x = $\frac{\pi}{6}$ and x = $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$ where x = $\in$ $[0, 2\pi]$

$\therefore$ General values of x are $\frac{\pi}{6} + 2n\pi$ and $\frac{5\pi}{6} + 2n\pi$, n $\in$ Z

Hence

Solution set = $\{\frac{\pi}{2} + 2n\pi\} \cup \{\frac{3\pi}{2} + 2n\pi\} \cup \{\frac{\pi}{6} + 2n\pi\} \cup \{\frac{5\pi}{6} + 2n\pi\}$

EXERCISE 14

1. Find the solution of the following equations which lie in $[0, 2\pi]$

i. $\sin x = \dfrac{-\sqrt{3}}{2}$ **Multan 2009**

Sol. Sinx is –ve in III & IV quadrant

and $x = \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{3}$

Therefore

$$x = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \text{ in III}$$

$$x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \text{ in IV}$$

ii. **Cosec θ = 2**

Sol. $\sin\theta = \dfrac{1}{2}$

$\sin\theta$ is +ve in I & II

Reference angle $= \theta = \sin^{-1}(1/2) = \dfrac{\pi}{6}$ in I

$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \text{ in II}$$

iii. **Secx = –2** **Multan 08, Guj 09, Rwl 09**

Sol. $\Rightarrow \cos x = -\dfrac{1}{2}$

Cos x is –ve in II & III quadrant

Reference angle $= x = \cos^{-1}\dfrac{1}{2} = \dfrac{\pi}{3}$

$$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \text{ in II}$$

$$x = \pi - \frac{\pi}{3} = \frac{4\pi}{3} \text{ in III}$$

iv. $\cot\theta = \dfrac{1}{\sqrt{3}}$ **Sargodha 2008**

Sol. $\Rightarrow \tan\theta = \sqrt{3}$

$\tan\theta$ is +ve in I & III quadrant

and $\theta = \tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$ in I

$$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \text{ in III}$$

2. Solve the following trigonometric equations.

i. $\tan^2\theta = \frac{1}{3}$ **Fsd 08, 09, Sgd 09 Federal**

Sol. $\Rightarrow \tan\theta = \pm\frac{1}{\sqrt{3}}$

When $\tan\theta = \frac{1}{\sqrt{3}}$

θ is +ve in I & III quadrant

$\theta = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$ in I

$\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ in III

When $\tan\theta = \frac{-1}{\sqrt{3}}$

Tan θ is –ve in II & IV quad

$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ in II

$\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$ in IV

So $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

iii. $\mathbf{Sec^2\theta = \frac{4}{3}}$ **Multan 08, Fsd 09**

Sol. $\Rightarrow \sec\theta = \pm\frac{2}{\sqrt{3}}$

$\Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2}$

When $\cos\theta = \frac{\sqrt{3}}{2}$

$\cos\theta$ is +ve in I & IV quadrant

$\theta = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$ in I

ii. $\mathbf{Cosec^2\theta = \frac{4}{3}}$ **Sgd 2011, Federal**

Sol. $\Rightarrow \text{Cosec}\,\theta = \pm\frac{2}{\sqrt{3}}$ or $\sin\theta = \pm\frac{\sqrt{3}}{2}$

When $\sin\theta = \frac{\sqrt{3}}{2}$

Sin is +ve in I & II and

$\theta = \sin\theta^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

$\theta = \frac{\pi}{3}$ in I

$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ in II

When $\sin\theta$ is –ve in III & IV

$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ in III

$\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ in IV

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

iv. $\mathbf{Cot^2\theta = \frac{1}{3}}$ **Lahore 2009**

Sol. $\Rightarrow \cot\theta = \pm\frac{1}{\sqrt{3}}$

Or $\tan\theta = \pm\sqrt{3}$

When $\tan\theta = \sqrt{3}$

θ is in I & III quadrant

$\theta = 2\pi - \dfrac{\pi}{6} = \dfrac{11\pi}{6}$ in IV

When $\text{Cos}\,\theta = -\dfrac{\sqrt{3}}{2}$

$\text{Cos}\,\theta$ is –ve in II & III quadrant

$\theta = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$ in III

$\theta = \pi + \dfrac{\pi}{6} = \dfrac{7\pi}{6}$ in IV

So $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

$\theta = \tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$ in I

$\theta = \pi + \pi/3 = \dfrac{4\pi}{3}$ in III

When $\tan\,\theta = -\sqrt{3}$

$\tan\theta$ is –ve = in II & IV

$\theta = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$ in II

$\theta = 2\pi - \dfrac{\pi}{3} = \dfrac{5\pi}{3}$ in IV

So $\theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

Find the values of θ satisfying the following equations:

3. $\mathbf{3\tan^2\theta + 2\sqrt{3}\tan\theta + 1 = 0}$

Sol. $(\sqrt{3}\tan\theta)^2 + 2\sqrt{3}\tan\theta + (1)^2 = 0$

$(\sqrt{3}\tan\theta + 1)^2 = 0$

$\sqrt{3}\tan\theta + 1 = 0 \Rightarrow \tan\theta = \dfrac{-1}{\sqrt{3}}$

Reference Angle $= \theta = \tan^{-1}\dfrac{1}{\sqrt{3}} = \dfrac{\pi}{6}$

$\theta = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$ in II and $\theta = 2\pi - \dfrac{\pi}{6} = \dfrac{11\pi}{6}$ in IV

4. $\mathbf{Tan^2\theta - Sec\theta - 1 = 0}$ **Federal**

Sol. or $\text{Sec}^2\theta - 1 - \text{Sec}\,\theta - 1 = 0$

$(\text{Sec}\,\theta - 1)(\text{Sec}\,\theta + 1) - (\text{Sec}\,\theta + 1) = 0$

$(\text{Sec}\,\theta + 1)[\text{Sec}\,\theta - 1 - 1] = 0 \Rightarrow (\text{Sec}\,\theta + 1)[\text{Sec}\,\theta - 2] = 0$

$\Rightarrow \text{Sec}\,\theta + 1 = 0$ or $\text{Sec}\,\theta - 2 = 0$

$\text{Sec}\,\theta = -1$ or $\text{Sec}\,\theta = 2$

$\text{Cos}\,\theta = -1$

$\theta = \pi$

$\text{Cos}\,\theta = \dfrac{1}{2}$

$\text{Cos}\,\theta$ is +ve in I & IV quadrant

$\theta = \text{Cos}^{-1}\dfrac{1}{2} = \dfrac{\pi}{3}$ in I

$\theta = 2\pi - \dfrac{\pi}{3} = \dfrac{5\pi}{3}$ in IV

5. $\mathbf{2\,Sin\,\theta + Cos^2\theta - 1 = 0}$

Sol. $2\text{Sin}\,\theta + \cancel{1} - \text{Sin}^2\theta - \cancel{1} = 0$

$2\text{Sin}\,\theta - \text{Sin}^2\theta = 0$

$\text{Sin}\,\theta\,(2 - \text{Sin}\,\theta) = 0$

$\text{Sin}\,\theta = 0$ or $2 - \text{Sin}\,\theta = 0$

$\theta = 0,\ \pi$ $\quad$ $\text{Sin}\,\theta = 2$ Not possible

6. $\mathbf{2Sin^2\theta - Sin\,\theta = 0 \Rightarrow Sin\,\theta\,(2Sin\,\theta - 1) = 0}$ **Multan 2007, Sargodha 2010**

$\text{Sin}\,\theta = 0$ or $2\text{Sin}\,\theta - 1 = 0$

$\theta = 0,\ \pi$ $\quad$ $\text{Sin}\,\theta = \dfrac{1}{2}$

$\text{Sin}\,\theta$ is +ve in I & II quadrant

$\theta = \text{Sin}^{-1}\dfrac{1}{2} = \dfrac{\pi}{2}$ in I

$\theta = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$ in II

Hence $\theta = 0,\ \pi,\ \dfrac{\pi}{6},\ \dfrac{5\pi}{6}$

7. $\mathbf{3Cos^2\theta - 2\sqrt{3}\,Sin\,\theta\,Cos\,\theta - 3Sin^2\theta = 0}$

Sol. $3\text{Cos}^2\theta - 2\sqrt{3}\,\text{Sin}\,\theta\,\text{Cos}\,\theta - 3\text{Sin}^2\theta = 0$ ('÷' by $\text{Sin}^2\theta$ we get)

$3\text{Cot}^2\theta - 2\sqrt{3}\,\text{Cot}\,\theta - 3 = 0$

Subtract and add $\sqrt{3}\,\text{Cot}\,\theta$

$3\text{Cot}^2\theta - 2\sqrt{3}\ \text{Cot}\theta - \sqrt{3}\ \text{Cot}\theta + \sqrt{3}\ \text{Cot}\theta - 3 = 0$

$3\text{Cot}^2\theta - 3\sqrt{3}\ \text{Cot}\theta + \sqrt{3}\ \text{Cot}\theta - \sqrt{3}\ \sqrt{3} = 0$

$3\text{Cot}\theta\ (\text{Cot}\theta - \sqrt{3}) + \sqrt{3}\ (\text{Cot}\theta - \sqrt{3}) = 0$

$(\text{Cot}\theta - \sqrt{3})\ (3\text{Cot}\theta + \sqrt{3}) = 0$

$\text{Cot}\theta - \sqrt{3} = 0$ or $3\text{Cot}\theta + \sqrt{3} = 0$

$\text{Cot}\theta = \sqrt{3}$

$\tan\theta = \dfrac{1}{\sqrt{3}}$

$\tan\theta$ is +ve in I & III

$\theta = \tan^{-1}\dfrac{1}{\sqrt{3}} = \dfrac{\pi}{6}$ in I

$\theta = \pi + \dfrac{\pi}{6} = \dfrac{7\pi}{6} =$ in III

or $\text{Cot}\theta = \dfrac{-1}{\sqrt{3}} = \dfrac{-\cancel{\sqrt{3}}}{\sqrt{3}\cancel{\sqrt{3}}}$

$\tan\theta = -\sqrt{3}$

$\tan\theta$ is –ve in II & IV

$\theta = \tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$

$\theta = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$ in II

$\theta = 2\pi - \dfrac{\pi}{3} = \dfrac{5\pi}{3}$ in IV

Hence $\theta = \dfrac{\pi}{6}, \dfrac{7\pi}{6}, \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

8. $\mathbf{4\text{Sin}^2\theta - 8\,\text{Cos}\theta + 1 = 0}$

Sol. $4(1 - \text{Cos}^2\theta) - 8\text{Cos}\theta + 1 = 0$

$4 - 4\text{Cos}^2\theta - 8\text{Cos}\theta + 1 = 0$

$-4\text{Cos}^2\theta - 8\text{Cos}\theta + 5 = 0$

$4\text{Cos}^2\theta + 8\text{Cos}\theta - 5 = 0$ (Multiplying by "–1")

$4\text{Cos}^2\theta + 10\text{Cos}\theta - 2\text{Cos}\theta - 5 = 0$

$2\text{Cos}\theta\ (2\text{Cos}\theta + 5) - 1\,(2\text{Cos}\theta + 5) = 0$

$(2\text{Cos}\theta + 5)\ (2\text{Cos}\theta - 1) = 0$

$2\text{Cos}\theta + 5 = 0$ or $2\text{Cos}\theta - 1 = 0$

$\text{Cos}\theta = \dfrac{-5}{2}$ Impassible or $\text{Cos}\theta = \dfrac{1}{2}$

$\theta = \text{Cos}^{-1}\dfrac{1}{2} = \dfrac{\pi}{3}$ in I

$$\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \text{ in IV}$$

So $\qquad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Find the solution set of the following equations.

9. $\sqrt{3}$ **tanx – Secx – 1 = 0** Note: Add $2n\pi$ in Cosx & Sinx and $n\pi$ in tanx. For sol

Sol. $\sqrt{3}\tan x - \sec x - 1 = 0$ —— I

$\sqrt{3}\tan x = \sec x + 1$

Squaring both sides

$3\tan^2 x = \sec^2 x + 2\sec x + 1$

$3(\sec^2 x - 1) = \sec^2 x + 2\sec x + 1$

$3\sec^2 x - 3 - \sec^2 x - 2\sec x - 1 = 0$

$2\sec^2 x - 2\sec x - 4 = 0 \Rightarrow \sec^2 x - \sec x - 2 = 0$ ($\div$ by 2)

$\sec^2 x - 2\sec x + \sec x - 2 = 0$

$\sec x(\sec x - 2) + 1(\sec x - 2) = 0$

$(\sec x - 2)(\sec x + 1) = 0$

$\sec x - 2 = 0$ or $\sec x + 1 = 0$

$\sec x = 2$ or $\sec x = -1$

$\cos x = \frac{1}{2}$ or $\cos x = -1$

Cosx is +ve I & IV

$x = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$ in I $\qquad x = \cos^{-1}(-1)$

$x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ in IV $\qquad x = \pi$

$5\pi/3$ Does not satisfies I equation

$$S.S = \left\{\frac{\pi}{3} + 2n\pi\right\} \cup \{\pi + 2n\pi\}, n \in Z$$

10. **Cos2x = Sin3x**

Sol. $1 - 2\text{Sin}^2x = 3\text{Sin}x - 4\text{Sin}^3x$

or $4\text{Sin}^3x - 2\text{Sin}^2x - 3\text{Sin}x + 1 = 0$

take Sinx = 1

$$\begin{array}{c|cccc} 1 & 4 & -2 & -3 & 1 \\ & & 4 & 2 & -1 \\ \hline & 4 & 2 & -1 & \boxed{0} \end{array}$$

$4\sin^3x - 2\sin^2x - 3\text{Sin}x+1 = (\text{Sin}x - 1)(4\text{Sin}^2x + 2\text{Sin}x - 1) = 0$

Sinx−1=0 or $4\text{Sin}^2x + 2\text{Sin}x - 1=0$

Sinx − 1 = 0	$\text{Sinx} = \dfrac{-2 \pm \sqrt{(2)-4(4)(-1)}}{2(4)}$
Sinx = 1	$= \dfrac{-2 \pm \sqrt{4+16}}{8} = \dfrac{-2 \pm \sqrt{20}}{8}$
$X = \dfrac{\pi}{2}$	$= \dfrac{-2 \pm 2\sqrt{5}}{8} = \dfrac{2(-1 \pm \sqrt{5})}{8}$
	$\text{Sinx} = \dfrac{-1-\sqrt{5}}{4} = -0.8090$
	$\text{Sinx} = \dfrac{-1+\sqrt{5}}{4} = 0.3090$

Sin x = 0.3090

Sinx +ve in I & II

$X = \text{Sin}^{-1}(0.3090)$

$X = 18^\circ = 18^\circ \times \dfrac{\pi}{180} = \dfrac{\pi}{10} \, in\, I$

$x = \pi - \dfrac{\pi}{10} = \dfrac{9\pi}{10}$ in II

Sinx = − 0.8090

Sinx is −ve in III & IV

$x = \text{Sin}^{-1}(0.8090)$

$x = \dfrac{3\pi}{10} = 54^\circ = 54 \times \dfrac{\pi}{180} = \dfrac{3\pi}{10}$

$x = \pi + \dfrac{3\pi}{10}$ in II

$x = 2\pi - \dfrac{3\pi}{10}\, x = \dfrac{17\pi}{10}$ in IV

$$S.S\left\{\frac{\pi}{10}+2n\pi\right\}U\left\{\frac{9\pi}{10}+2n\pi\right\}U\left\{\frac{17\pi}{10}+2n\pi\right\}U\left\{\frac{3\pi}{10}+2n\pi\right\}, n \in Z$$

11. **$\text{Sec}3\theta = \text{Sec}\theta$**

Sol. $\text{Sec}3\theta = \text{Sec}\theta \Rightarrow \text{Cos}3\theta = \text{Cos}\theta$

or $\text{Cos}3\theta - \text{Cos}\theta = 0$

$-2\text{Sin}\,\dfrac{3\theta+\theta}{2}\,\text{Sin}\,\dfrac{3\theta-\theta}{2} = 0$

$\Rightarrow \text{Sin}2\theta\ \text{Sin}\theta = 0$

$\Rightarrow \text{Sin}2\theta = 0$ or $\text{Sin}\theta = 0$

$2\theta = n\pi$ or $\theta = n\pi$

$\Rightarrow \theta = \dfrac{n\pi}{2}$

$S.S = \{n\pi\} \cup \left\{\dfrac{n\pi}{2}\right\}, n \in Z$

12. **$\tan2\theta + \text{Cot}\theta = 0$** **Multan 2008, Federal**

$\tan2\theta + \text{Cot}\theta = 0$

$$\frac{Sin2\theta}{Cos2\theta} + \frac{Cos\theta}{Sin\theta} = 0$$

$$\frac{Sin2\theta\ Sin\theta + Cos2\theta\ Cos\theta}{Sin\theta\ Cos2\theta} = 0$$

$\Rightarrow \quad \text{Cos}2\theta\ \text{Cos}\theta + \text{Sin}2\theta\ \text{Sin}\theta = 0$

$\Rightarrow \quad \text{Cos}(2\theta - \theta) = 0 \Rightarrow \text{Cos}\theta = 0 \Rightarrow \theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

$$S.S = \left\{\frac{\pi}{2} + 2n\pi\right\} \cup \left\{\frac{3\pi}{2} + 2n\pi\right\}, n \in Z$$

13. **Sin 2x + Sinx = 0** **Sargodha 2011, Federal**

Sol. or 2Sinx Cosx + Sinx = 0

Sinx (2Cosx + 1) = 0 $\Rightarrow$ Sinx = 0 or 2Cosx + 1 = 0 ;

$x = n\pi$ or Cosx = −1/2

Cosx is – ve in II & III

$$x = \text{Cos}^{-1}\frac{1}{2} = \frac{\pi}{3}$$

$$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \text{ in II}, \quad x = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \text{ in III}$$

$$S.S = \{n\pi\} \cup \left\{\frac{2\pi}{3} + 2n\pi\right\} \cup \left\{\frac{4\pi}{3} + 2n\pi\right\} \quad n \in Z$$

14. **Sin 4x – Sin 2x = Cos 3x**

Sol. $2\text{Cos}\frac{4x+2x}{2}\text{Sin}\frac{4x-2x}{2} = \text{Cos } 3x$

$2\text{Cos } 3x \text{ Sin}x - \text{Cos } 3x = 0 \Rightarrow \text{Cos } 3x\,(2\text{Sin}x - 1) = 0$

$\text{Cos } 3x = 0$ or $2\text{Sin}x - 1 = 0$

$3x = \frac{\pi}{2},\ 3x = \frac{3\pi}{2}$ or $\text{Sin}x = \frac{1}{2}$

$3x = \frac{\pi}{2} + 2n\pi,\ 3x = \frac{3\pi}{2} + 2n\pi$ or Sinx is + ve in I & II

$x = \frac{\pi}{6} + \frac{2n\pi}{3},\ x = \frac{\pi}{2} + \frac{2n\pi}{3}$ or $x = \text{Sin}^{-1}\frac{1}{2} = \frac{\pi}{6}$ in I

or $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ in II

$$S.S = \left\{\frac{\pi}{6} + \frac{2n\pi}{3}\right\} \cup \left\{\frac{\pi}{2} + \frac{2n\pi}{3}\right\} \cup \left\{\frac{\pi}{6} + 2n\pi\right\} \cup \left\{\frac{5\pi}{6} + 2n\pi\right\} \quad n \in Z$$

15. **Sinx + Cos3x = Cos5x** **Multan 2007**

Sol. or $\text{Cos}5x - \text{Cos}3x - \text{Sin}x = 0$

$$-2\text{Sin}\frac{5x+3x}{2}\text{Sin}\frac{5x-3x}{2} - \text{Sin}x = 0$$

$-2\text{Sin } 4x \text{ Sin}x - \text{Sin}x = 0 \Rightarrow -1\,[2\sin 4x \sin x + \text{Sin}x] = 0$

$\text{Sin}x\,(2\text{Sin}4x + 1) = 0$

$\text{Sin}x = 0$ or $2\text{Sin } 4x + 1 = 0$

$x = 0, \pi$ or $2\text{Sin } 4x = -1$

$$\text{Sin } 4x = \frac{-1}{2} \Rightarrow 4x = \text{Sin}^{-1}\frac{1}{2} = \frac{\pi}{6}$$

Sinx is – ve in III & IV quadrant

$$4x = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \text{ in III} \Rightarrow 4x = \frac{7\pi}{6} + 2n\pi \Rightarrow x = \frac{7\pi}{24} + \frac{n\pi}{2}$$

$$4x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} + 2n\pi \Rightarrow x = \frac{11\pi}{24} + \frac{n\pi}{2} \text{ in IV}$$

$$S.S = \{0 + 2n\pi\} \cup \{\pi + 2n\pi\} \cup \left\{\frac{7\pi}{24} + \frac{n\pi}{2}\right\} \cup \left\{\frac{11\pi}{24} + \frac{n\pi}{2}\right\}$$

16. **sin 3x + sin 2x + sin x = 0** **Faisalabad 2008**

Sol. Sin 3x + Sin 2x + Sinx = 0 or Sin 3x + Sinx + Sin2x = 0

$$2\text{Sin}\frac{3x+x}{2}\text{Cos}\frac{3x-x}{2} + \text{Sin } 2x = 0 \Rightarrow 2\text{Sin } 2x \text{ Cos } x + \text{Sin}2x = 0$$

$$\text{Sin } 2x\,(2\text{Cos}x + 1) = 0 \Rightarrow 2\text{ Cos}x + 1 = 0 \text{ or } \quad \text{Sin } 2x = 0$$

$$\text{If Sin } 2x = 0 \Rightarrow 2x = 0, \pi \Rightarrow 2x = 0 + 2n\pi \ \& \ 2x = \pi + 2n\pi \Rightarrow x = n\pi \ \& \ x = \frac{\pi}{2} + n\pi$$

If 2Cosx + 1 = 0 ; Cosx = –1/2

$$\text{Cos } x \text{ is – ve in II \& III } x = \text{Cos}^{-1}\frac{1}{2} = \frac{\pi}{3}$$

$$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \text{ in II \& } x = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \text{ in III}$$

$$S.S = \{n\pi\} \cup \left\{\frac{\pi}{2} + n\pi\right\} \cup \left\{\frac{4\pi}{3} + 2n\pi\right\} \cup \left\{\frac{2\pi}{3} + 2n\pi\right\} \ n \in Z$$

17. **Sin 7x – Sinx = Sin 3x**

Sol. $$2\text{Cos}\frac{7x+x}{3}\text{ Sin}\frac{7x-x}{3} - \text{Sin } 3x = 0$$

$$2\text{Cos } 4x \text{ Sin } 3x - \text{Sin } 3x = 0 \Rightarrow \text{Sin } 3x\,(2\text{Cos } 4x - 1) = 0$$

Sin3x = 0 or 2Cos 4x – 1=0

$$\text{If Sin } 3x = 0 \quad \Rightarrow 3x = 0, \pi \Rightarrow 3x = 0 + 2n\pi, 3x = \pi\, 2n\pi$$

$$x = \frac{2n\pi}{3} \text{ or } x = \frac{\pi}{3} + \frac{2n\pi}{3}$$

$2\text{Cos }4x - 1 = 0 \Rightarrow 4x = \text{Cos}^{-1}\dfrac{1}{2} = \dfrac{\pi}{3}$

$4x = \dfrac{\pi}{3} + 2n\pi$

$x = \dfrac{\pi}{12} + \dfrac{n\pi}{2}$ in I

$4x = 2\pi - \dfrac{\pi}{3} + 2n\pi$

$4x = \dfrac{5\pi}{3} + 2n\pi$

$x = \dfrac{5\pi}{12} + \dfrac{n\pi}{2}$ in IV

$$S.S = \left\{\frac{2n\pi}{3}\right\} \cup \left\{\frac{\pi}{3} + \frac{2n\pi}{3}\right\} \cup \left\{\frac{\pi}{12} + \frac{n\pi}{2}\right\} \cup \left\{\frac{5\pi}{12} + \frac{n\pi}{2}\right\}, n \in Z$$

18. **Sinx + Sin 3x + Sin 5x = 0**

Sol. Sinx + Sin 3x + Sin 5x = 0 or Sin 5x + Sinx + Sin 3x = 0

$2\text{Sin}\dfrac{5x+x}{2}\text{Cos}\dfrac{5x-x}{2} + \text{Sin }3x = 0 \Rightarrow 2\text{Sin }3x\text{ Cos}2x + \text{Sin }3x = 0$

Sin3x (2cos 2x + 1) = 0

Sin3x = 0	or 2Cos x + 1 = 0
$\Rightarrow 3x = 0, 3x = \pi$	$\text{Cos}x = \dfrac{-1}{2}$
$3x = 0 + 2n\pi$ & $3x = \pi + 2n\pi$	Cos x is – ve in II & III
$x = \dfrac{2n\pi}{3}$ & $x = \dfrac{\pi}{3} + \dfrac{2n\pi}{3}$	$2x = \text{Cos}^{-1}\dfrac{1}{2} = \dfrac{\pi}{3}$
	$2x = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3} + 2n\pi$
	$x = \dfrac{\pi}{3} + n\pi$ in II
	also $2x = \pi + \dfrac{\pi}{3} = \dfrac{4\pi}{3} + 2n\pi$
	$x = \dfrac{2\pi}{3} + n\pi$ in III

$$S.S = \left\{\frac{2n\pi}{3}\right\} \cup \left\{\frac{\pi}{3} + \frac{2n\pi}{3}\right\} \cup \left\{\frac{\pi}{3} + n\pi\right\} \cup \left\{\frac{2\pi}{3} + n\pi\right\}, n \in Z$$

19. $\mathbf{Sin\,\theta + Sin\,3\theta + Sin\,5\theta + Sin\,7\theta = 0}$

Sol. $\text{Sin}\,7\theta + \text{Sin}\,\theta + \text{Sin}5\theta + \text{Sin}3\theta = 0$

$$2\text{Sin}\left(\frac{7\theta+\theta}{2}\right)\text{Cos}\left(\frac{7\theta-\theta}{2}\right) + 2\text{Sin}\left(\frac{5\theta+3\theta}{2}\right)\text{Cos}\left(\frac{5\theta-3\theta}{2}\right)$$

$$2\text{Sin}\,4\theta\,\text{Cos}\,3\theta + 2\text{Sin}\,4\theta\,\text{Cos}\,\theta = 0 \Rightarrow 2\text{Sin}\,4\theta\,(\text{Cos}\,3\theta + \text{Cos}\,\theta) = 0$$

$$2\text{Sin}\,4\theta\left(2Cos\frac{3\theta+\theta}{2}\,Cos\frac{3\theta-\theta}{2}\right) = 0 \Rightarrow 4\text{Sin}\,4\theta\,\text{Cos}\,2\theta\,\text{Cos}\,\theta = 0$$

$\text{Sin}\,4\theta = 0,\qquad \text{Cos}2\theta = 0,\qquad \text{Cos}\,\theta = 0$

If $\text{Sin}\,4\theta = 0 \Rightarrow 4\theta = 0,\ \pi \Rightarrow 4\theta = 2n\pi\ \&\ 4\theta = \pi + 2n\pi \Rightarrow \theta = \frac{n\pi}{2}\ \&\ \theta = \frac{\pi}{4} + \frac{n\pi}{2}$

If $\text{Cos}\,2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} + 2n\pi\ \&\ 2\theta = \frac{3\pi}{2} + 2n\pi \Rightarrow \theta = \frac{\pi}{4} + n\pi\ \&\ \theta = \frac{3\pi}{4} + n\pi$

If $\text{Cos}\,\theta = 0 \Rightarrow \theta = \frac{\pi}{2} + 2n\pi\ \&\ \theta = \frac{3\pi}{2} + 2n\pi$

$$S.S = \left\{\frac{n\pi}{2}\right\} \cup \left\{\frac{\pi}{4} + \frac{n\pi}{2}\right\} \cup \left\{\frac{\pi}{4} + n\pi\right\} \cup \left\{\frac{3\pi}{4} + n\pi\right\} \cup \left\{\frac{\pi}{2} + 2n\pi\right\} \cup \left\{\frac{3\pi}{2} + 2n\pi\right\}, n \in Z$$

20. $\mathbf{Cos\,\theta + Cos\,3\theta + Cos\,5\theta + Cos\,7\theta = 0}$

Sol. $\text{Cos}7\theta + \text{Cos}\,\theta + \text{Cos}5\theta + \text{Cos}\,3\theta = 0$

$$2\text{Cos}\left(\frac{7\theta+\theta}{2}\right)\text{Cos}\left(\frac{7\theta-\theta}{2}\right) + 2\text{Cos}\left(\frac{5\theta+3\theta}{2}\right)\text{Cos}\left(\frac{5\theta-3\theta}{2}\right) = 0$$

$$2\text{Cos}\,4\theta\,\text{Cos}\,3\theta + 2\text{Cos}\,4\theta\,\text{Cos}\,\theta = 0 \Rightarrow 2\text{Cos}\,4\theta\,(\text{Cos}\,3\theta + \text{Cos}\,\theta)$$

$$2\text{Cos}\,4\theta = 0 \text{ or } 2\text{Cos}\,\frac{3\theta+\theta}{2}\,\text{Cos}\,\frac{3\theta-\theta}{2} = 0 \Rightarrow 2\text{Cos}\,2\theta\,\text{Cos}\,\theta = 0$$

$\text{Cos}\,4\theta = 0\qquad,\qquad \text{Cos}\,2\theta = 0\qquad,\qquad \text{Cos}\,\theta = 0$

If $\text{Cos}4\theta = 0 \Rightarrow 4\theta = \frac{\pi}{2} + 2n\pi\ \&4\theta = \frac{3\pi}{2} + 2n\pi \Rightarrow \theta = \frac{\pi}{8} + \frac{n\pi}{2}\ \&\theta = \frac{3\pi}{8} + \frac{n\pi}{2}$

If $\text{Cos}\,2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} + 2n\pi\ \&\ 2\theta = \frac{3\pi}{2} + 2n\pi \Rightarrow \theta = \frac{\pi}{4} + n\pi\,\&\theta = \frac{3\pi}{4} + n\pi$

If $\text{Cos}\,\theta = 0 \Rightarrow \theta = \frac{\pi}{2} + 2n\pi\ \&\ \theta = \frac{3\pi}{2} + 2n\pi$

$$S.S = \left\{\frac{\pi}{2} + 2n\pi\right\} \cup \left\{\frac{3\pi}{2} + 2n\pi\right\} \cup \left\{\frac{\pi}{4} + n\pi\right\} \cup \left\{\frac{3\pi}{4} + n\pi\right\} \cup \left\{\frac{\pi}{8} + \frac{n\pi}{2}\right\} \cup \left\{\frac{3\pi}{8} + \frac{n\pi}{2}\right\} n \in Z$$

TEST YOUR SKILLS Marks: 25

Q # 1. Select the Correct Option

i. Solution Set of $1+Cosx=0$ is

a) $\left\{\frac{\pi}{2}+2n\pi\right\}, n \in z$ b) $\{\pi+2n\pi\}, n \in z$

c) $\left\{\frac{\pi}{3}+2n\pi\right\}, n \in z$ d) None of these

ii. $Sinx=\frac{1}{2}$, x is equal to

a) $\frac{\pi}{2}$ b) $\frac{\pi}{6}$ c) $\frac{\pi}{4}$ d) $\frac{\pi}{3}$

iii. Number of solutions of trigonometric function is:

a) Finite b) Infinite

c) Only one d) None

iv. Number of solution of $1+Cosx=0$ are in $[0,2\pi]$:

a) 1 b) 2

c) Infinite d) 3

Q # 2. Short Questions:

i. Solve $Sin^2x=\frac{3}{4}$ in $[0,2\pi]$

ii. Solve $1+Cosx=0$

iii. Find solution set of $2Sin^2\theta-Sin\theta=0$

iv. Define trigonometric equations

v. Solve $Sinx=\frac{1}{2}$

vi. Solve $\tan x=\frac{1}{\sqrt{3}}$

vii. Solve $Cotx=\frac{1}{\sqrt{3}}, \theta \in [0,2\pi]$

viii. Solve $Sinx+Cosx=0$

ix. Solve $Sin2x+Sinx=0$

x. Find solution set of $Sin2x=Cosx$